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THE NUMERICAL SOLUTIONS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The main object of the paper is to study the behavior of the numerical solutions of the Caputo-Fabrizio partial differential equations (CFPDEs) by using modified Adomian decomposition method (MADM). Moreover, we discuss some new existence, uniqueness, and convergence results. Finally, some examples are included to demonstrate the validity and applicability of the proposed technique.

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1. Introduction

Fractional calculus (FC) has many applications in mathematics and has attracted many researchers. The publicity of FC is due to the variety of the fractional derivatives operators, such us the Caputo fractional derivative (CFD), the Riemann-Liouville fractional derivative (RLFD), the Atangana-Baleanu fractional derivative (ABFD), the Caputo-Fabrizio fractional derivative (CFFD), and many others. The ABFD and CFFD were recently used in modeling physical phenomena. The ABFD and CFFD are great compromises in modeling real world phenomena [1–5].

In recent years, a many of approximate and analytical methods have been utilized to solve the PDEs with LFDOs such as the FADM [6–8], FVIM [9–15], FRDTM [15–18], FDTM [19–21], FSEM [22,23], FSTM [24], FLTM [25], FLVIM [26–31], FHAM [32], FLHPM [33], FSHPM [34], FSHAM [35], FLDM [36], FFDM [37], FSVIM [38,39] and another methods [40–70].

The main objectiveof the paper is to study thebehavior of the solution that can be formally determined by analyticalapproximated method as the MADM. Furthermore, we proved the existence, uniqueness results (EUR), and convergence of the solution of the CFPDEs. The rest of the paper is organized as

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follows: In Section 2, some preliminaries related to FC are recalled. In Section 3, modified ADM is constructed for solving the CFPDEs. In Section 4, the EUR of the solutions have been proved. In Section 5, the two examples are presented to illustrate the accuracy of MADM. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

Definition 2.1. let $u \in H^1(a,b)$, $a > b$, $a \in (-\infty, t)$, $0 < \alpha < 1$. The fractional derivative of $u(t)$ in the Caputo-Fabrizio sense is [2,6,37]

$$ {}_{a}^{CF}D_t^{\alpha}u(t) = \frac{\beta(\alpha)}{(1-\alpha)} \int_a^t u^{'}(s)\, exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds \tag{2.1} $$

where $\beta(\alpha)$ is a normalization function satisfying $\beta(0) = \beta(1) = 1$

Hence, we have the following properties: [40,41]

- ${}_{a}^{CF}D_t^{\alpha}u = u$, where $\alpha = 0$.
- ${}_{a}^{CF}D_t^{\alpha}[u(t) + v(t)] = {}_{a}^{CF} D_t^{\alpha}u(t) + {}_{a}^{CF} D_t^{\alpha}v(t)$.
- ${}_{a}^{CF}D_t^{\alpha}(c) = 0$, where c is constant.

Definition 2.2. let $u \in H^1(a,b)$, $a > b$, $a \in (-\infty, t)$, $0 < \alpha < 1$. The fractional integral of order α of $u(t)$ is [2,7,37]

$$ {}_{a}^{CF}I_t^{\alpha}u(t) = \frac{1-\alpha}{\beta(\alpha)}u(t) + \frac{\alpha}{\beta(\alpha)} \int_a^t u(s)ds. \tag{2.2} $$

Hence, we have the following properties: [40,41]

- ${}_{a}^{CF}I_t^{\alpha}u(t) = u(t)$, where $\alpha = 0$.
- ${}_{a}^{CF}I_t^{\alpha}[u(t) + v(t)] = {}_{a}^{CF} I_t^{\alpha}u(t) + {}_{a}^{CF} I_t^{\alpha}v(t)$.
- ${}_{a}^{CF}I_t^{a}\, {}_{a}^{CF}D_t^{a}\, u(t) = u(t) - u(a)$.

Definition 2.3. Over the set of functions

$$ A = \{u(t) | \exists M, \tau_1, \tau_2 > 0, |u(t)| < Me^{|t|/\tau_j}, if t \in (-1)^j \times [0, \infty)\}, $$

the Sumudu transform (ST) is defined by [42]

$$ G(z) = S[u(t)] = \int_0^{\infty} u(zt)e^{-t}dt \tag{2.3} $$

where $z \in (\tau_1, \tau_2)$

Some ST properties: [42]

- $S[c] = c$, c is constant
- $S[t^{n\alpha}] = \Gamma(1 + n\alpha)\, z^{n\alpha}$

Proposition 2.1. If $S\{f(t)\}$ is the ST of $f(t)$, then the ST of the fractional derivative with CFFD of $f(t)$ of order $\alpha \in (0,1)$ is

$$S\{^{CF}D_t^{\alpha} f(t)\} = \frac{1}{1-\alpha+\alpha z}\left[S\{f(t)\} - f(0)\right] \tag{2.4}$$

