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THE HOMOGENEOUS q-SHIFT OPERATOR FOR CIGLER'S POLYNOMIALS

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Abstract. We define a homogeneous q-shift operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$. Subsequently, we utilise this operator to derive various polynomial q-identities, including the generating function, Rogers-type formula, as well as Mehler's formula along with it's extension and two mixed generating functions for Cigler's polynomials $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$. Also, we obtain a transformational identity involving generating functions of $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$. We provide some special values for $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ identities in order to establish identities for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ and Al-Salam-Carlitz polynomials $U_n(x,y,a|q)$.

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1.. Introduction

This study will adhere to standard notations and definitions regarding the q-series utilised in [14]. Note that $0<|q|<1$. Let $b\in\mathbb{C}$, the q-shifted factorial is defined as: [10,11,14]

$$(b;q)_0=1,\quad (b;q)_n=(1-b)(1-bq)...(1-bq^{n-1}),\quad (b;q)_\infty=\prod_{k=0}^{\infty}(1-bq^k)$$

the multiple q-shifted factorials, along with:

$$(b_1,b_2,\ldots,b_r;q)_n=(b_1;q)_n(b_2;q)_n\cdots(b_r;q)_n,$$

where $n\in\mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [14]:

$${}_r\phi_k\left(\begin{array}{c}b_1,\ldots,b_r\\ c_1,\ldots,c_k\end{array};q,x\right)=\sum_{n=0}^{\infty}\frac{(b_1,\ldots,b_r;q)_n}{(q,c_1,\ldots,c_k;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+k-r}x^n,$$

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where $q \neq 0$ when $r > k+1$. Note that

$$ {}_{r+1}\phi_r\left(\begin{matrix} b_1,\ldots,b_{r+1} \\ c_1,\ldots,c_r \end{matrix};q,x\right) = \sum_{n=0}^{\infty} \frac{(b_1,\ldots,b_{r+1};q)_n}{(q,c_1,\ldots,c_r;q)_n} x^n, \quad |x|<1. $$

The q-binomial coefficient is defined as follows [14]:

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \quad of \quad 0 \leqslant k \leqslant n, $$

where $n,k \in \mathbb{N}$.

$$ \begin{bmatrix} \alpha \\ k \end{bmatrix} = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\, q^{\alpha k}(-1)^k q^{-\binom{k}{2}}, \quad \alpha \in \mathbb{R}. $$

The Cauchy identity is expressed as follows [14,22]:

$$ \sum_{m=0}^{\infty} \frac{(b;q)_m}{(q;q)_m}\, a^m = \frac{(ba;q)_\infty}{(a;q)_\infty}, \quad |a|<1. $$

When $b=0$, the Cauchy identity reduces to Euler's identity [14]:

$$ \sum_{m=0}^{\infty} \frac{a^m}{(q;q)_m} = \frac{1}{(a;q)_\infty}, \quad |a|<1, \tag{1.1} $$

The q-Chu-Vandermonde sum is represented as follows [14]:

$$ {}_2\phi_1(q^{-n},a;c;q,q) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.2} $$

Jackson's transformation of ${}_2\phi_1$ series [14, Appendix III, equation (III.4)] is:

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left(\begin{matrix} a,c/b \\ c,az \end{matrix};q,bz\right), \quad \max\{|z|,|bz|\}<1. \tag{1.3} $$

Setting $a=0$, $b \longrightarrow c/b$ and $z = bd/c$ in equation (1.3) leads to

$$ {}_1\phi_1\left(\begin{matrix} b \\ c \end{matrix};q,d\right) = (bd/c;q)_\infty\, {}_2\phi_1\left(\begin{matrix} 0,b \\ c \end{matrix};q,bd/c\right), \quad |bd/c|<1. \tag{1.4} $$

transformation of ${}_2\phi_1$ series by Heine [14, Appendix III] is:

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b,z \\ az \end{matrix};q,b\right), \quad \max\{|z|,|b|\}<1. \tag{1.5} $$

Transformation of ${}_3\phi_2$ series [14, Appendix III] is:

$$ {}_3\phi_2\left(\begin{matrix} q^{-n},b,c \\ d,e \end{matrix};q,q\right) = \frac{(e/c;q)_n}{(e;q)_n} c^n\, {}_3\phi_2\left(\begin{matrix} q^{-n},c,d/b \\ d,q^{1-n}c/e \end{matrix};q,bq/e\right). \tag{1.6} $$

We make use of the following identities in this work [14]:

$$ (aq^{-n};q)_n = (q/a;q)_n(-a)^n q^{-n-\binom{n}{2}}. \tag{1.7} $$

$$ (q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{1.8} $$

The following polynomials were defined by Hahn In 1949 [16]:

$$\phi_n^{(\alpha)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha;q)_k \, x^k. \tag{1.9}$$

The following result was formulated by Al-Salam and Carlitz in 1965 [6,20,21] for $\phi_n^{(a)}(x)$:

- Mehler's formula for $\phi_n^{(a)}(x)$ is [5]

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n^{(a)}(x)\phi_n^{(b)}(y)\frac{\tau^n}{(q;q)_n} &= \frac{(ax, by\tau;q)_\infty}{(\tau, x, y\tau;q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} \tau, y\tau, a \\ q/x, by\tau \end{matrix} ;q,q \right), \max\{|\tau|, |x|, |y\tau|\} < 1. \\ &= \frac{(ax\tau, by\tau;q)_\infty}{(\tau, x\tau, y\tau;q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} a, b, \tau \\ ax\tau, by\tau \end{matrix} ;q, xy\tau \right), \end{aligned} \tag{1.10}$$

  where $\max\{|\tau|, |x\tau|, |y\tau|, |xy\tau|\} < 1$.

Rewrite (1.10) by using $y \to y/\tau$ and $b \to b/y$ as follows :

$${}_3\phi_2 \left(\begin{matrix} \tau, y, a \\ q/x, b \end{matrix} ;q,q \right) = \frac{(ax\tau, x;q)_\infty}{(ax, x\tau;q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} a, b/y, \tau \\ ax\tau, b \end{matrix} ;q, xy \right), \tag{1.11}$$

where $\max\{|ax|, |x\tau|, |xy|\} < 1$.

The generating function for $\phi_n^{(a)}(x)$ was proposed by Srivastava and Agarwal in 1989: [27]

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)(\lambda;q)_n \frac{\tau^n}{(q;q)_n} = \frac{(\lambda\tau;q)_\infty}{(\tau;q)_\infty}\, {}_2\phi_1 \left(\begin{matrix} \lambda, a \\ \lambda\tau \end{matrix} ;q, x\tau \right), \quad \max\{|\tau|, |x\tau|\} < 1. \tag{1.12}$$

The q-differential operator was defined by [9,18]

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The Leibniz rule for D_q is [19]

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.13}$$

The$q$-exponential operator was introduced by Chen and Liu [9] introduced the following

$$T(aD_q) = \sum_{k=0}^{\infty} \frac{(aD_q)^k}{(q;q)_k}.$$

It is easy to prove that [1]

$$T(D_q)\{\tau^\nu\} = \sum_{\kappa=0}^{\nu} \begin{bmatrix} \nu \\ \kappa \end{bmatrix} \tau^{\nu-\kappa}.$$

The Cauchy polynomials are defined as follows [2,23,24,26]:

$$P_n(a,b) = (b/a;q)_n \, a^n = (a-b)(a-qb)\cdots(a-q^{n-1}b),$$

which have the generating function [3,4,8,17]

$$\sum_{n=0}^{\infty} P_n(a,b)\frac{t^n}{(q;q)_n} = \frac{(bt;q)_\infty}{(at;q)_\infty}, \quad |at|<1. \tag{1.14}$$

In 2003, Chen et al [8] introduced the homogeneous q-difference operator D_{xy}, which operates on functions of two variables as follows:

$$D_{xy}\left\{g(x,y)\right\} = \frac{g(x,q^{-1}y)-g(qx,y)}{x-q^{-1}y}.$$

The homogeneous q-shift operator was constructed as follows:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}.$$

Proposition 1.1. [8]. *The following result is established*

$$D_{xy}^k\left\{P_n(x,y)\right\} = \frac{(q;q)_n}{(q;q)_{n-k}}\,P_{n-k}(x,y).$$
$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k\frac{(yt;q)_\infty}{(xt;q)_\infty}.$$

The bivariate Rogers–Szegö polynomials were defined by Chen et al [8]. in the following manner:

$$h_n(x;y|q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix} P_j(x,y) \tag{1.15}$$

and provided the generating function for $h_n(x;y|q)$ as follows:

$$\sum_{n=0}^{\infty} h_n(x;y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \quad \max\{|t|,|xt|\}<1. \tag{1.16}$$

In 2007 Chen et al [12] provided the following findings for $h_n(x;y|q)$:

- Rogers-type formula on $h_n(x;y|q)$ is

$$\sum_{n=0}^{\infty}\sum_{n=0}^{\infty} h_{n+m}(x;y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix};q,t\right), \tag{1.17}$$

where $\max\{|s|,|t|,|xs|,|xt|\}<1$.

- Mehler's formula for $h_n(x;y|q)$ is

$$\sum_{n=0}^{\infty} h_n(u;v|q)h_n(x;y|q)\frac{t^n}{(q;q)_n} = \frac{(vxt,yt;q)_\infty}{(xt,t,uxt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} y, xt, v/u \\ yt, vxt \end{matrix};q,ut\right), \tag{1.18}$$

where $\max\{|t,|xt|,|ut|,|uxt|\}<1$.

The Al-Salam-Carlitz polynomials can be denoted as follows [13]:

$$U_n(x,y,a;q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(-1)^j q^{\binom{j}{2}}a^j\,P_{n-j}(x;y). \tag{1.19}$$

Chen et al. [13] presented Rogers formula for the Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ utilising the q-exponential operator $T(bD_q)$. Consequently, they derive the inverse linearisation formula for $U_n(x,y,a;q)$.

