Asia Pac. J. Math. 2025 **12**:81

ANALYTICAL CONSTRUCTION AND THEORETICAL PROPERTIES OF FUZZY ARCHIMEDEAN AND ELLIPTICAL COPULAS

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Received Jul. 8, 2025

Abstract. In this article, we propose an approach to modeling uncertain dependence through the construction of fuzzy copulas. We develop fuzzy Archimedean copulas (Clayton and Frank), fuzzy elliptical copulas (Gaussian), and a fuzzy extreme value copula (Gumbel). A comparative analysis with their classical counterparts is conducted. Finally, we introduce fuzzy association measures, such as fuzzy tau of Kendall based on fuzzy generator functions, to better represent the uncertainty present in the data.

2020 Mathematics Subject Classification. 03E72; 60A10; 62A86; 62H05.

Key words and phrases: copula; fuzzy set; fuzzy copula; Archimedean.

1. Introduction

Over the last few decades, copulas have gained considerable popularity, with applications in finance, the environment [11] and many other fields. A copula [1] is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform,

$$C(u) = P(U \leq u), \tag{1}$$

with $U = (U_1, ..., U_n)$, $u \in [0,1]^n$.

We can also refer to copulas as functions that couple multivariate functions to their one-dimensional marginal distribution functions. Using Sklar's theorem [1], any multivariate joint distribution can be expressed in terms of univariate marginal distribution functions and a copula that describes the dependency structure between variables,

$$H(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \tag{2}$$

Archimedean copulas are an important class of copulas that can be used to construct multivariate distributions from one-dimensional generating functions.

DOI: 10.28924/APJM/12-81

An *Archimedean copula* $C : [0,1]^n \to [0,1]$ is defined using a generator function Ψ, which is continuous, strictly decreasing, and convex, such that:

$$\Psi : [0, +\infty) \to [0,1], \quad \Psi(0) = +\infty, \quad \Psi(1) = 0, \tag{3}$$

and whose generalized pseudo-inverse Ψ^{-1} is completely monotonic on $[0, \infty)$. The copula C is then given by:

$$C(u_1, u_2, \dots, u_n) = \Psi^{-1}\left(\Psi(u_1) + \Psi(u_2) + \cdots + \Psi(u_n)\right), \tag{4}$$

for all $(u_1, u_2, \dots, u_n) \in [0,1]^n$ [10].

The usefulness of Archimedean copulas in particular lies in their simplicity, flexibility in modeling extreme and asymmetric dependencies, and ease of estimation. They also offer a more intuitive interpretation, while remaining powerful enough for applications in diverse fields such as finance, insurance and data science. For many practical situations where the dependence between variables is complex or asymmetric, archimedean copulas are an attractive solution. Archimedean copulas can capture both positive and negative dependencies between random variables. Now, suppose the random variables and factors are fuzzy; classical copulas are not suited to dealing with uncertainty and imprecision.

It is in this sense that we construct fuzzy archimedean copulas to take account of the fuzziness of the data. Researchers such as G. Hesamian [4], S . Giakoumakis and B.Papadopoulos [6] have constructed fuzzy copulas in general ways, but the construction of archimedian fuzzy copulas is relatively new. Researchers such as V. Ranjbar and G. Hesamian [3] have also tackled the notion of fuzzy correlation coefficient and other measures of association. But we construct the fuzzy correlation coefficient and the fuzzy Kendall's thau from the fuzzy generating functions of some fuzzy archimedean copulas. Recently, in 2020, some researchers have proposed aggregation operators based on archimedian copulas and cocopulas to deal with hesitant fuzzy information in multicriteria decision-making problems [19]. The limitation of their work, in relation to the construction of archimedian fuzzy copulas, lies in the fact that it focuses mainly on the use of classical archimedian copulas to define aggregation operators in a hesitant fuzzy framework, without delving into the specific construction of truly fuzzy copulas, which would require fuzzification of the parameters, variables or generating function itself.

Our contribution is distinguished by the construction of fuzzy parametric copulas by fuzzifying variables and parameters and combining fuzzy inccuracy with the flexibility of archimedean copulas. This opens up new perspectives for modeling compplex and uncertain phenomena. These constructed fuzzy copulas could be applied in epidemiology, given the nature of the data, in the environment, in multi-criteria optimization and in many other fields. In addition, the other original feature is the construction of fuzzy association measures from the fuzzy generating functions of archimedean copulas. The paper is organized as follows: In Section 2, we define the basic concepts required to understand the paper. In Section 3, we present our results, namely the construction of fuzzy Archimedean copulas, fuzzy elliptic copulas and the fuzzy extreme value copula, while making a comparative study between the classical copula and the fuzzy copula. We also construct the fuzzy Kendal tau from the generating functions of the fuzzy archimedean copulas. In Section 4, we end with a conclusion and outlook.

2. Preliminaries

2.1. Bivariate copula.

Definition 2.1. [1] *Any function C defined on* $[0,1]^2$ *to* $[0,1]$ *and having the following properties is called a bivariate copula:*

$$C(u,0) = C(0,v) = 0, \tag{5}$$

$$C(u,1) = u \qquad \text{and} \qquad C(1,v) = v, \tag{6}$$

$$C(u_1,v_1) + C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) \geq 0, \tag{7}$$

with $u_1 \leq v_1, u_2 \leq v_2$.

2.2. Archimedean copula.

Definition 2.2. [15] *A two-dimensional archimedean copula C is a function C:* $[0,1]^2 \to [0,1]$ *and defined by*

$$C(u,v) = \psi^{-1}(\psi(u) + \psi(v)), \tag{8}$$

where $\psi : [0,1] \to [0,+\infty]$ *is a function called the generator of the copula* C.

Definition 2.3. [1] *A copula* C *is convex if* $\forall (a,b),(c,d) \in [0,1]^2$ *and* $\forall \lambda \in [0,1]$,

$$C(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d) \leq \lambda C(a,b) + (1-\lambda)C(c,d). \tag{9}$$

Definition 2.4. [20] *Let* f *be a function defined from* A *into* $\mathbb{R}$. f *is said to be convex if* $\forall x,y \in A$ *and for* $\lambda \in [0,1]$ *we have:*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{10}$$

The convexity of a function can also be defined as follows:

Definition 2.5. [20] *Let* f *be a function defined from* I *to* $\mathbb{R}$ *twice derivable.* f *is convex if and only if* $f''(x) \geq 0, \forall x \in A$.

Definition 2.6. [1] *Let* $\psi : [0,1] \to [0,+\infty]$ *the generating function of an archimedean copula. We have:*

1. ψ *is convex.*
2. ψ *is continuous and strictly decreasing.*

(3) $\psi(1) = 0$ *et* $\psi(0) = +\infty$.

(4) ψ^{-1} *is the reciprocal of* ψ.

2.3. **Fuzzy set.**

Definition 2.7. [5] *Let* X *be a set called universe. A fuzzy set* $\tilde{A}$ *of* X *is defined by*

$$\tilde{A} = \{(x, u_{\tilde{A}}(x)), x \in X\}, \tag{11}$$

where $u_{\tilde{A}} : X \to [0,1]$ *is called the membership function of the fuzzy subset* $\tilde{A}$.

A subset of level $alpha$ noted $\tilde{A}^{[\alpha]}$ ([8]) or α cut is the set defined by

$$\tilde{A}^{[\alpha]} = \{x \in X \diagup u_{\tilde{A}}(x) \geq \alpha\}. \tag{12}$$

Definition 2.8. [12] *A fuzzy number is a convex and normalized fuzzy set of the set of real numbers.The set of fuzzy numbers is denoted by* $F(\mathbb{R})$. *To each fuzzy set at level* α, *we associate a fuzzy interval denoted by :*

$$\tilde{A}^{[\alpha]} = [\tilde{A}^L_\alpha, \tilde{A}^R_\alpha]. \tag{13}$$

Definition 2.9. [6] *Let* $\tilde{A}$ *be a fuzzy number and* $\alpha \in [0,1]$. *The* α *- pessimistic of* $\tilde{A}$ *denoted* $\tilde{A}_\alpha$ *is defined by*

$$\tilde{A}_\alpha = \begin{cases} \tilde{A}^L_{2\alpha} & \textit{if} \quad 0 \leq \alpha \leq \frac{1}{2}, \\ \tilde{A}^R_{2(1-\alpha)} & \textit{if} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{14}$$

Definition 2.10. [9] *Let* $\tilde{u}, \tilde{v} \in F(\mathbb{R})$, *the gH-difference is the fuzzy quantity* $\tilde{w} \in F(\mathbb{R})$ *if it exists, such that*

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Longleftrightarrow \begin{cases} \tilde{u} = \tilde{v} + \tilde{w}, \\ \textit{or} \quad \tilde{v} = \tilde{u} + (-1)\tilde{w}. \end{cases} \tag{15}$$

Definition 2.11. ([6]), [9] *Let* $\tilde{A}$ *and* $\tilde{B} \in F(\mathbb{R})$ *and* $\tilde{A}^{[\alpha]} = [\tilde{A}^L_\alpha, \tilde{A}^R_\alpha]$, $\tilde{B}^{[\alpha]} = [\tilde{B}^L_\alpha, \tilde{B}^R_\alpha]$:

(1)

$$(\tilde{A} \oplus \tilde{B})^{[\alpha]} = [\tilde{A}^L_\alpha + \tilde{B}^L_\alpha, \tilde{A}^R_\alpha + \tilde{B}^R_\alpha],$$

(2)

$$(\tilde{A} \ominus_{gH} \tilde{B})^{[\alpha]} = [\min\{\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^R_\alpha - \tilde{B}^R_\alpha\}, \max\{\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^R_\alpha - \tilde{B}^R_\alpha\}].$$

Definition 2.12. [14] *A fuzzy number* $\tilde{a}$ *is said to be triangular if it is of the form* $\tilde{a} = (a_1, a_2, a_3)$ *where* a_1, a_2 *and* a_3 *are real numbers and has its membership function defined by*

$$u_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \textit{if} \quad a_1 \leq x \leq a_2, \\ \dfrac{a_3 - x}{a_3 - a_2} & \textit{if} \quad a_2 \leq x \leq a_3, \\ 0 \quad \textit{else}. \end{cases} \tag{16}$$

Definition 2.13. [14] *Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. We have:*

(1) $\tilde{a} \approx \tilde{b} \Longleftrightarrow a_i = b_i, \forall i = 1, 2, 3$;

(2) $\tilde{a} \succeq \tilde{b} \Longleftrightarrow a_i \geq b_i, \forall i = 1, 2, 3$;

(3) $\tilde{a} \preceq \tilde{b} \Longleftrightarrow \begin{cases} a_1 \leq b_1, \\ a_1 - a_2 \leq b_1 - b_2, \\ a_1 + a_3 \leq b_1 + b_3. \end{cases}$

Definition 2.14. [17] *We say that $\tilde{f}(t)$ is a fuzzy function of $\mathbb{R}$ with value in $F(\mathbb{R})$ if it is a fuzzy number for all $t \in \mathbb{R}$.*

Definition 2.15. [13] *Let $\tilde{f} : [a, b] \to F(\mathbb{R})$ and $t_0 \in [a, b]$.*
We say that $\tilde{f}$ is differentiable in the generalized Hukuhara sense (gH-differentiable) in t_0 if there exists an element $\tilde{f}'(t_0) \in F(\mathbb{R})$ such that

$$\tilde{f}'(t_0) = \lim_{h \to 0^-} \frac{\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0)}{h}. \tag{17}$$