- The generating function for $U_n(x,y,a;q)$ is

$$\sum_{n=0}^{\infty}U_n(x,y,a;q)\frac{t^n}{(q;q)_n}=\frac{(at,yt;q)_\infty}{(xt;q)_\infty},\quad |xt|<1. \tag{1.20}$$

- The Rogers-type formula for $U_n(x,y,a;q)$ [13] is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}U_{n+m}(x,y,a;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(as,ys;q)_\infty}{(xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(xs;q)_k(at)^k}{(q;q)_k(as,ys;q)_k}\,{}_2\phi_1\left(\begin{matrix}y/x,0\\ ysq^k\end{matrix};q,xt\right), \tag{1.21}$$

provided that $\max\{|xs|,|xt|\}<1$.

- The inverse linearisation formula for $U_n(x,y,a;q)$ is [13]

$$U_{n+m}(x,y,a;q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(-1)^j q^{\binom{j}{2}}(aq^m)^j P_{n-j}(x,y)U_m(x,yq^{n-j},a;q). \tag{1.22}$$

In 2013, Saad and Sukhi [25] provided Mehler's formula for $P_n(x,y)$ as follows:

$$\sum_{n=0}^{\infty}P_n(x;y)P_n(u;v)\frac{t^n}{(q;q)_n}=\frac{(xvt;q)_\infty}{(xut;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}v/u\\ vxt\end{matrix};q,yut\right), \tag{1.23}$$

where $|xut|<1$.

In 2018, Wang and Cao [28] research the following Cigler's polynomials:

$$\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)=\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\\kappa\end{bmatrix}\begin{bmatrix}\alpha\\\kappa\end{bmatrix}(q;q)_\kappa(-1)^\kappa q^{\binom{\kappa}{2}}\rho^\kappa P_{\nu-\kappa}(\chi,\psi). \tag{1.24}$$

- Letting $\chi=1,\psi=0$, $\alpha=\log_q(x/y)$, $\rho=y$ in (1.24), the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ defined in (1.15) is derived.
- Letting $\alpha\to\infty$, $\chi=x$, $\psi=y$ and $\rho=a$ in (1.24), Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ defined in (1.19) is derived.

Wang and Cao [28] found the following results, for Cigler's polynomials, by the method of homogeneous q-difference equations.

- The generating function for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ is

$$\sum_{\nu=0}^{\infty}\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)\frac{t^\nu}{(q;q)_\nu}=\frac{(\psi t;q)_\infty(\rho t;q)_\alpha}{(\chi t;q)_\infty},\quad \max\{|\chi t|,|\rho t q^\alpha|\}<1. \tag{1.25}$$

- An extension of the generating function for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ is

$$\sum_{\nu=0}^{\infty}\mathcal{C}_{\nu+\sigma}^{(\alpha-\nu-\sigma)}(\chi,\psi,\rho|q)\frac{t^\nu}{(q;q)_\nu}$$
$$=t^{-\sigma}\frac{(\psi t;q)_\infty(\rho t,q)_\alpha}{(\chi t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q^{-\sigma},\chi t,\rho t q^\alpha\\ \psi t,\rho t\end{matrix};q,q\right),\quad |\chi t|<1. \tag{1.26}$$

- Srivastava-Agarwal type [28] generating function for $C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ is

$$\sum_{\nu=0}^{\infty}C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)(\lambda;q)_\nu\frac{t^\nu}{(q;q)_\nu}=\frac{(\lambda,\rho t,\psi t;q)_\infty}{(\rho t q^\alpha,\chi t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\rho t q^\alpha,\chi t,0\\ \rho t,\psi t\end{matrix};q,\lambda\right), \tag{1.27}$$

  where $\max\{|\chi t|,|\rho t q^\alpha|,|\lambda|\}<1$.
- Transformational identity involving generating functions for $C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ is

  Let the coefficients $A(\nu)$ and $B(\nu)$ satisfy the following relationship:

$$\sum_{\nu=0}^{\infty}A(\nu)P_\nu(\chi,\psi)=\sum_{\nu=0}^{\infty}B(\nu)\frac{(q^\nu\psi t;q)_\infty}{(q^\nu\chi t;q)_\infty}. \tag{1.28}$$

  Then

$$\sum_{\nu=0}^{\infty}A(\nu)\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho)=\sum_{\nu=0}^{\infty}B(\nu)\frac{(\psi q^\nu t,\rho t q^\nu;q)_\infty}{(\chi q^\nu t,\rho t q^\alpha q^\nu;q)_\infty}, \tag{1.29}$$

  where each series in (1.28) and (1.29) are absolutely convergent.

The paper is organised as follows: In section 2, we'll introduce the homogeneous q-shift operator $\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})$. Afterwards, we determine several associated identities. Section 3 discusses the generating function and its extension for Cigler's polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$, by applying operator technique (2.2). Section 4 looks at the Rogers-type formula and its extension, plus the inverse linearisation formula. Section 5 describes Mehler's formula and its extension. In section 6, for the polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$, we develop two mixed generating functions for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$. We use the homogeneous q-shift operator $\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})$ in section 7 to obtain the transformational identity for the polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$.

2.. The Homogeneous q-Shift Operator and Certain Operator Identities

Within this section, we introduce the homogeneous q-shift operator $\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})$. We shall next proceed to determine some of its identities.

Definition 2.1. *The homogeneous q-shift operator is defined as follows:*

$$\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)=\sum_{\kappa=0}^{\infty}\begin{bmatrix}\alpha\\ \kappa\end{bmatrix}(-1)^{\kappa}q^{\binom{\kappa}{2}}(\rho D_{\chi\psi})^{\kappa},\quad \alpha\in\mathcal{R}. \tag{2.1}$$

Specific values assigned to the homogeneous q-shift operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$, yield numerous previously mentioned operators; for further details, refer to [7,8,13].

There is an easy way to prove the following lemma:

Lemma 2.2. *Assuming that the operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$ defined as in (2.1), then*

$$\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\{P_{\nu}(\chi,\psi)\}=\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q). \tag{2.2}$$

$$\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\right\}=\frac{(\psi t,\rho t;q)_{\infty}}{(\chi t,\rho q^{\alpha}t;q)_{\infty}},\quad |\chi t|<1. \tag{2.3}$$

Lemma 2.3. *Define the operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$ as shown in (2.1), then*

$$\begin{aligned}&\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{P_{\kappa}(\chi t,\psi t)}{(\psi t;q)_{\kappa}}\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\right\}\\ &=\frac{(\psi t;q)_{\infty}(\rho t;q)_{\alpha}}{(\chi t;q)_{\infty}}\sum_{\iota=0}^{\kappa}\frac{(q^{-\kappa},\chi t,\rho tq^{\alpha};q)_{\iota}}{(q,\psi t,\rho t;q)_{\iota}}\,q^{\iota},\quad \max\{|\rho t|,|\chi t|\}<1.\end{aligned} \tag{2.4}$$

Proof. Using q-Chu-Vandermonde sum (1.2), we obtain

$$\begin{aligned}\frac{P_{\kappa}(\chi t,\psi t)}{(\psi t;q)_{\kappa}}\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}&=\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\frac{(\psi t/\chi t;q)_{\kappa}(\chi t)^{\kappa}}{(\psi t;q)_{\kappa}}\\ &=\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\,{}_2\phi_1(q^{-\kappa},\chi t;\psi t;q,q)\\ &=\sum_{\iota=0}^{\kappa}\frac{(q^{-\kappa};q)_{\iota}}{(q;q)_{\iota}}\frac{(q^{\iota}\psi t;q)_{\infty}}{(q^{\iota}\chi t;q)_{\infty}}q^{\iota}.\end{aligned} \tag{2.5}$$

Applying the operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$ on both side of (2.5), we get

$$\begin{aligned}&\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{P_{k}(\chi t,\psi t)}{(\psi t;q)_{\kappa}}\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\right\}\\ &=\sum_{\iota=0}^{\kappa}\frac{(q^{-\kappa};q)_{\iota}}{(q;q)_{\iota}}q^{\iota}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{(q^{\iota}\psi t;q)_{\infty}}{(q^{\iota}\chi t;q)_{\infty}}\right\}\\ &=\sum_{\iota=0}^{\kappa}\frac{(q^{-\kappa};q)_{\iota},}{(q;q)_{\iota}}q^{\iota}\frac{(\psi tq^{\iota},\rho tq^{\iota};q)_{\infty}}{(\chi tq^{\iota},\rho tq^{\alpha+\iota};q)_{\infty}}\quad\text{(by using (2.3))}\\ &=\frac{(\psi t,\rho t;q)_{\infty}}{(\chi t,\rho tq^{\alpha};q)_{\infty}}\sum_{\iota=0}^{\kappa}\frac{(q^{-\kappa},\chi t,\rho tq^{\alpha};q)_{\iota}}{(q,\psi t,\rho t;q)_{\iota}}q^{\iota}.\end{aligned}$$

□

3.. Generating Function for $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$

In this section, we derive an extension of the generating function of the polynomials $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ via the operator technique (2.2). So, to derive a generating function and extend both polynomials $h_n(x,y|q)$ and $U_n(x,y,a;q)$, firstly, we use the operator $\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)$ to prove an extension to generating function for $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ (equation (1.26)).