Definition 2.16. [18] *A fuzzy measure $\tilde{\mu}$ on a measurable space (X, M) is a fuzzy positive function defined on all sets of M that satisfies the following two conditions following conditions:*

(1) $\tilde{\mu}(\emptyset) = \tilde{0}$,

(2) *for any enumerable sequence $\{E_i\}$ of measurable sets disjoint sets,*

$$\tilde{\mu}(\cup_{i=1}^{\infty}) = \oplus_{i=1}^{\infty} \tilde{\mu}(E_i). \tag{18}$$

Definition 2.17. [18] *Let $(\Omega, \mathcal{A}, \mu)$be a measured space σ, $\widetilde{M} : \mathcal{A} \to F(\mathbb{R})$ a fuzzy measure, $S_{\widetilde{M}_\alpha}$ the set of all measurable selections of $\widetilde{M}_\alpha$ and $\tilde{F} : \Omega \to F(\mathbb{R})$ a function μ-integrable. The integral of a fuzzy function with respect to the fuzzy measure is defined by*

$$I(A) = \int_A \tilde{F} \, d\widetilde{M}, \tag{19}$$

and its $\alpha - cuts$ are defined as follows:

$$I_\alpha(A) = \int_A \tilde{F}_\alpha \, d\widetilde{M}_\alpha = \left\{ \int_A f \, dm : m \in S_{\widetilde{M}_\alpha} \text{and } f \in S_{\tilde{F}_\alpha} \right\}. \tag{20}$$

2.4. **Fuzzy random variable.**

Definition 2.18. [2] *Let $(\Omega, \mathbf{A}, P)$ be a probability space. A fuzzy random variable is an application of Ω to the set of fuzzy numbers $F(\mathbb{R})$.*

Definition 2.19. [7] *We call fuzzy $L - R$ random variable, the variable defined by*

$$\tilde{X}(w) = (\underline{X}(w), \overline{X}(w), a, b); \forall \omega \in \Omega, \tag{21}$$

where $\underline{X}$ and $\overline{X}$ are real random variables defined by $\underline{X}(\omega) = \inf \tilde{X}(\omega)$ and $\overline{X}(\omega) = \sup \tilde{X}(\omega)$, respectively.

3. Main results

3.1. Fuzzy archimedean copula.

Definition 3.1. *Let $\tilde{\theta} \in F(\mathbb{R})$ be a fuzzy number with membership function $\mu_{\tilde{\theta}}(\theta)$, and let $\varphi_\theta : [0,1] \to [0,\infty]$ be a classical Archimedean generator depending on a parameter θ.*

Then, the fuzzy generator $\tilde{\varphi}$ is defined as a family of fuzzy functions:

$$\tilde{\varphi}(t) = \left\{ \left(\varphi_\theta(t), \mu_{\tilde{\theta}}(\theta) \right) : \theta \in F(\mathbb{R}) \right\}, \quad \forall t \in [0,1] \tag{22}$$

Proposition 3.1. *Let $\alpha \in [0,1]$. Let $\tilde{C} : I \times I \longrightarrow F(\mathbb{R})$ be a function defined by:*

$$\tilde{C}(u,v)_\alpha = \tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha), \tag{23}$$

Then $\tilde{C}$ is a fuzzy copula called a fuzzy archimedean copula and $\tilde{\psi}$ is the fuzzy generating function of the fuzzy Archimedean copula $\tilde{C}$.

Proof. Let $\tilde{\theta}$ be a *fuzzy parameter* with membership function $\mu_{\tilde{\theta}}(\theta)$, and let $\psi_\theta : [0,1] \to [0,\infty]$ be a *classical Archimedean generator*, strictly decreasing, continuous, such that $\psi_\theta(1) = 0$ and $\psi_\theta(0) = \infty$.

The *fuzzy generator* is then defined by:

$$\tilde{\psi}(t) = \left\{ \left(\psi_\theta(t), \mu_{\tilde{\theta}}(\theta) \right) : \theta \in \Theta \right\}, \quad \forall t \in [0,1],$$

and its α-cut is:

$$[\tilde{\psi}(t)]^\alpha = \left\{ \psi_\theta(t) : \theta \in [\theta_\alpha^L, \theta_\alpha^U] \right\},$$

where $[\theta_\alpha^L, \theta_\alpha^U]$ is the α-cut of $\tilde{\theta}$.

We aim to construct a fuzzy copula $\tilde{C}$ via:

$$\tilde{C}(u,v)_\alpha = \tilde{\psi}^{-1}\left(\psi(u)_\alpha \oplus \psi(v)_\alpha \right),$$

where $\oplus$ denotes the fuzzy sum of intervals, defined by:

$$\psi(u)_\alpha \oplus \psi(v)_\alpha = \left[\min_{\theta \in [\theta_\alpha^L, \theta_\alpha^U]} \psi_\theta(u) + \psi_\theta(v), \ \max_{\theta \in [\theta_\alpha^L, \theta_\alpha^U]} \psi_\theta(u) + \psi_\theta(v) \right].$$

Since each ψ_θ is strictly decreasing, continuous, and invertible, the fuzzy inverse $\tilde{\psi}^{-1}$ is well defined. Therefore, we obtain:

$$\tilde{C}(u,v)_\alpha = \left\{ \psi_\theta^{-1}\left(\psi_\theta(u) + \psi_\theta(v) \right) : \theta \in [\theta_\alpha^L, \theta_\alpha^U] \right\},$$

which is the α-cut of a well-defined fuzzy set on $[0,1]$, for all $u, v \in [0,1]$. This ensures the existence of a *fuzzy Archimedean copula* $\tilde{C}(u,v)$ constructed from the fuzzy generator $\tilde{\psi}$.

Now, the aim is to show that the function $\tilde{C}$ verifies the properties of a copula.

$$\begin{aligned}
\tilde{C}(0,v)_\alpha &= \tilde{\psi}^{-1}(\psi(0)_\alpha \oplus \psi(v)_\alpha) \\
&= \tilde{\psi}^{-1}(+\infty \oplus \psi(v)_\alpha) \\
&= \tilde{\psi}^{-1}(+\infty) \\
&= \tilde{0}_\alpha.
\end{aligned}$$

$$\begin{aligned}
\tilde{C}(u,0)_\alpha &= \tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(0)_\alpha) \\
&= \tilde{\psi}^{-1}(\psi(u)_\alpha \oplus +\infty) \\
&= \tilde{\psi}^{-1}(+\infty) \\
&= \tilde{0}_\alpha.
\end{aligned}$$

$$\begin{aligned}
\tilde{C}(u,1)_\alpha &= \tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \phi(1)_\alpha) \\
&= \tilde{\psi}^{-1}((\psi(u)_\alpha \oplus \tilde{0}) \\
&= \tilde{\psi}^{-1}(\psi(u)_\alpha) \\
&= \tilde{u}_\alpha.
\end{aligned}$$

$$\begin{aligned}
\tilde{C}(1,v)_\alpha &= \tilde{\psi}^{-1}(\psi(1)_\alpha \oplus \phi(v)_\alpha) \\
&= \tilde{\psi}^{-1}((\tilde{0}_\alpha \oplus \phi(v)_\alpha) \\
&= \tilde{\psi}^{-1}(\psi(v)_\alpha) \\
&= \tilde{v}_\alpha.
\end{aligned}$$

Let's now show that $\tilde{C}(u,v)_\alpha$ is 2-increasing.

$$\begin{aligned}
\frac{\partial \tilde{C}(u,v)_\alpha}{\partial u} &= \frac{\partial}{\partial u}(\tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha)) \\
&= (\tilde{\psi}^{-1})'(\tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha)).\tilde{\psi}'(u)_\alpha \\
&= \frac{\tilde{\psi}'(u)_\alpha}{\psi'(\tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha))}
\end{aligned}$$

In additive, we have:

$$\begin{aligned}
\frac{\partial^2 \tilde{C}(u,v)_\alpha}{\partial v \partial u} &= \frac{\partial}{\partial v}\left(\frac{\tilde{\psi}'(u)_\alpha}{\psi'(\tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha))}\right) \\
&= \frac{-\tilde{\psi}'(u)_\alpha \tilde{\psi}'(v)_\alpha \tilde{\psi}''(\tilde{\psi}^{-1}(\tilde{\psi}(u)_\alpha \oplus \tilde{\psi}(v)_\alpha))}{(\tilde{\psi}'(\tilde{\psi}^{-1}(\tilde{\psi}(u)_\alpha \oplus \tilde{\psi}(v)_\alpha))^3}.
\end{aligned}$$

The function $\tilde{\psi}$ being strictly decreasing and convex, we have: $\psi'(t) < 0$ and $\psi''(t) > 0$. We deduce that

$$\frac{\partial^2 \tilde{C}(u,v)_\alpha}{\partial v \partial u} \geq 0.$$

Consequently, $\tilde{C}(u,v)_\alpha$ is 2- increasing. □

Definition 3.2. *For any $\alpha \in [0,1]$, the α-cut of the fuzzy archimedean copula $\tilde{C}$ is defined by*

$$\tilde{C}^{[\alpha]}(u,v) = [\tilde{\psi}^{-1}(\psi(u)_{\frac{\alpha}{2}} \oplus \psi(v)_{\frac{\alpha}{2}}, \tilde{\psi}^{-1}(\psi(u)_{1-\frac{\alpha}{2}} \oplus \psi(v)_{1-\frac{\alpha}{2}}). \tag{24}$$

In this subsection, we construct fuzzy archimedean copulas associated with the usual archimedean copulas.

3.1.1. *Fuzzy Clayton copula.*

exemple 3.1. *Let $\alpha \in [0,1]$ and $\tilde{\theta} \in \mathbb{F}(\mathbb{R})$. The function defined by $\tilde{\psi}(t,\tilde{\theta}) = t^{-\tilde{\theta}} - 1$ with values in $\mathbb{F}(\mathbb{R})$ is a strictly decreasing convex fuzzy function. Let $\tilde{C} : I \times I \longrightarrow F(\mathbb{R})$ defined by*

$$\tilde{C}(u,v)_\alpha = \frac{(\tilde{u}_\alpha^{-\tilde{\theta}} \oplus \tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1})^{-\frac{1}{\theta}} \oplus d_H((\tilde{u}_\alpha^{-\tilde{\theta}} \oplus \tilde{v}_\alpha^{-\tilde{\theta}}, \tilde{1})^{-\frac{1}{\theta}})}{2}. \tag{25}$$

Ten $\tilde{C}$ is a fuzzy Clayton copula.

Proof. Let's show that $\tilde{C}$ is well defined.