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\mathcal{C}_{\nu+\sigma}^{(\alpha-\nu-\sigma)}(\chi,\psi,\rho|q)\frac{t^{\nu}}{(q;q)_{\nu}}\\
&=t^{-\sigma}\sum_{\nu=0}^{\infty}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)P_{\nu+\sigma}(\chi,\psi)\frac{t^{\nu+\sigma}}{(q;q)_{\nu}}\quad\text{(by using (2.2))}\\
&=t^{-\sigma}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\sum_{\nu=0}^{\infty}P_{\nu+\sigma}(\chi,\psi)\frac{t^{\nu+\sigma}}{(q;q)_{\nu}}\\
&=t^{-\sigma}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)P_{\sigma}(\chi t,\psi t)\sum_{\nu=0}^{\infty}P_{\nu}(\chi,q^{\sigma}\psi)\frac{t^{\nu}}{(q;q)_{\nu}}\\
&=t^{-\sigma}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)P_{\sigma}(\chi t,\psi t)\frac{(q^{\sigma}\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\quad\text{(by using (1.14))}\\
&=t^{-\sigma}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{P_{\sigma}(\chi t,\psi t)}{(\psi t;q)_{\sigma}}\frac{(\psi t;q)_{\infty}}{(\chi t;q)_{\infty}}\right\}\\
&=t^{-\sigma}\frac{(\psi t;q)_{\infty}(\rho t,q)_{\alpha}}{(\chi t;q)_{\infty}}\sum_{\iota=0}^{\sigma}\frac{(q^{-\sigma},\chi t,\rho t q^{\alpha};q)_{\iota}}{(q,\psi t,\rho t;q)_{\iota}}\,q^{\iota}\quad\text{(by using (2.4))}\\
&=t^{-\sigma}\frac{(\psi t;q)_{\infty}(\rho t,q)_{\alpha}}{(\chi t;q)_{\infty}}\;{}_3\phi_2\left(\begin{matrix}q^{-\sigma},\chi t,\rho t q^{\alpha}\\ \psi t,\rho t\end{matrix};q,q\right).
\end{aligned}$$

□

- Letting $\sigma=0$ in equation (1.26), the generating function for $\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ was obtained (equation (1.25)).
  - ∗ Letting $\chi=1$, $\psi=0$, $\alpha=\log_q(x/y)$, $\rho=y$ in (1.25), the generating function to a bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ was obtained (equation (1.16)).
  - ∗ Letting $\alpha\to\infty$, $\chi=x$, $\psi=y$ and $\rho=a$ in (1.25), we have regained the generating function for the polynomials $U_n(x,y,a;q)$. (equation (1.20)).
- By taking $\chi=1$, $\psi=0$,$\sigma=s$, $\alpha\to\log_q(x/y)$ and $\rho\to y$ in equation (1.26), we have obtained an extension to generating function for $h_n(x,y|q)$

Corollary 3.1. *The following identity holds:*

$$\sum_{j=0}^{\infty}h_{j+s}(x,y|q)\frac{t^{j+s}}{(q;q)_j}=\frac{(yt;q)_{\infty}}{(t,xt;q)_{\infty}}\;{}_3\phi_2\left(\begin{matrix}q^{-s},t,xt\\ 0,yt\end{matrix};q,q\right),$$

Assume $\max\{|t|, |xt|\} < 1$.

- For taking $\chi = x$, $\psi = y$, $\rho = a$, $\alpha \to \infty$, $\sigma = s$ in equation (1.26), an extension for the generating function for $U_n(x, y, a; q)$ was derived.

Corollary 3.2. *The following result was established:*

$$\sum_{j=0}^{\infty} U_{j+s}(x, y, a; q)\frac{t^{j+s}}{(q;q)_j} = \frac{(yt, at; q)_\infty}{(xt; q)_\infty}\ {}_3\phi_2\left(\begin{matrix} q^{-s}, xt, 0 \\ yt, at \end{matrix}; q, q\right),$$

provided $\max\{|t|, |xt|\} < 1$.

4.. Rogers-type Formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$

In this section, we are investigating the Rogers-type formula and its extension, as well as the inverse linearisation formula for the polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$ within the framework of operators. Rogers-type formula and it's extension, as well as the inverse formula of the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ and the generalised Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ are derived by using specific values for the variables of $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$.

We are provided two proofs for Roger's formula regarding $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$.

Theorem 4.1. (Roger's Formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$).

$$\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty} \mathcal{C}_{\nu+\kappa}^{(\alpha-\nu-\kappa)}(\chi, \psi, \rho|q)\frac{\tau^\nu}{(q;q)_\nu}\frac{\sigma^\kappa}{(q;q)_\kappa}$$

$$= \frac{(\psi\sigma; q)_\infty(\rho\sigma; q)_\alpha}{(\chi\sigma, \tau/\sigma; q)_\infty}\ {}_4\phi_3\left(\begin{matrix} \chi\sigma, \rho\sigma q^\alpha, 0, 0 \\ q\sigma/\tau, \rho\sigma, \psi\sigma \end{matrix}; q, q\right), \quad \max\{|\chi\sigma|, |q^{-\iota}\tau/\sigma|\} < 1. \tag{4.1}$$

$$= \frac{(\psi\sigma, \rho\sigma; q)_\infty}{(\rho\sigma q^\alpha, \chi\sigma; q)_\infty}\sum_{\kappa=0}^{\infty}\frac{(\psi q^{-\alpha}/\rho; q)_\kappa}{(q;q)_\kappa}\frac{(\chi\sigma; q)_\kappa(\tau\rho q^\alpha)^\kappa}{(\rho\sigma; q)_\kappa(\psi\sigma; q)_\kappa}\ {}_2\phi_1\left(\begin{matrix} \rho/\chi, 0 \\ \rho\sigma q^\kappa \end{matrix}; q, \chi\tau\right), \tag{4.2}$$

where $\max\{|\chi\sigma|, \chi\tau|\} < 1$.

<u>First proof</u>

Proof.

$$\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\infty} \mathcal{C}_{\nu+\kappa}^{(\alpha-\nu-\kappa)}(\chi, \psi, \rho|q)\frac{\tau^\nu}{(q;q)_\nu}\frac{\sigma^\kappa}{(q;q)_\kappa}$$

$$= \sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\infty} \mathcal{T}\left(q^{-\alpha}; q, \rho D_{\chi\psi}\right)\ \frac{P_{\nu+\kappa}(\chi, \psi)\tau^\nu}{(q;q)_\nu}\frac{\sigma^\kappa}{(q;q)_\kappa} \quad \text{(by using (2.2))}$$

$$= \mathcal{T}\left(q^{-\alpha}; q, \rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty} P_\nu(\chi, \psi)\frac{\tau^\nu}{(q;q)_\nu}\sum_{\kappa=0}^{\infty} P_\kappa(\chi, q^\nu\psi)\frac{\sigma^\kappa}{(q;q)_\kappa}\right\} \quad \text{(by using (1.14))}$$

$$= \mathcal{T}\left(q^{-\alpha}; q, \rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty} P_\nu(\chi,\psi)\frac{(q^\nu\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\frac{\tau^\nu}{(q;q)_\nu}\right\}$$

$$= \sum_{\nu=0}^{\infty}\frac{\tau^\nu}{(q;q)_\nu}\mathcal{T}\left(q^{-\alpha}; q, \rho D_{\chi\psi}\right)\left\{\frac{P_\nu(\chi,\psi)}{(\psi\sigma;q)_\nu}\frac{(\psi\sigma;q)_\infty}{(\chi\sigma;q)_\infty}\right\}$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma;q)_\infty}\sum_{\nu=0}^{\infty}\frac{\tau^\nu\sigma^{-\nu}}{(q;q)_\nu}\sum_{\iota=0}^{\nu}\frac{(q^{-\nu},\chi\sigma,\rho\sigma q^\alpha;q)_\iota}{(q,\psi\sigma,\rho\sigma;q)_\iota}q^\iota \quad \text{(by using (2.4))}$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma;q)_\infty}\sum_{\iota=0}^{\infty}\sum_{\nu=\iota}^{\infty}\frac{\tau^\nu\sigma^{-\nu}}{(q;q)_{\nu-\iota}}\frac{(-1)^\iota q^{\binom{\iota}{2}}q^{-\nu\iota}(\chi\sigma,\rho\sigma q^\alpha;q)_\iota}{(q,\psi\sigma,\rho\sigma;q)_\iota}q^\iota \quad \text{(by using (1.8))}$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma;q)_\infty}\sum_{\iota=0}^{\infty}\frac{\tau^\iota\sigma^{-\iota}(-1)^\iota q^{\binom{\iota}{2}}q^{-\iota^2}(\chi\sigma,\rho\sigma q^\alpha;q)_\iota}{(q,\psi\sigma,\rho\sigma;q)_\iota}q^\iota\sum_{\nu=0}^{\infty}\frac{(q^{-\iota}\tau/\sigma)^\nu}{(q;q)_\nu}$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma;q)_\infty}\sum_{\iota=0}^{\infty}\frac{\tau^\iota\sigma^{-\iota}(-1)^\iota q^{\binom{\iota}{2}}q^{-\iota^2}(\chi\sigma,\rho\sigma q^\alpha;q)_\iota}{(q,\psi\sigma,\rho\sigma;q)_\iota}q^\iota\frac{1}{(q^{-\iota}\tau/\sigma;q)_\infty}$$

(by using (1.1))

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma,\tau/\sigma;q)_\infty}\sum_{\iota=0}^{\infty}\frac{\tau^\iota\sigma^{-\iota}(-1)^\iota q^{-\binom{\iota}{2}}(\chi\sigma,\rho\sigma q^\alpha;q)_\iota}{(-1)^\iota(\tau/\sigma)^\iota q^{-\binom{\iota}{2}-\iota}(q\sigma/\tau,q,\psi\sigma,\rho\sigma;q)_\iota} \quad \text{(by using (1.7))}$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma,\tau/\sigma;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\chi\sigma,\rho\sigma q^\alpha,0,0\\ q\sigma/\tau,\rho\sigma,\psi\sigma\end{matrix};q,q\right).$$

□

Second proof

Proof.