From $\tilde{\psi}(t) = t^{-\theta} - 1$, we derive that $\tilde{\psi}^{-1}(t) = (t+1)^{-\frac{1}{\theta}}$.

$$\begin{aligned}
\tilde{C}(u,v)_\alpha &= \max(0, \tilde{\psi}^{-1}(\psi(u)_\alpha \oplus \psi(v)_\alpha)) \\
&= \max(0, (u^{-\theta} \ominus_{gH} \tilde{1}_\alpha \oplus v^{-\theta} \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{1}_\alpha)) \\
&= \max(0, (\tilde{u}_\alpha^{-\tilde{\theta}} \oplus \tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1}_\alpha)^{-\frac{1}{\theta}}) \\
&= \frac{(\tilde{u}_\alpha^{-\tilde{\theta}} \oplus \tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1})^{-\frac{1}{\theta}} \oplus d_H((\tilde{u}_\alpha^{-\tilde{\theta}} \oplus \tilde{v}_\alpha^{-\tilde{\theta}}, \tilde{1})^{-\frac{1}{\theta}})}{2}.
\end{aligned}$$

So $\tilde{C}$ is well defined. Let's now check that $\tilde{C}$ verifies the properties of a copula.

$$\begin{aligned}
\tilde{C}(0,v)_\alpha &= \frac{\tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1}_\alpha)^{-\frac{1}{\theta}} \oplus d_H(\tilde{v}_\alpha^{-\tilde{\theta}}, \tilde{1}_\alpha)^{-\frac{1}{\theta}})}{2} \\
&= \frac{(\tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1}_\alpha)^{-\frac{1}{\theta}} \ominus_{gH} (\tilde{v}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1})^{-\frac{1}{\theta}}}{2} \\
&= \tilde{0}_\alpha.
\end{aligned}$$

Similarly, by symmetry, we show that $\tilde{C}(u,0)_\alpha = \tilde{0}_\alpha$.

$$\begin{aligned}
\tilde{C}(u,1)_\alpha &= \frac{\tilde{u}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{1})^{-\frac{1}{\theta}} \oplus |\tilde{u}_\alpha^{-\tilde{\theta}} \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{1})^{-\frac{1}{\theta}}|}{2} \\
&= \tilde{u}_\alpha.
\end{aligned}$$

By analogy, $\tilde{C}(1, v)_\alpha = \tilde{v}_\alpha$.

Since $\tilde{C}$ is convex, we deduce that $\tilde{C}$ is 2-increasing. □

The α-cuts for some α values are shown in Figure 1.

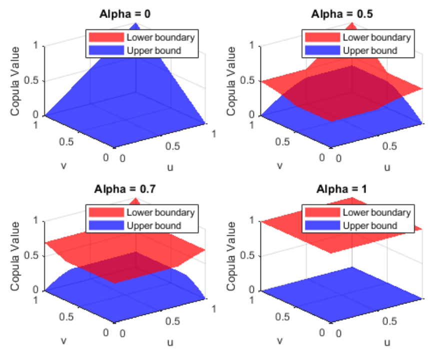

FIGURE 1. α-cuts of CLayton's fuzzy copula

The Figure 1 illustrates the subgraphs of a fuzzy Clayton copula for different levels of the copula (0, 0.5, 0.7, 1). Each subgraph represents two surfaces that delimit the lower and upper bounds of the copula values for a given value of α. The axis u and v represent the marginal variables, and the z axis corresponds to the value of the copula.

When $\alpha = 0$, the surfaces include all possible uncertainty, offering the widest ranges. As α increases, the surfaces become tighter, reducing uncertainty and converging towards deterministic values for $\alpha = 1$.

3.1.2. *Fuzzy Frank copula.*

Definition 3.3. *Let $\alpha \in [0, 1]$ and $\tilde{\theta} \in \mathbb{F}(\mathbb{R})$. The fuzzy copula with values in $\mathbb{F}(\mathbb{R})$ defined by*

$$\tilde{C}(u, v)_\alpha = -\frac{1}{\theta} \ln \left(1 \oplus \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right), \tag{26}$$

is Frank's fuzzy copula.

Remark 3.1. *the logarithm function ln is strictly increasing on $]0, +\infty[$, which allows it to be applied directly to the bounds of the α- cuts if the argument remains positive. It is therefore essential to ensure that the expression inside the logarithm stays strictly positive for each α, in order to guarantee the validity of the computation.*

3.1.3. *Comparative study.* In this paragraph, we want to make a comparative study between classical copulas and fuzzy copulas.

To do this, we fix u and v and calculate the different copulas for $\theta = 3$ and vary α. The results are shown in Table 1.

TABLE 1. Comparing values for $u = 0.5$ and $v = 0.7$.

α	θ^L	θ^R	Classical copula ($\theta = 3$)	Fuzzy copula
0	2	4	0.366	[0.307,0.412]
0.2	2.2	3.8	0.366	[0.315,0.400]
0.5	2.5	3.5	0.366	[0.328,0.382]
0.8	2.8	3.2	0.366	[0.339,0?370]
1	3	3	0.366	0.366

The Table 1 shows that the classical value ($\theta = 3$) is included in the fuzzy bounds for all levels of α. We can also see that the bounds tighten around the classical value for α tending towards 1. We can therefore conclude that the fuzzy copula is better suited to phenomena where the θ parameter is uncertain. The graphical representation of the two copulas is provided in Figure 2.

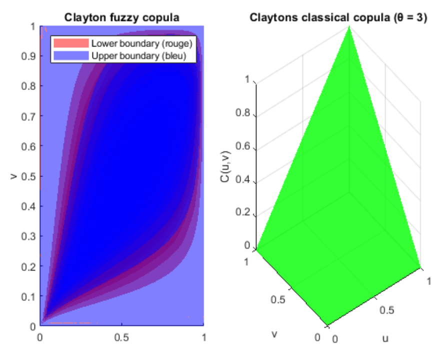

FIGURE 2. Clayton's classical copula and CLayton's fuzzy copula

The graph on the right represents the classic Clayton copula for $\theta = 3$. This surface is unique and describes a fixed dependency relationship between the variables u and v. The convex form reveals an increasing positive dependence, characteristic of the Clayton copula.

The graph on the left illustrates the lower (red) and upper (blue) bounds for the fuzzy Clayton copula, constructed from a triangular fuzzy.

The region between the bounds represents the uncertainty associated with the fuzzy parameter. This uncertainty decreases with increasing levels, gradually converging towards the classical copula when α approaches 1.

The fuzzy copula encompasses the surface of the classical copula, validating the robustness of the fuzzy model. This shows that the fuzzy copula can be seen as a generalization of the classical version for applications where the parameter is uncertain.

The fuzzy model is particularly useful in contexts where the parameter is not precisely known, for example in epidemiology or financial applications. The alignment of the bounds on the properties of the classical copula for high levels confirms the consistency and accuracy of the fuzzy construction method. This study validates the effectiveness of fuzzy copulas for modeling uncertain dependencies and underlines their applicability in fields requiring robust and flexible approaches.

Now we'll do a quantitative study by calculating metrics such as the mean deviation defined by

$$e = \frac{1}{n}\sum_{i=1}^{n}|x_i - \hat{x}_i|), \tag{27}$$

the root-mean-square distance defined by

$$DMQ = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{x}_i)^2)}, \tag{28}$$

and the tail dependence index for the clayton copula given by

$$\lambda = 2^{-\dfrac{1}{\theta}}. \tag{29}$$

The results of the calculations for the various metrics are shown in Table 2.

Table 2. Comparison of dependency indices and quantitative metrics between the classical and fuzzy Clayton copula.

Type of copula	λ^L	λ^R	**e**	DMQ^L	DMQ^R
Clayton's classical copula ($\theta = 3$)	$2^{-\frac{1}{3}} \approx 0.7937$	0	N/A	N/A	N/A
C^L	$2^{-\frac{1}{2}} \approx 0.7071$	0	0.243	0.152	0.215
C^R	$2^{-\frac{1}{4}} \approx 0.8409$	0	0.243	0.152	0.215

The λ^L index of fuzzy copulas varies according to the θ bounds, reflecting the flexibility of the fuzzy copula. The mean deviation $e = 0.243$ shows a moderate variation between the bounds of the fuzzy copulas. The values of DMQ indicate that the upper bound of the fuzzy copula is slightly closer to the classical copula than the lower bound.

Analysis of the metrics shows that the fuzzy copula is more flexible than the classical copula, offering intervals (upper and lower bounds) to better manage uncertainty in the data. The low values of the mean deviations e and DMQ between the fuzzy copula and the classical copula confirm that the fuzzy copula is capable of closely approximating the classical copula while adding a dimension of uncertainty. As a result, the fuzzy copula is better suited to contexts where parameters or data are imprecise or unclear. This adaptability confers superiority on fuzzy copulas in specific cases where uncertainty plays a crucial role.

3.2. **Fuzzy elliptical copula.** A fuzzy elliptic copula is a fuzzy extension of elliptic copulas, which themselves are derived from elliptic distributions such as the multivariate normal distribution or Student's law.

3.2.1. *Fuzzy Gaussian copula.*

Definition 3.4. *Let $I = [0, 1]$ and $\alpha \in I$.The function $\tilde{C} : I \times I \to F(\mathbb{R})$ definned by*

$$\tilde{C}_\alpha(u, v) = \tilde{\Phi}_\rho(\tilde{\Phi}^{-1}(u), \tilde{\Phi}^{-1}(v)), \tag{30}$$

is the fuzzy gaussian copula where $\tilde{\Phi}_\rho$ is the fuzzy bivariate distribution function of the normal distribution and $\tilde{\Phi}$ is the fuzzy distribution function of the reduced centered normal distribution.

The graphical representation of the fuzzy Gaussian copula is given in Figure 3.

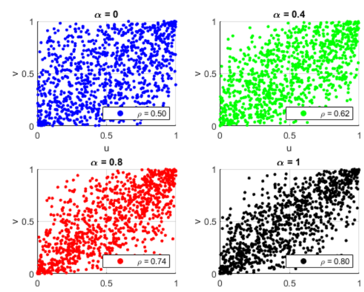

Figure 3. Gaussian copula for different values of α

The figure 4 illustrate the evolution of dependence in a fuzzy Gaussian copula as a function of the value of the parameter α. The latter modulates the ρ correlation between the u and v variables, evolving from $\rho = 0.5$ for $\alpha = 0$ to $\rho = 0.8$ for $\alpha = 1$. When $\alpha = 0$, the dependence is relatively weak, resulting in a more dispersed cloud of points. As α increases, we observe a progressive tightening of the points along the ascending diagonal, indicating an increasing correlation. For $\alpha = 1$, the dependency is maximal and the points are clearly aligned, showing a stronger relationship between the variables. This evolution highlights the role of the α parameter in managing fuzzy uncertainty: a low value of translates into higher variability of dependence, while a high value imposes a more rigid structure between variables.

In conclusion, the fuzzy Gaussian copula offers a flexible model of dependence, useful in epidemiology (for example, correlation between diseases and risk factors) or finance (dependence between assets under uncertainty). This approach makes it possible to incorporate controlled variability into the dependency structure between random variables.

3.3. **Fuzzy extreme value copulas.** A fuzzy extreme-value copula is a fuzzy extension of extreme-value copulas, which can be used to model asymptotic dependence between random variables in an uncertain setting.