$$\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\mathcal{C}_{\nu+\kappa}^{(\alpha-\nu-\kappa)}(\chi,\psi,\rho|q)\frac{\tau^\nu}{(q;q)_\nu}\frac{\sigma^\kappa}{(q;q)_\kappa}$$

$$= \sum_{\kappa=0}^{\infty}\sum_{\nu=\kappa}^{\infty}\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)\frac{(q;q)_\nu\tau^{\nu-\kappa}}{(q;q)_\nu(q;q)_{\nu-\kappa}}\frac{\sigma^\kappa}{(q;q)_\kappa}$$

$$= \sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)\frac{(q;q)_\nu\tau^{\nu-\kappa}}{(q;q)_\kappa(q;q)_{\nu-\kappa}}\frac{\sigma^\kappa}{(q;q)_\nu}$$

$$= \sum_{\nu=0}^{\infty}\frac{\mathcal{C}_{\nu}^{(\alpha-\nu)}(\chi,\psi,\rho|q)}{(q;q)_\nu}T(\sigma D_q)\{\tau^\nu\}$$

$$= T(\sigma D_q)\left\{\frac{(\psi\tau,\rho\tau;q)_\infty}{(\chi\tau,\rho\tau q^\alpha;q)_\infty}\right\}$$

$$= \sum_{\nu=0}^{\infty}\frac{\sigma^\nu}{(q;q)_\nu}\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}q^{\kappa(\kappa-\nu)}D_q^\kappa\frac{(\psi\tau;q)_\infty}{(\rho\tau q^\alpha;q)_\infty}D_q^{\nu-\kappa}\frac{(\rho\tau q^\kappa;q)_\infty}{(\chi\tau q^\kappa;q)_\infty} \quad \text{(the Leibniz rule)}$$

$$= \sum_{\nu=\kappa}^{\infty}\frac{\sigma^\nu}{(q;q)_\nu}\sum_{\kappa=0}^{\infty}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}(\psi q^{-\alpha}/\rho;q)_\kappa(\rho q^\alpha)^\kappa\frac{(\psi\tau q^\kappa;q)_\infty}{(\rho\tau q^\alpha;q)_\infty}(\frac{\rho}{\chi};q)_{\nu-\kappa}\chi^{\nu-\kappa}\frac{(\rho\tau q^\nu;q)_\infty}{(\chi\tau q^\kappa;q)_\infty} \tag{4.3}$$

$$= \sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\sigma^{\nu+\kappa}(\psi q^{-\alpha}/\rho;q)_\kappa(\frac{\rho}{\chi};q)_\nu}{(q;q)_\nu(q;q)_\kappa} \frac{(\rho\tau q^{\nu+\kappa},\psi\tau q^\kappa;q)_\infty \chi^\nu(\rho q^\alpha)^\kappa}{(\rho\tau q^\alpha,\chi\tau q^\kappa;q)_\infty}$$

$$= \frac{(\psi\tau,\rho\tau;q)_\infty}{(\rho\tau q^\alpha,\chi\tau;q)_\infty} \sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{(\psi q^{-\alpha}/\rho;q)_\kappa(\frac{\rho}{\chi};q)_\nu}{(q;q)_\nu(q;q)_\kappa} \frac{(\chi\tau;q)_\kappa(\chi\sigma)^\nu(\sigma\rho q^\alpha)^\kappa}{(\rho\tau;q)_{\nu+\kappa}(\psi\tau;q)_\kappa}$$

$$= \frac{(\psi\tau,\rho\tau;q)_\infty}{(\rho\tau q^\alpha,\chi\tau;q)_\infty} \sum_{\kappa=0}^{\infty} \frac{(\psi q^{-\alpha}/\rho;q)_\kappa}{(q;q)_\kappa} \frac{(\chi\tau;q)_\kappa(\sigma\rho q^\alpha)^\kappa}{(\rho\tau;q)_\kappa(\psi\tau;q)_\kappa}\, {}_2\phi_1\left(\begin{matrix}\rho/\chi,0\\ \rho\tau q^\kappa\end{matrix};q,\chi\sigma\right)$$

$$= \frac{(\psi\sigma,\rho\sigma;q)_\infty}{(\rho\sigma q^\alpha,\chi\sigma;q)_\infty} \sum_{\kappa=0}^{\infty} \frac{(\psi q^{-\alpha}/\rho;q)_\kappa}{(q;q)_\kappa} \frac{(\chi\sigma;q)_\kappa(\tau\rho q^\alpha)^\kappa}{(\rho\sigma;q)_\kappa(\psi\sigma;q)_\kappa}\, {}_2\phi_1\left(\begin{matrix}\rho/\chi,0\\ \rho\sigma q^\kappa\end{matrix};q,\chi\tau\right).$$

□

From equations (4.1) and (4.2), the following transformation is derived:

Corollary 4.2. *The following identity holds:*

$${}_4\phi_3\left(\begin{matrix}\chi\sigma,\rho\sigma q^\alpha,0,0\\ q\sigma/\tau,\rho\sigma,\psi\sigma\end{matrix};q,q\right) = (\tau/\sigma;q)_\infty \sum_{\kappa=0}^{\infty} \frac{(\psi q^{-\alpha}/\rho;q)_\kappa}{(q;q)_\kappa} \frac{(\chi\sigma;q)_\kappa(\tau\rho q^\alpha)^\kappa}{(\rho\sigma;q)_\kappa(\psi\sigma;q)_\kappa}$$
$$\times\, {}_2\phi_1\left(\begin{matrix}\rho/\chi,0\\ \rho\sigma q^\kappa\end{matrix};q,\chi\tau\right), \quad \max\{|\chi\tau|,|\tau/\sigma|\}<1.$$

- By taking $\alpha=\log_q(x/y)$, $\chi=1$, $\psi=0$, $\tau=t$, $\sigma=s$ and $\rho=y$ in equation (4.1), the Rogers-type formula for $h_n(x,y|q)$ is derived (equation (1.17)).

  Proof.

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} h_{n+j}(x|q)\frac{t^n}{(q;q)_n}\frac{s^j}{(q;q)_j} = \frac{(ys;q)_\infty}{(xs,s,t/s;q)_\infty}\, {}_3\phi_2\left(\begin{matrix}s,xs,0\\ qs/t,ys\end{matrix};q,q\right)$$
$$= \frac{(ys;q)_\infty}{(xs,s,xt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}y,xs\\ ys\end{matrix};q,t\right), \quad \text{(by using (1.11))}$$

  where $\max\{|xs|,|s|,|xt|,|t|\}<1$. □

- For taking $\alpha\to\infty$, $\chi=x$, $\psi=a$, $\tau=t$, $\sigma=s$ and $\rho=y$ in equation (4.2), the Rogers-type formula for $U_n(x,y,a;q)$ is derived (equation (1.21)).

The aforementioned Rogers-type formula yields the inverse linearisation formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$.

Theorem 4.3. *For ℓ,ν are nonnegative integers, we are possessing*

$$\mathcal{C}_{\ell+\nu}^{(\alpha-\ell-\nu)}(\chi,\psi,\rho|q) = \sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix} P_\kappa(q^\alpha\rho,\psi)P_{\nu-\kappa}(\chi,\rho)\mathcal{C}_\ell^{(\alpha-\nu-\ell)}(\chi,\psi,\rho q^{\nu-\kappa}|q)q^{\ell\kappa}. \tag{4.4}$$

Proof. Rewrite equation (4.3) as follows:

$$\begin{aligned}
&\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\mathcal{C}_{\nu+\kappa}^{(\alpha-\nu-\kappa)}(\chi,\psi,\rho|q)\frac{\tau^{\nu}}{(q;q)_{\nu}}\frac{\sigma^{\kappa}}{(q;q)_{\kappa}}\\
&\quad=\sum_{\kappa=0}^{\infty}\sum_{\nu=\kappa}^{\infty}\frac{\sigma^{\nu}}{(q;q)_{\nu}}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}(\psi q^{-\alpha}/\rho;q)_{\kappa}(\rho q^{\alpha})^{\kappa}\frac{(\psi\tau q^{\kappa};q)_{\infty}}{(\rho\tau q^{\alpha};q)_{\infty}}(\frac{\rho}{\chi};q)_{\nu-\kappa}\chi^{\nu-\kappa}\frac{(\rho\tau q^{\nu};q)_{\infty}}{(\chi\tau q^{\kappa};q)_{\infty}}\\
&\quad=\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\frac{\sigma^{\nu}}{(q;q)_{\nu-\kappa}(q;q)_{\kappa}}P_{k}(q^{\alpha}\rho,\psi)P_{\nu-\kappa}(\chi,\rho)\frac{(\psi\tau q^{\kappa};q)_{\infty}}{(\rho\tau q^{\alpha};q)_{\infty}}\frac{(\rho\tau q^{\nu};q)_{\infty}}{(\chi\tau q^{\kappa};q)_{\infty}}\\
&\quad=\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\frac{\sigma^{\nu}}{(q;q)_{\nu-\kappa}(q;q)_{\kappa}}P_{k}(q^{\alpha}\rho,\psi)P_{\nu-\kappa}(\chi,\rho)\sum_{\ell=0}^{\infty}\frac{\mathcal{C}_{\ell}^{(\alpha-\nu-\ell)}(\chi q^{\kappa},\psi q^{\kappa},\rho q^{\nu}|q)\tau^{\ell}}{(q;q)_{\ell}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by using (1.25))}\\
&\quad=\sum_{\ell=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\frac{1}{(q;q)_{\nu-\kappa}(q;q)_{\kappa}}P_{k}(q^{\alpha}\rho,\psi)P_{\nu-\kappa}(\chi,\rho)\frac{\mathcal{C}_{\ell}^{(\alpha-\nu-\ell)}(\chi q^{\kappa},\psi q^{\kappa},\rho q^{\nu}|q)\sigma^{\nu}\tau^{\ell}}{(q;q)_{\ell}}\\
&\quad=\sum_{\ell=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\frac{1}{(q;q)_{\nu-\kappa}(q;q)_{\kappa}}P_{k}(q^{\alpha}\rho,\psi)P_{\nu-\kappa}(\chi,\rho)\frac{\mathcal{C}_{\ell}^{(\alpha-\nu-\ell)}(\chi,\psi,\rho q^{\nu-\kappa}|q)q^{\ell\kappa}\sigma^{\nu}\tau^{\ell}}{(q;q)_{\ell}}.
\end{aligned}$$

By equating the coefficients of $\sigma^{\nu}\tau^{\ell}$ in the aforementioned equation, the required identity is derived. □

- Letting $\alpha=\log_q(x/y)$, $\chi=1$, $\psi=0$, and $\rho=y$ in equation (4.4), the inverse linearization formula for $h_n(x,y|q)$ was derived.

  Corollary 4.4. *The following identity holds:*

  $$h_{\ell+n}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}(y;q)_{n-k}h_{\ell}(xq^{-k},yq^{n-k}|q)(xq^{\ell})^{k}.$$

- Letting $\alpha\to\infty$, $\chi=x$, $\psi=a$, and $\rho=y$ in equation (4.4), the inverse linearisation formula for $U_n(x,y,a;q)$ was regained (equation (1.22)).

Theorem 4.5. (An Extension of Roger's formula for $\mathcal{C}_{\nu+\kappa}^{(\alpha-\nu-\kappa)}(\chi,\psi,\rho|q)$).

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{\kappa=0}^{\infty}\mathcal{C}_{\nu+\mu+\kappa}^{(\alpha-\nu-\mu-\kappa)}(\chi,\psi,\rho|q)\frac{t^{\nu}}{(q;q)_{\nu+\mu}}\frac{\tau^{\mu}}{(q;q)_{\mu}}\frac{\sigma^{\kappa}}{(q;q)_{\kappa}}\\
&\quad=\frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma,\tau/t,t/\sigma;q)_{\infty}}\,{}_4\phi_3\left(\begin{matrix}\chi\sigma,\rho\sigma q^{\alpha},0,0\\ \rho\sigma,q\sigma/t,\psi\sigma\end{matrix};q,q\right),
\end{aligned}\tag{4.5}$$

where $\max\{|\chi\sigma|,|\tau/t|,|t/\sigma|\}<1$.

Proof.

$$\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{\kappa=0}^{\infty}\mathcal{C}_{\nu+\mu+\kappa}^{(\alpha-\nu-\mu-\kappa)}(\chi,\psi,\rho|q)\frac{t^{\nu}}{(q;q)_{\nu+\mu}}\frac{\tau^{\mu}}{(q;q)_{\mu}}\frac{\sigma^{\kappa}}{(q;q)_{\kappa}}$$

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{\kappa=0}^{\infty}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)P_{\nu+\mu+\kappa}(\chi,\psi)\frac{t^{\nu}}{(q;q)_{\nu+\mu}}\frac{\tau^{\mu}}{(q;q)_{\mu}}\frac{\sigma^{k}}{(q;q)_{\kappa}}$$

$$= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}P_{\nu+\mu}(\chi,\psi)\frac{t^{\nu}}{(q;q)_{\nu+\mu}}\frac{\tau^{\mu}}{(q;q)_{\mu}}\sum_{\kappa=0}^{\infty}P_{\kappa}(\chi,q^{\nu+\mu}\psi)\frac{\sigma^{\kappa}}{(q;q)_{\kappa}}\right\}$$

$$= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}P_{\nu+\mu}(\chi,\psi)\frac{t^{\nu}}{(q;q)_{\mu+\nu}}\frac{\tau^{\mu}}{(q;q)_{m}}\frac{(q^{\nu+\mu}\psi\sigma,q)_{\infty}}{(\chi\sigma;q)_{\infty}}\right\}$$

(by using (1.14))

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{t^{\nu}}{(q;q)_{\mu+\nu}}\frac{\tau^{\mu}}{(q;q)_{m}}\mathcal{T}\left(q^{-\alpha};q;\rho D_{\chi\psi}\right)\left\{P_{\nu+\mu}(\chi,\psi)\frac{(q^{\nu+\mu}\psi\sigma,q)_{\infty}}{(\chi\sigma;q)_{\infty}}\right\}$$

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{t^{\nu}}{(q;q)_{\mu+\nu}}\frac{\tau^{\mu}}{(q;q)_{\mu}}\frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sigma^{-(\mu+\nu)}\sum_{\iota=0}^{\mu+\nu}\frac{(q^{-\mu+\nu},\chi\sigma,\rho\sigma q^{\alpha};q)_{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}q^{\iota}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{t^{\nu}}{(q;q)_{\nu+\mu}}\frac{\tau^{\mu}\sigma^{-(\nu+\mu)}}{(q;q)_{\mu}}\sum_{\iota=0}^{\nu+\mu}\frac{(q^{-(\nu+\mu)},\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}$$

(by using (2.4))

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\iota=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{(t/\sigma)^{\nu}}{(q;q)_{\nu+\mu}}\frac{(\tau/\sigma)^{\mu}}{(q;q)_{\mu}}\sum_{\nu+\mu=\iota}^{\infty}\frac{(q^{-(\nu+\mu)},\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\iota=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{j=\iota}^{\infty}\frac{(t/\sigma)^{j-\mu}}{(q;q)_{j}}\frac{(\tau/\sigma)^{\mu}}{(q;q)_{\mu}}\frac{(q^{-j},\chi\sigma,\rho\sigma q^{\alpha};q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\iota=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{j+\iota=\iota}^{\infty}\frac{(t/\sigma)^{j+\iota-\mu}}{(q;q)_{j+\iota-\iota}}\frac{(\tau/\sigma)^{\mu}}{(q;q)_{\mu}}\frac{(-1)^{\iota}q^{-\iota(j+\iota)}q^{\binom{\iota}{2}}(\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}$$

(by using (1.8))

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\iota=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t/\sigma)^{j}}{(q;q)_{j}}\frac{(\tau/t)^{\mu}(t/\sigma)^{\iota}}{(q;q)_{\mu}}\frac{(-1)^{\iota}q^{-\iota(j+\iota)}q^{\binom{\iota}{2}}(\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma;q)_{\iota}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma;q)_{\infty}}\sum_{\mu=0}^{\infty}\frac{(\tau/t)^{\mu}}{(q;q)_{\mu}}\sum_{\iota=0}^{\infty}\frac{(-1)^{\iota}q^{-\binom{\iota}{2}}(t/\sigma)^{k}}{(q;q)_{\iota}}\frac{(\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}}{(\psi\sigma,\rho\sigma;q)_{\iota}}\sum_{j=0}^{\infty}\frac{(q^{-\iota}t/\sigma)^{j}}{(q;q)_{j}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma,\tau/t;q)_{\infty}}\sum_{\iota=0}^{\infty}\frac{(-1)^{\iota}q^{-\binom{\iota}{2}}(t/\sigma)^{\iota}}{(q;q)_{\iota}}\frac{(\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}}{(\psi\sigma,\rho\sigma;q)_{\iota}}\frac{1}{(q^{-\iota}t/\sigma;q)_{\infty}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma,\tau/t;q)_{\infty}}\sum_{\iota=0}^{\infty}\frac{(-1)^{\iota}q^{-\binom{\iota}{2}}(t/\sigma)^{\iota}}{(q;q)_{\iota}}\frac{(\chi\sigma,\rho\sigma q^{\alpha},q)_{\iota}}{(\psi\sigma,\rho\sigma q^{\alpha},q^{-\iota}t/\sigma;q)_{\iota}}$$

$$= \frac{(\psi\sigma;q)_{\infty}(\rho\sigma;q)_{\alpha}}{(\chi\sigma,\tau/t,t/\sigma;q)_{\infty}}\sum_{\iota=0}^{\infty}\frac{(\chi\sigma,\rho\sigma q^{\alpha};q)_{\iota}q^{\iota}}{(q,\psi\sigma,\rho\sigma,q\sigma/t;q)_{\iota}}\quad\text{(by using (1.7))}.$$

$$= \frac{(\psi\sigma;q)_\infty(\rho\sigma;q)_\alpha}{(\chi\sigma,\tau/t,t/\sigma;q)_\infty}\,{}_4\phi_3\left(\begin{array}{c}\chi\sigma,\rho\sigma q^\alpha,0,0\\ \rho\sigma,q\sigma/t,\psi\sigma\end{array};q,q\right).$$

□

- By taking $\tau=0, t\to\tau$ in equation (4.5), the Rogers-type formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ is derived (equation (4.2)).
- By taking $\alpha=\log_q(x/y)$, $\chi=1$, $\psi=0$,$\sigma=s$ and $\rho=y$ in equation (4.5) and using (1.11), an extension of Roger-type formula for polynomials $h_n(x,y|q)$ is derived.

Corollary 4.6. *The following identity holds:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}h_{n+m+k}(x,y|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$= \frac{(ys;q)_\infty}{(s,xs,\tau/t,xt;q)_\infty}\,{}_2\phi_1\left(\begin{array}{c}y,xs\\ 0,ys\end{array};q,t\right),$$

where $\max\{|s|,|xs|,|\tau/t|,|xt|,|t|\}<1$.

- By taking $\alpha\to\infty$, $\chi=x$,$\psi=y$, $\sigma=s$,$\rho=a$ in equation (4.5), the extension of Roger's formula for polynomials $U_n(x,y,a;q)$ was obtained.

Corollary 4.7. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}U_{n+m+k}(x,y,a;q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{s^k}{(q;q)_k}$$
$$= \frac{(ys;q)_\infty(as;q)_\infty}{(xs,\tau/t,t/s;q)_\infty}\,{}_4\phi_3\left(\begin{array}{c}xs,0,0,0\\ as,qs/t,ys\end{array};q,q\right),$$

where $\max\{|xs|,|\tau/t|,|t/s|\}<1$.

5.. Mehler's Formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$

This section presents Mehler's formula and it's extension for the polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ utilising the operator representation (2.2). The special values for the variables in Mehler's formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ are used to establish Miller's formula and it's extension for both $h_n(x,y|q)$ and $U(x,y,a;q)$.

Theorem 5.1. (Extension of Mehler's formula for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$). *We have*

$$\sum_{\nu=0}^{\infty}\mathcal{C}_{\nu+\mu}^{(\alpha-\nu-\mu)}(\chi,\psi,\rho|q)\mathcal{C}_\nu^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^\nu}{(q;q)_\nu}$$
$$= \frac{(\zeta t)^{-\mu}(\psi\zeta t;q)_\infty(\rho\zeta t;q)_\alpha}{(\chi t\zeta;q)_\infty}\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\omega/\zeta)^\nu(-1)^\nu q^{\binom{\nu}{2}}P_\kappa(\sigma q^\beta/\zeta,\sigma/\zeta)}{(q;q)_\nu(q;q)_\kappa}$$

$$\times\ {}_3\phi_2\left(\begin{array}{c} q^{-(\nu+\kappa-\mu)}, \chi\zeta t, \rho t\zeta q^{\alpha} \\ \psi\zeta t, \rho t\zeta \end{array}; q; q\right), \quad |\chi t\zeta| < 1. \tag{5.1}$$