Definition 3.5. *Let $\alpha \in [0,1]$. The function $\tilde{C} : I \times I \longrightarrow F(\mathbb{R})$ is said a fuzzy extreme value copula if*

$$\tilde{C}(u^n, v^n)_\alpha = \tilde{C}^n(u,v)_\alpha \tag{31}$$

3.3.1. *Fuzzy Gumbel copula.*

Proposition 3.2. *Let $\alpha \in [0,1]$ and $\tilde{\theta} \in \mathbb{F}(\mathbb{R})$. The fuzzy copula $\tilde{C} : I \times I \longrightarrow F(\mathbb{R})$ defined by*

$$\tilde{C}(u,v)_\alpha = \exp(-[(-\ln u)^{\tilde{\theta}} \oplus (-\ln v)^{\tilde{\theta}})]^{\frac{1}{\theta}}), \tag{32}$$

is a fuzzy extreme value copula.It is the fuzzy copula of the Gumbel family.

Proof. Firstly, the Gumbel copula is a fuzzy archimedian, so it is a fuzzy copula. Secondly, we'll show that it is a fuzzy extreme value copula.
We have:

$$\begin{aligned}
\tilde{C}(u^n, v^n) &= \exp(-\left[(-\ln u^n)^{\tilde{\theta}} \oplus (-\ln v^n)^{\tilde{\theta}})\right]^{\frac{1}{\theta}}) \\
&= \exp\left(-\left[(-n\ln u)^{\tilde{\theta}} \oplus (-n\ln v)^{\tilde{\theta}})\right]^{\frac{1}{\theta}}\right) \\
&= \exp\left(-\left[n^{\theta}((-\ln u)^{\tilde{\theta}} \oplus (-\ln v)^{\tilde{\theta}}))\right]^{\frac{1}{\theta}}\right) \\
&= \exp\left(-n\left[(-\ln u)^{\tilde{\theta}} \oplus (-\ln v)^{\tilde{\theta}})\right]^{\frac{1}{\theta}}\right)
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{C}(u,v))^n &= \left(\exp(-\left[(-\ln u)^{\tilde{\theta}} \oplus (-\ln v)^{\tilde{\theta}})\right]^{\frac{1}{\theta}})\right)^n \\
&= \exp\left(-n\left[(-\ln u)^{\tilde{\theta}} \oplus (-\ln v)^{\tilde{\theta}})\right]^{\frac{1}{\theta}}\right).
\end{aligned}$$

We deduce that

$$\tilde{C}(u^n, v^n) = \tilde{C}^n(u,v).$$

Therefore, $\tilde{C}$ is extreme fuzzy value copula. □

The graphical representation of the fuzzy density of the Gumbel family for $\alpha = 0.3$ is given in Figure 4.

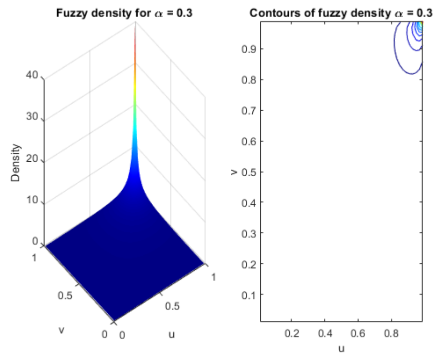

FIGURE 4. Fuzzy density of the Gumbel copula for $\alpha = 0.3$

The Figure 4 shows the fuzzy density and contours of the Gumbel copula for $\alpha = 0.3$. The fuzzy density (left) reveals a strong dependence between u and v in the areas close to the diagonal, but especially in the right-hand tails of the graphs, where the values of u and v are high. This extreme dependence is characteristic of the Gumbel copula, which models extreme events. The contours (right) clearly show these areas of high dependence, with contour lines closer together in regions where u and v are close to 1. The shape of the right tail, indicated by the tight curves in the contours, highlights the dependence in extreme values, where simultaneous high events for u and v are more likely to occur together.

3.4. **Fuzzy Kendall's tau.**

Definition 3.6. *Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy random variables with fuzzy copula $\tilde{C}$ whose fuzzy generating function is $\tilde{\psi}$. The fuzzy Kendall tau of $\tilde{X}$ and $\tilde{Y}$ is defined by*

$$\tilde{\tau}_\alpha = \tilde{1}_\alpha \oplus 4 \otimes \int_I \frac{\tilde{\psi}_\alpha(t)}{\tilde{\psi}'_\alpha(t)} dt. \tag{33}$$

Definition 3.7. *The α- Kendall's tau cut is defined by*

$$\tilde{\tau}^{[\alpha]} = [\tilde{\tau}^L_\alpha, \tilde{\tau}^R_\alpha]. \tag{34}$$

where

$$\tilde{\tau}^L_\alpha = inf\{1 + 4\int_I \frac{\tilde{\psi}_\alpha(t)}{\tilde{\psi}'_\alpha(t)} dt\},$$
$$\tilde{\tau}^R_\alpha = \sup\{1 + 4\int_I \frac{\tilde{\psi}_\alpha(t)}{\tilde{\psi}'_\alpha(t)} dt\}.$$

The generator for the crisp Clayton copula with parameter $\theta > 0$ is $\psi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$. We have:

$$\int_0^1 \frac{\psi_\theta(t)}{\psi'_\theta(t)} dt = \int_0^1 -\frac{1}{\theta}(t - t^{\theta+1}) dt$$

$$= -\frac{1}{\theta}\left[\frac{t^2}{2} - \frac{t^{\theta+2}}{\theta+2}\right]_0^1$$
$$= -\frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{\theta+2}\right) = -\frac{1}{\theta}\left(\frac{\theta+2-2}{2(\theta+2)}\right) = -\frac{1}{2(\theta+2)}.$$

Kendall's tau for the crisp copula is then given by the standard formula:

$$\tau_C = 1 + 4\int_0^1 \frac{\psi_\theta(t)}{\psi'_\theta(t)}dt = 1 + 4\left(-\frac{1}{2(\theta+2)}\right) = 1 - \frac{2}{\theta+2} = \frac{\theta}{\theta+2}.$$