Proof. From (2.2), we have

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \mathcal{C}_{\nu+\mu}^{(\alpha-\nu-\mu)}(\chi,\psi,\rho|q)\mathcal{C}_{\nu}^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^{\nu}}{(q;q)_{\nu}} \\
&= \sum_{\nu=0}^{\infty} \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{P_{\nu+\mu}(\chi,\psi)\mathcal{C}_{\nu}^{(\beta-\nu)}\left(\zeta,\omega,\sigma|q\right)\frac{t^{\nu}}{(q;q)_{\nu}}\right\} \quad \text{(by using (2.2))} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty} P_{\nu+\mu}(\chi,\psi)\frac{t^{\nu}}{(q;q)_{\nu}}\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}\begin{bmatrix}\beta\\ \kappa\end{bmatrix}(q;q)_{\kappa}(-1)^{\kappa}q^{\binom{\kappa}{2}}\sigma^{\kappa}P_{\nu-\kappa}(\zeta,\omega)\right\} \\
&\qquad \text{(by using (1.24))} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\kappa=0}^{\infty}\sum_{\nu=\kappa}^{\infty} P_{\nu+\mu}(\chi,\psi)\frac{t^{\nu}}{(q;q)_{\kappa}(q;q)_{\nu-\kappa}}P_{\kappa}(\sigma q^{\beta},\sigma)P_{\nu-\kappa}(\zeta,\omega)\right\} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\kappa=0}^{\infty}\frac{t^{\kappa}P_{\kappa}(\sigma q^{\beta},\sigma)}{(q;q)_{\kappa}}\sum_{\nu=0}^{\infty}\frac{P_{\nu+\kappa+\mu}(\chi,\psi)t^{\nu}P_{\nu}(\zeta,\omega)}{(q;q)_{\nu}}\right\} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\kappa=0}^{\infty}\frac{t^{\kappa}P_{\kappa+\mu}(\chi,\psi)P_{\kappa}(\sigma q^{\beta},\sigma)}{(q;q)_{\kappa}}\ {}_2\phi_1\left(\begin{array}{c}\frac{q^{\kappa+\mu}\psi}{\chi}, \omega/\zeta \\ 0\end{array}; q, \chi\zeta t\right)\right\} \\
&= \mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\sum_{\kappa=0}^{\infty}\frac{t^{\kappa}P_{\kappa+\mu}(\chi,\psi)P_{\kappa}(\sigma q^{\beta},\sigma)}{(q;q)_{\kappa}}\frac{(q^{\kappa+\mu}\psi\zeta t;q)_{\infty}}{(\chi\zeta t;q)_{\infty}}\right. \\
&\quad \left.\times\ {}_2\phi_2\left(\begin{array}{c}\frac{q^{\kappa+\mu}\psi}{\chi}, 0 \\ 0, q^{\kappa+\mu}\psi\zeta t\end{array}; q, \omega\chi t\right)\right\} \quad \text{(by using (1.3))} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{t^{\kappa}P_{\kappa+\mu}(\chi,\psi)P_{\kappa}(\sigma q^{\beta},\sigma)(-1)^{\nu}q^{\binom{\nu}{2}}P_{\nu}(\chi,q^{\kappa+\mu}\psi)}{(q;q)_{\kappa}(q,q^{\kappa+\mu}\psi\zeta t;q)_{\nu}}\right. \\
&\quad \left.\times\frac{(q^{\kappa+\mu}\psi\zeta t;q)_{\infty}(\omega t)^{\nu}}{(\chi\zeta t;q)_{\infty}}\right\} \\
&= \mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{t^{\kappa}(\omega t)^{\nu}P_{\kappa}(\sigma q^{\beta},\sigma)(-1)^{\nu}q^{\binom{\nu}{2}}P_{\nu+\kappa+\mu}(\chi,\psi)}{(q;q)_{\kappa}(q;q)_{\nu}(\psi\zeta t;q)_{\kappa+\mu}}\right. \\
&\quad \left.\times\frac{(\psi\zeta t;q)_{\infty}}{(q^{\kappa+\mu}\psi\zeta t;q)_{\nu}(\chi\zeta t;q)_{\infty}}\right\} \\
&= \sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{t^{\kappa}(\omega t)^{\nu}P_{\kappa}(\sigma q^{\beta},\sigma)(-1)^{\nu}q^{\binom{\nu}{2}}}{(q;q)_{\kappa}(q;q)_{\nu}}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{P_{\nu+\kappa+\mu}(\chi,\psi)(\psi\zeta t;q)_{\infty}}{(\psi\zeta t;q)_{\kappa+\nu+\kappa}(\chi\zeta t,q)_{\infty}}\right\} \\
&= \sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{j=0}^{\kappa+\nu+\mu}\frac{t^{\kappa}(\omega t)^{\nu}P_{\kappa}(\sigma q^{\beta},\sigma)(-1)^{\nu}q^{\binom{\nu}{2}}}{(q;q)_{\kappa}(q;q)_{\nu}}\frac{(\psi\zeta t;q)_{\infty}(\rho\zeta t;q)_{\alpha}(\zeta t)^{-\nu-\kappa-\mu}}{(\chi t\zeta;q)_{\infty}} \\
&\quad \times\frac{(q^{-\nu-\kappa-\mu},\chi\zeta t,\rho t\zeta q^{\alpha};q)_{j}}{(q,\psi\zeta t,\rho t\zeta;q)_{j}}\ q^{j}
\end{aligned}$$

$$= \frac{(\zeta t)^{-\mu}(\psi\zeta t;q)_\infty(\rho\zeta t,q)_\alpha}{(\chi t\zeta;q)_\infty}\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\omega/\zeta)^\nu(-1)^\nu q^{\binom{\nu}{2}}P_\kappa(\sigma q^\beta/\zeta,\sigma/\zeta)}{(q;q)_\nu(q;q)_\kappa}$$

$$\times \sum_{j=0}^{\kappa+\nu+\mu}\frac{(q^{-(\nu+\kappa+\mu)},\chi\zeta t,\rho t\zeta q^\alpha;q)_j}{(q,\psi\zeta t,\rho t\zeta;q)_j}\,q^j$$

$$= \frac{(\zeta t)^{-\mu}(\psi\zeta t;q)_\infty(\rho\zeta t;q)_\alpha}{(\chi t\zeta;q)_\infty}\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\omega/\zeta)^\nu(-1)^\nu q^{\binom{\nu}{2}}P_\kappa(\sigma q^\beta/\zeta,\sigma/\zeta)}{(q;q)_\nu(q;q)_\kappa}$$

$$\times\ {}_3\phi_2\left(\begin{matrix} q^{-(\nu+\kappa-\mu)},\chi\zeta t,\rho t\zeta q^\alpha \\ \psi\zeta t,\rho t\zeta\end{matrix};q;q\right).$$

□

- By substituting $\mu = 0$ in equation (5.1), Mehler's formula for Cigler's polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ was obtained.

Corollary 5.2. *The following identity was established*:

$$\sum_{\nu=0}^{\infty}\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)\mathcal{C}_\nu^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^\nu}{(q;q)_\nu}$$

$$= \frac{(\psi\zeta t;q)_\infty(\rho\zeta t;q)_\alpha}{(\chi t\zeta;q)_\infty}\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\omega/\zeta)^\nu(-1)^\nu q^{\binom{\nu}{2}}P_\kappa(\sigma q^\beta/\zeta,\sigma/\zeta)}{(q;q)_\nu(q;q)_\kappa}$$

$$\times\ {}_3\phi_2\left(\begin{matrix} q^{-(\nu+\kappa)},\chi\zeta t,\rho t\zeta q^\alpha \\ \psi\zeta t,\rho t\zeta\end{matrix};q,q\right),\quad \max\{|\chi t\zeta|,|\chi t|\}<1. \tag{5.2}$$

∗ By taking $\chi = 1$, $\psi = 0$, $\alpha = \log_q(x/y)$, $\rho = y$, $\zeta = 1$, $\omega = 0$, $\beta = \log_q(u/v)$, $\zeta = u$ and $\sigma = v$ in equation (5.2), we re-derived Mehler's formula for the polynomials $h_n(x,y|q)$ (equation (1.18)).

Proof.

$$\sum_{n=0}^{\infty}h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}$$

$$= \frac{(yt;q)_\infty}{(t,tx;q)_\infty}\sum_{k=0}^{\infty}\frac{P_k(u,v)}{(q;q)_k}\sum_{j=0}^{k}\frac{(q^{-k},t,tx;q)_j}{(q;q)_j(yt;q)_j}\,q^j$$

$$= \frac{(yt;q)_\infty}{(t,tx;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{P_k(u,v)}{(q;q)_{k-j}}\frac{(-1)^j q^{\binom{j}{2}}q^{-kj}(t,tx;q)_j}{(q;q)_j(yt;q)_j}\,q^j$$

$$= \frac{(yt;q)_\infty}{(t,tx;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{P_{k+j}(u,v)}{}\frac{(-1)^j q^{\binom{j}{2}}q^{j-j^2}(t,tx;q)_j}{(q;q)_j(yt;q)_j}\,q^{-kj}$$

$$= \frac{(yt;q)_\infty}{(t,tx;q)_\infty}\sum_{j=0}^{\infty}\frac{P_j(u,v)}{(q;q)_j}\frac{(-1)^j q^{-\binom{j}{2}}(t,tx;q)_j}{(yt;q)_j}\sum_{k=0}^{\infty}\frac{P_k(u,q^jv)q^{-kj}}{(q;q)_k}$$

$$= \frac{(yt;q)_\infty}{(t,tx;q)_\infty} \sum_{j=0}^{\infty} \frac{P_j(u,v)}{(q;q)_j} \frac{(-1)^j q^{-\binom{j}{2}}(t,tx;q)_j}{(yt;q)_j} \frac{(vq^j q^{-j};q)_\infty}{(uq^{-j};q)_\infty}$$

$$= \frac{(yt,v;q)_\infty}{(t,tx;q)_\infty} \sum_{j=0}^{\infty} \frac{P_j(u,v)}{(q;q)_j} \frac{(t,tx;q)_j}{(yt;q)_j} \frac{(-1)^j q^{-\binom{j}{2}}}{(uq^{-j};q)_\infty}$$

$$= \frac{(yt,v;q)_\infty}{(t,tx,u;q)_\infty} \sum_{j=0}^{\infty} \frac{(v/u;q)_j u^j}{(q;q)_j} \frac{(t,tx;q)_j}{(yt;q)_j} \frac{(q/u)^j}{(q/u;q)_j}$$

$$= \frac{(yt,v;q)_\infty}{(t,tx,u;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} t,tx,v/u \\ q/u,yt \end{matrix} ;q,q \right).$$

$$= \frac{(yt,v;q)_\infty}{(t,tx,u;q)_\infty} \frac{(vxt,u;q)_\infty}{(v,uxt;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} v/u,y,tx \\ vxt,yt \end{matrix} ;q,ut \right) \quad \text{(by using (1.11))}$$

$$= \frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} y,xt,v/u \\ yt,vxt \end{matrix} ;q,ut \right).$$

□

∗ By substituting $\alpha \to \infty$, $\chi = x$, $\psi = y$, $\rho = a$, $\beta \to \infty$, $\zeta = z$, $\omega = w$, $\sigma = b$ in equation (5.2) and using an equation (1.6), Mehler's formula for $U_n(x,y,a;q)$ was proposed.