To obtain Kendall's fuzzy tau $\tilde{\tau}_C$ for the fuzzy copula $C_{\tilde{\theta}}$, we apply Zadeh's Extension Principle to the derived crisp relationship $\tau_C = h(\theta) = \frac{\theta}{\theta+2}$. This involves replacing the crisp parameter θ with the fuzzy number $\tilde{\theta}$, the crisp number 2 with the fuzzy number $\tilde{2}$, and the arithmetic operations (addition, division) with their fuzzy counterparts ($\oplus$, fuzzy division):

$$\tilde{\tau}_C = h(\tilde{\theta}) = \frac{\tilde{\theta}}{\tilde{\theta} \oplus \tilde{2}}. \tag{35}$$

This result derived via the Extension Principle applied to the final crisp formula is consistent with applying the Extension Principle directly to the integral definition (**??**). The fuzzy integral $\int_0^1 \frac{\tilde{\psi}_{\tilde{\theta}}(t)}{\tilde{\psi}'_{\tilde{\theta}}(t)}dt$ evaluates, via the Extension Principle, to the fuzzy number obtained by applying the function $f(\theta) = -\frac{1}{2(\theta+2)}$ to $\tilde{\theta}$, denoted $f(\tilde{\theta})$. Substituting this into Eq. (**??**) gives $\tilde{\tau}_C = \tilde{1} \oplus 4 \otimes f(\tilde{\theta})$. This operation corresponds precisely to applying the function $h(\theta) = 1 + 4f(\theta) = 1 + 4(-\frac{1}{2(\theta+2)}) = \frac{\theta}{\theta+2}$ to the fuzzy number $\tilde{\theta}$, which yields $h(\tilde{\theta})$ as in Eq. (35).

Thus, we confirm that Kendall's fuzzy tau associated with Clayton's fuzzy copula is given by:

$$\tilde{\tau}_C = \tilde{1} \oplus 4 \otimes \int_0^1 \frac{\tilde{\psi}_{\tilde{\theta}}(t)}{\tilde{\psi}'_{\tilde{\theta}}(t)}dt = \frac{\tilde{\theta}}{\tilde{\theta} \oplus \tilde{2}}.$$

The α-cut $\tilde{\tau}$ bounds for some α values are recorded in Table 3.

TABLE 3. α-cutting τ for some α values

α	0	0.1	0.2	0.3	0.5	0.6	0.7	0.9	1
$\tilde{\tau}^L$	0.3333	0.375	0.41176	0.4444	0.5	0.52381	0.54545	0.5833	0.6
$\tilde{\tau}^R$	0.71429	0.70588	0.69697	0.6875	0.66667	0.65517	0.64286	0.61538	0.6

The Table 3 gives the upper and lower bounds of the α-cuts of τ for some α values. The table shows the evolution of the bounds as a function of the α level. We can see that when α=0, we have a wide interval ($[0.33, 0, 71]$. As α becomes larger until it reaches 1, the interval narrows to the singleton $\{0.6\}$. The graphical representation of these α-cuts is shown in Figure 4.

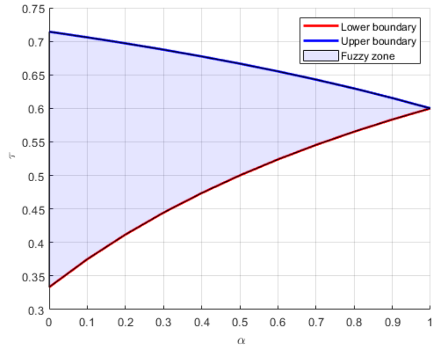

FIGURE 5. Upper and lower limits of τ for each α-cut

The Figure 4 shows the lower and upper bounds of $\tau = \frac{\theta}{\theta+2}$ calculated for different α values of the triangular fuzzy number cuts defined for θ. The lower (red) and upper (blue) bounds show how the possible values of $alpha$ become tighter as $alpha$ increases. The blurred area, visualized as a shaded band, illustrates the uncertainty associated with each level of uncertainty. For $\alpha = 0$, the uncertainty is maximal and corresponds to the complete interval $[\tau_{\text{min}}, \tau_{\text{max}}]$. For $alpha = 1$, the bounds converge to a single value, corresponding to the peak of the triangular fuzzy number for θ. This figure highlights the progressive reduction of uncertainty in $taigaas$alpha increases, reflecting the increasing granularity of the cuts. We therefore note that the α -cut of the fuzzy Kendall tau verifies the following properties:

$$\tilde{\tau}^{[\alpha]}(\tilde{X}, \tilde{Y}) = \tilde{\tau}^{[\alpha]}(\tilde{Y}, \tilde{X}), \tag{36}$$

$$-1 \leq \tilde{\tau}^{[\alpha]}(\tilde{X}, \tilde{Y}) \leq 1. \tag{37}$$

4. CONCLUSION

In this article, we discuss the notion of fuzzy sets and copulas. We have constructed fuzzy Archimedean copulas generated by continuous, strictly decreasing and convex fuzzy functions. In this way, Clayton's fuzzy copulas and Franc's fuzzy copulas were constructed. An analytical and quantitative comparison between the classical Clayton copula and the fuzzy Clayton copula validates the effectiveness of fuzzy copulas for modeling uncertain dependencies, and underlines their applicability in domains requiring robust and flexible approaches. Elliptic and extreme value copulas have also been constructed. We have also used the fuzzy generating functions to construct measures of fuzzy association such as Kendall's fuzzy thau between two fuzzy random variables. Looking ahead, given the fuzzy nature of some health data, we plan to explore the application of fuzzy copulas to diabetes. This would make it possible to model the complex dependence between risk factors such as obesity, physical activity and family history, offering tools to better understand and predict the interactions influencing the onset of the disease.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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