Corollary 5.3. *The following identity holds:*

$$\sum_{n=0}^{\infty} U_n(x,y,a;q) U_n(z,w,b;q) \frac{t^n}{(q;q)_n}$$
$$= \frac{(yzt;q)_\infty (azt;q)_\infty}{(xtz;q)_\infty} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(abt)^k (wta)^n q^{\binom{n}{2}} q^{\binom{k}{2}} q^{\binom{n+k}{2}}}{(q;q)_n (q;q)_k}$$
$$\times\ {}_2\phi_1\left(\begin{matrix} q^{-(n+k)}, y/x \\ yzt \end{matrix} ;q;xq/a \right), \quad \max\{|xtz|, xq/a\} < 1.$$

- By substituting $q^\alpha \to x/y$, $\chi = 1$, $\psi = 0$, $\rho = y$, $q^\beta \to z/w$, $\zeta = 1$, $\omega = 0$ and $\sigma = w$ in equation (5.1), an extension of Mehler's formula for $h_n(x,y|q)$ was derived.

Corollary 5.4. *The following identity holds:*

$$\sum_{n=0}^{\infty} h_{n+m}(x,y|q)\, h_n(z,w|q) \frac{t^n}{(q;q)_n}$$
$$= \frac{t^{-m}(yt;q)_\infty (azt;q)_\infty}{(t,xt;q)_\infty} \sum_{k=0}^{\infty} \frac{P_k(z,w)}{(q;q)_k}\, {}_3\phi_2\left(\begin{matrix} q^{-(m+k)}, t, tx \\ 0, yt \end{matrix} ;q;q \right),$$

provided that $\max\{|t|, |xt|\} < 1$.

- By substituting $\alpha \to \infty$, $\chi = x$, $\psi = y$, $\rho = a$, $\beta \to \infty$, $\zeta = z$, $\omega = w$, $\sigma = b$ in equation (5.1) and using equation (1.6), an extension of Mehler's formula for $U_n(x,y,a;q)$ was obtained.

Corollary 5.5. *We are possessing*

$$\begin{aligned}
&\sum_{n=0}^{\infty} U_{n+m}(x,y,a;q)U_n(z,w,b;q)\frac{t^n}{(q;q)_n}\\
&\quad=(za)^m\frac{(yzt;q)_\infty(azt;q)_\infty}{(xtz;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m q^{\binom{n}{2}}q^{\binom{k}{2}}q^{\binom{n+k+m}{2}}(wta)^n(abt)^k}{(q;q)_n(q;q)_k}\\
&\qquad\times\ {}_2\phi_1\left(\begin{matrix} q^{-(n+m+k)}, y/x \\ yzt\end{matrix};q;xq/a\right),\quad \max\{|xtz|, xq/a\}<1.
\end{aligned}$$

6.. Mixed Generating Functions for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$

In this section, by an operator approach, we are creating mixed generating functions for Cigler's polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$. We are specified parameter values for Cigler's polynomials, in order to determine Mehler's formula for q-Hahn polynomials $\phi_n^{(a)}(x|q)$, as well as, by employing specified parameter values to establish a mixed generating function for both $h_n(x,y|q)$ and $U(x,y,a;q)$.

Theorem 6.1. (Mixed generating functions of $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$). *We have*

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\phi_\nu^{(\alpha)}(\chi|q)\mathcal{C}_\nu^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^\nu}{(q;q)_\nu}\\
&\quad=\frac{(\omega t,\alpha\chi;q)_\infty(\sigma t,q)_\beta}{(\zeta t,\chi;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}\zeta t,\sigma t q^\beta,\alpha,0\\ \omega t,\sigma t, q/\chi\end{matrix};q,q\right),
\end{aligned}\tag{6.1}$$

provided $\max\{|\zeta t|,|\chi|\}<1$.

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\phi_\nu^{(\alpha)}(\chi|q)\mathcal{C}_\nu^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^\nu}{(q;q)_\nu}\\
&=\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}(\alpha;q)_\kappa\chi^\kappa\mathcal{C}_\nu^{(\beta-\nu)}(\zeta,\omega,\sigma|q)\frac{t^\nu}{(q;q)_\nu}\quad(\text{by using (1.9)})\\
&=\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}(\alpha;q)_\kappa\chi^\kappa\,\mathcal{T}(q^{-\beta};q,\sigma D_{\zeta\omega})\left\{P_\nu(\zeta,\omega)\right\}\frac{t^\nu}{(q;q)_\nu}\quad(\text{by using (2.2)})\\
&=\mathcal{T}(q^{-\beta};q,\sigma D_{\zeta\omega})P_\nu(\zeta,\omega)\sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\nu}\begin{bmatrix}\nu\\ \kappa\end{bmatrix}(\alpha;q)_\kappa\chi^\kappa\,\frac{t^\nu}{(q;q)_\nu}\\
&=\mathcal{T}(q^{-\beta};q,\sigma D_{\zeta\omega})\left\{\sum_{\kappa=0}^{\infty}\sum_{\nu=\kappa}^{\infty}\frac{(\alpha;q)_\kappa\chi^\kappa P_\nu(\zeta,\omega)t^\nu}{(q;q)_\kappa(q;q)_{\nu-\kappa}}\right\}\\
&=\mathcal{T}(q^{-\beta};q,\sigma D_{\zeta\omega})\left\{\sum_{\kappa=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\alpha;q)_\kappa\chi^k P_{\nu+\kappa}(\zeta,\omega)t^{\kappa+\nu}}{(q;q)_\kappa(q;q)_\nu}\right\}\\
&=\mathcal{T}(q^{-\beta};q,\sigma D_{\zeta\omega})\left\{\sum_{\kappa=0}^{\infty}P_\kappa(\zeta,\omega)\frac{(\alpha;q)_\kappa(\chi t)^\kappa}{(q;q)_\kappa}\sum_{\nu=0}^{\infty}P_\nu(\zeta,q^\kappa\omega)\frac{t^\nu}{(q;q)_\nu}\right\}
\end{aligned}$$

$$= \mathcal{T}(q^{-\beta}; q, \sigma D_{\zeta\omega})\left\{ \sum_{\kappa=0}^{\infty} P_\kappa(\zeta,\omega) \frac{(\alpha;q)_\kappa (\chi t)^\kappa}{(q;q)_\kappa} \frac{(q^\kappa \omega t;q)_\infty}{(\zeta t;q)_\infty} \right\} \quad \text{(by using (1.14))}$$

$$= \sum_{\kappa=0}^{\infty} \frac{(\alpha;q)_\kappa \chi^\kappa}{(q;q)_\kappa} \mathcal{T}(q^{-\beta}; q, \sigma D_{\zeta\omega})\left\{ \frac{P_\kappa(\zeta t,\omega t)}{(\omega t;q)_\kappa} \frac{(\omega t;q)_\infty}{(\zeta t;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha;q)_\kappa \chi^\kappa}{(q;q)_\kappa} \frac{(\omega t;q)_\infty (\sigma t,q)_\beta}{(\zeta t;q)_\infty} \sum_{j=0}^{\kappa} \frac{(q^{-\kappa}, \zeta t, \sigma t q^\beta; q)_j}{(q,\omega t, \sigma t;q)_j} q^j \quad \text{(by using (2.4))}$$

$$= \frac{(\omega t;q)_\infty (\sigma t;q)_\beta}{(\zeta t;q)_\infty} \sum_{j=0}^{\infty} \sum_{\kappa+j=j}^{\infty} \frac{(\alpha;q)_{j+\kappa} \chi^{\kappa+j} (-1)^j q^{\binom{j}{2}} q^{-j(j+\kappa)}}{(q;q)_\kappa} \frac{(\zeta t, \sigma t q^\beta;q)_j}{(q,\omega t,\sigma t;q)_j} q^j$$

$$= \frac{(\omega t;q)_\infty (\sigma t;q)_\beta}{(\zeta t;q)_\infty} \sum_{j=0}^{\infty} (\alpha;q)_j \chi^j (-1)^j q^{\binom{j}{2}} q^{-j^2} \frac{(\zeta t, \sigma t q^\beta;q)_j}{(q,\omega t,\sigma t;q)_j} \sum_{\kappa=0}^{\infty} \frac{(\alpha q^j;q)_\kappa \chi^\kappa q^{-j\kappa} q^j}{(q;q)_\kappa}$$

$$= \frac{(\omega t;q)_\infty (\sigma t;q)_\beta}{(\zeta t;q)_\infty} \sum_{j=0}^{\infty} (\alpha;q)_j \chi^j (-1)^j q^{\binom{j}{2}} q^{-j^2} \frac{(\zeta t, \sigma t q^\beta;q)_j}{(q,\omega t,\sigma t;q)_j} \frac{(\alpha\chi;q)_\infty q^j}{(\chi q^{-j};q)_\infty}$$

$$= \frac{(\omega t, \alpha\chi;q)_\infty (\sigma t;q)_\beta}{(\zeta t,\chi;q)_\infty} \sum_{j=0}^{\infty} \frac{(\zeta t, \sigma t q^\beta;q)_j}{(q,\omega t,\sigma t;q)_j} \frac{(\alpha;q)_j q^j}{(q/\chi;q)_j} \quad \text{(by using (1.7))}$$

$$= \frac{(\omega t, \alpha\chi;q)_\infty (\sigma t;q)_\beta}{(\zeta t,\chi;q)_\infty} \, {}_4\phi_3 \left(\begin{matrix} \zeta t, \sigma t q^\beta, \alpha, 0 \\ \omega t, \sigma t, q/\chi \end{matrix} ; q, q \right).$$

□

- Taking $\zeta = 1$, $\omega = 0$, $\chi = x$, $\alpha = a$, $\beta = \log_q(b^{-1})$, $\sigma = by$ and $t = \tau$ in equation (6.1), we are retaining Mehler's formula for $\phi_n^{(a)}(x|q)$ (equation (1.10)).
- Taking $\chi = x$, $\alpha = a$, $\zeta = z$, $\omega = w$, $\beta \to \infty$ and $\sigma = b$ in equation (6.1), we are obtaining mixed generating functions formula for $U_n(x,y,a;q)$.

Corollary 6.2. *The following result holds:*

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q) U_n(z,w,b;q) \frac{t^n}{(q;q)_n} = \frac{(wt,ax,bt;q)_\infty}{(zt,x;q)_\infty} \, {}_4\phi_3 \left(\begin{matrix} zt, a, 0, 0 \\ wt, bt, q/x \end{matrix} ; q, q \right),$$

provided $\max\{|zt|, |x|\} < 1$.

- Letting $\chi = x$, $\alpha = a$, $\zeta = 1$, $\omega = 0$, $\beta \to \log_q(z/w)$ and $\sigma \to w$ in equation (6.1), then using the transformation (1.11), the mixed generating functions formula for $h_n(x,y|q)$ was derived.

Corollary 6.3. *The following identity holds:*

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q) h_n(z,w|q) \frac{t^n}{(q;q)_n} = \frac{(wt,axzt;q)_\infty}{(t,zt,xzt;q)_\infty} \, {}_3\phi_2 \left(\begin{matrix} a, w, zt \\ axzt, wt \end{matrix} ; q, xt \right),$$

provided $\max\{|zt|, |t|, |xzt|, |xt|\} < 1$.

Theorem 6.4. (Another mixed generating functions for $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$). *We have*

$$\sum_{\nu=0}^{\infty}\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)P_\nu(\zeta,\omega)\frac{t^\nu}{(q;q)_\nu}$$
$$=\frac{(\omega/\zeta,\rho\zeta t,\psi\zeta t;q)_\infty}{(\rho\zeta tq^\alpha,\chi\zeta t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\rho\zeta tq^\alpha,\chi\zeta t,0\\ \rho\zeta t,\psi\zeta t\end{matrix};q,\omega/\zeta\right),\quad |\chi\zeta t|<1. \tag{6.2}$$

Proof.

$$\sum_{\nu=0}^{\infty}\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho)P_\nu(\zeta,\omega)\frac{t^\nu}{(q;q)_\nu}$$
$$=\sum_{\nu=0}^{\infty}\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{P_\nu(\chi,\psi)P_\nu(\zeta,\omega)\frac{t^\nu}{(q;q)_\nu}\right\}$$
$$=\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty}P_\nu(\chi,\psi)P_\nu(\zeta,\omega)\frac{t^\nu}{(q;q)_\nu}\right\}$$
$$=\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\sum_{\nu=0}^{\infty}(\psi/\chi;q)_\nu(\omega/\zeta;q)_\nu\frac{(\chi\zeta t)^\nu}{(q;q)_\nu}\right\}$$
$$=\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{{}_2\phi_1\left(\begin{matrix}\psi/\chi,\omega/\zeta\\ 0\end{matrix};q,\chi\zeta t\right)\right\}$$
$$=\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\frac{(\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}\psi/\chi,0\\ 0,\psi\zeta t\end{matrix};q,\omega\chi t\right)\right\}\quad\text{(by using (1.3))}$$
$$=\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\frac{(\psi\zeta t;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}\psi/\chi\\ \psi\zeta t\end{matrix};q,\omega\chi t\right)\right\}$$
$$=\mathcal{T}\left(q^{-\alpha};q,\rho D_{\chi\psi}\right)\left\{\frac{(\psi\zeta t,\omega/\zeta;q)_\infty}{(\chi\zeta t;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0,\zeta\chi t\\ \psi\zeta t\end{matrix};q,\omega/\zeta\right)\right\}\quad\text{(by using (1.4))}$$
$$=(\omega/\zeta;q)_\infty\sum_{\nu=0}^{\infty}\frac{(\rho\zeta tq^\nu,\psi\zeta tq^\nu;q)_\infty}{(\rho\zeta tq^\nu q^\alpha,\chi\zeta tq^\nu;q)_\infty}\frac{(\omega/\zeta)^\nu}{(q;q)_\nu}\quad\text{(by using (2.3))}$$
$$=\frac{(\omega/\zeta,\rho\zeta t,\psi\zeta t;q)_\infty}{(\rho\zeta tq^\alpha,\chi\zeta t;q)_\infty}\sum_{\nu=0}^{\infty}\frac{(\rho\zeta tq^\alpha,\chi\zeta t;q)_\nu}{(q,\rho\zeta t,\psi\zeta t;q)_\nu}(\omega/\zeta)^\nu$$
$$=\frac{(\omega/\zeta,\rho\zeta t,\psi\zeta t;q)_\infty}{(\rho\zeta tq^\alpha,\chi\zeta t;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\rho\zeta tq^\alpha,\chi\zeta t,0\\ \rho\zeta t,\psi\zeta t\end{matrix};q,\omega/\zeta\right).$$

□

- Assuming $\chi=x$, $\psi=y$, $\zeta=z$, $\rho=a$, $\alpha\to\infty$, $\zeta=z$ and $\omega=w$ in equation (6.2), we are obtaining another mixed generating functions formula for $U_n(x,y,a;q)$.

Corollary 6.5. *The following identity holds:*

$$\sum_{n=0}^{\infty} U_n(x,y,a;q)P_n(z,w)\frac{t^n}{(q;q)_n} = \frac{(w/z,azt,yzt;q)_\infty}{(xzt;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} xzt,0,0 \\ azt,yzt \end{matrix};q,w/z\right),$$

provided $\max\{|w/z|,|xzt|\}<1$.

- Taking $\chi=1$, $\psi=0$, $\alpha\to\log_q(x/y)$, $\rho=y$,$\zeta=z$ and $\omega=w$ in equation (6.2), then using the transformation (1.3), another mixed generating functions formula for $h_n(x,y|q)$ was obtained.

Corollary 6.6. *The following identity was established*:

$$\sum_{n=0}^{\infty} h_n(x,y|q)P_n(z,w)\frac{t^n}{(q;q)_n} = \frac{(yzt,wxt,y;q)_\infty}{(ztx,zt;q)_\infty}\ {}_2\phi_2\left(\begin{matrix} ztx,y \\ yzt,wtx \end{matrix};q,wt\right),$$

provided $\max\{|zt|,|xzt|,|wt|\}<1$.

- Letting $\omega/\zeta=\lambda$ and $\zeta\to 1$ in equation (6.2), Srivastava-Agarwal type generating function formula for $C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ (equation (1.27)) was formulated.
- Letting $\chi=1$, $\psi=0$, $\alpha\longrightarrow\log_q(a^{-1})$, $\rho\to ax$ and $t=\tau$ in equation (6.2) and using Heine's transformation (1.5), Srivastava-Agarwal type generating function formula for $\phi_n^{(a)}(x)$ (equation (1.12)) was formulated.

7.. A Transformational Identity Involving Generating Functions for $C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$

Here, we demonstrate the polynomials' $C_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q)$ transformational identity (equation (1.29)), assuming that equation (1.28) was satisfied, by using the homogeneous operator $\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})$.

Proof. Let $f(\chi,\psi,\rho)$ denote the left-hand side of (1.29). Using (1.28) and (2.2), we determine that

$$\begin{aligned} f(\chi,\psi,\rho) &= \sum_{\nu=0}^{\infty} A(\nu)\mathcal{C}_\nu^{(\alpha-\nu)}(\chi,\psi,\rho|q) \\ &= \mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\sum_{\nu=0}^{\infty} A(\nu)P_\nu(\chi,\psi)\right\} \\ &= \mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\sum_{\nu=0}^{\infty} B(\nu)\frac{(q^\nu\psi t;q)_\infty}{(q^\nu\chi t;q)_\infty}\right\} \\ &= \sum_{\nu=0}^{\infty} B(\nu)\mathcal{T}(q^{-\alpha};q,\rho D_{\chi\psi})\left\{\frac{(\psi q^\nu t;q)_\infty}{(\chi q^\nu t;q)_\infty}\right\} \\ &= \sum_{\nu=0}^{\infty} B(\nu)\frac{(\psi q^\nu t,\rho t q^\nu;q)_\infty}{(\chi q^\nu t,\rho t q^\alpha q^\nu;q)_\infty} \quad \text{(by using (2.3))} \end{aligned}$$

It is exactly (1.29)'s right-hand side. equation (1.29) has been fully proved. □

- Letting $\chi=1$, $\psi=0$, $\alpha\to\log_q(x/y)$ and $\rho=y$ in equations (1.28) and (1.29), the transformational identity for $h_n(x,y|q)$ was formulated.

Corollary 7.1. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relation -ship*:

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t; q)_\infty}. \tag{7.1}$$

Then

$$\sum_{n=0}^{\infty} A(n) h_n(x, y|q) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t; q)_\infty} \frac{(ytq^n; q)_\infty}{(xtq^n; q)_\infty}, \tag{7.2}$$

both series in (7.1) and (7.2) are absolutely convergent.

- Letting $\alpha \to \infty$, $\chi = x$, $\psi = y$ and $\rho = a$ in equations (1.28) and (1.29), the transformational identity for $U_n(x, y, a; q)$ is obtained.

Corollary 7.2. *Suppose the coefficients $A(m)$ and $B(m)$ fulfil the following relationship*:

$$\sum_{m=0}^{\infty} A(m) P_m(x, y) = \sum_{m=0}^{\infty} \frac{B(m)}{(q^m t; q)_\infty}. \tag{7.3}$$

Then

$$\sum_{m=0}^{\infty} A(m) U_m(x, y, a; q) = \sum_{m=0}^{\infty} B(m) \frac{(ytq^m, atq^m; q)_\infty}{(xtq^m; q)_\infty}, \tag{7.4}$$

Both series in (7.3) and (7.4) are absolutely convergent.

8.. Conclusions

(1) By supplying special values to the generalised homogeneous q-shift operator $\mathcal{T}\left(q^{-\alpha}; q, \rho D_{\chi\psi}\right)$, it is possible to obtain a number of operators that were previously specified. for more information, see [7,8,13].

(2) The bivariate Rogers–Szegö polynomials and Al-Salam-Carlitz polynomials are specific instances for Cigler's polynomials $\mathcal{C}_\nu^{(\alpha-\nu)}(\chi, \psi, \rho|q)$.

(3) The identities for Cigler's polynomials are generalization for the polynomials identities for both the bivariate Rogers–Szegö polynomials and Al-Salam-Carlitz polynomials.

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