Proof. By using CFFD of (2.1) we have

$$\begin{aligned} S\{^{CF}D_t^{\alpha} f(t)\} &= S\left\{\frac{1}{1-\alpha}\int_0^t \frac{\partial f(s)}{\partial s} exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds\right\} \\ &= \frac{1}{1-\alpha} S\left\{\int_0^t \frac{\partial f(s)}{\partial s} exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds\right\} \end{aligned}$$

By using properties of ST we get

$$S\{^{CF}D_t^{\alpha} f(t)\} = \frac{1}{1-\alpha} z S\left\{\frac{\partial f(t)}{\partial t}\right\} S\left\{exp\left(-\frac{\alpha}{1-\alpha}t\right)\right\}$$

By using properties of ST we get

$$\begin{aligned} S\{^{CF}D_t^{\alpha} f(t)\} &= \frac{1}{1-\alpha} z\left[(z^{-1}S\{f(t)\} - z^{-1}f(0))\right]\left[\frac{1}{1+\frac{\alpha}{1-\alpha}z}\right] \\ &= \frac{1}{1-\alpha+\alpha z}[S\{f(t)\} - f(0)] \end{aligned}$$

□

3. Analysis of MADM

Let us consider the CFPDE:

$$^{CF}D_t^{\alpha} u(x,t) + R(u(x,t)) + N(u(x,t)) = g(x,t) \tag{3.1}$$

With the initial condition

$$u(x,0) = u_0(x), \tag{3.2}$$

where $^{CF}D_t^{\alpha}$ is CFFD, R is a linear operator , N is an nonlinear operator , g is a source term and $0 < \alpha < 1$.

Taking ST to (3.1) we obtain

$$S\{^{CF}D_t^{\alpha} u(x,t)\} + S\{R(u) + N(u)\} = S\{g\} \tag{3.3}$$

or

$$\frac{1}{1-\alpha+\alpha z}[S\{u(x,t)\} - u_0(x)] = S\{g(x,t) - R(u(x,t)) - N(u(x,t))\} \tag{3.4}$$

By substituting (3.2) in (3.4) we get

$$\begin{aligned} S\{u(x,t)\} &= u_0(x) + (1-\alpha+\alpha z)S\{g(x,t) - R(u) - N(u)\} \\ &= u_0(x) + (1-\alpha+\alpha z)S\{g\} \end{aligned}$$

$$-(1-\alpha+\alpha z)S\{R(u)+N(u)\} \tag{3.5}$$

By using inverse ST to both sides of (3.5) we get

$$\begin{aligned} u(x,t) &= S^{-1}\{u_0(x)\}+S^{-1}\{(1-\alpha+\alpha z)S\{g\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha z)S\{R(u)+N(u)\}\} \end{aligned} \tag{3.6}$$

Now, we represent solution as an infinite series given below

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.7}$$

and the nonlinear term can be decomposed as

$$N(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,u_2,\cdots) \tag{3.8}$$

Where

$$A_n(u_0,u_1,\cdots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial\epsilon^n}\left[N\left(\sum_{i=0}^{\infty}\epsilon^i u_i\right)\right]_{\epsilon=0}.$$

By substituting (3.7)and (3.8) in (3.6) we get

$$\begin{aligned} \sum_{n=o}^{\infty}u_n(x,t) &= S^{-1}\{u_0(x)\}+S^{-1}\{(1-\alpha+\alpha z)S\{g\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha z)S\{\sum_{n=o}^{\infty}u_n(x,t)\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha z)S\{\sum_{n=o}^{\infty}A_n(u)\}\} \\ &= u_0(x)+S^{-1}\{(1-\alpha+\alpha z)S\{g\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha t)S\{\sum_{n=o}^{\infty}u_n\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha z)S\{\sum_{n=o}^{\infty}A_n(u)\}\} \end{aligned} \tag{3.9}$$

On comparing both sides of the Eq. (3.9), we get

$$\begin{aligned} u_0 &= u_0(x)+S^{-1}\{(1-\alpha+\alpha z)S\{g(x,t)\}\} \\ u_{n+1} &= -S^{-1}\{(1-\alpha+\alpha z)S\{u_n(x,t)\}\} \\ &\quad -S^{-1}\{(1-\alpha+\alpha z)S\{A_n(u)\}\} \end{aligned} \tag{3.10}$$

Finally, we approximate the analytical solution (3.1) by truncated series:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \tag{3.11}$$

4. Main Results

In this section, we shall give an existence and uniqueness results of following initial value problem

$$ {}^{CF}D_t^{\alpha}u(x,t) = g(x,t,u(x,t)), u(x,0) = u_0(x), \tag{4.1}$$

and prove it.

Theorem 4.1. Let $0 < \alpha < 1$, $g : [a,b] \times [0,\infty) \times H^1([a,b] \times [0,\infty)) \longrightarrow R$ be a continuous map, then the solution of the following IVP:

$$\begin{aligned} &{}^{CF}D_t^{\alpha}u(x,t) = g(x,t,u(x,t)) - g(x,0,u(x,0))exp\left(-\frac{\alpha}{1-\alpha}t\right) \\ &u(x,0) = u_0(x) \end{aligned} \tag{4.2}$$

is given by

$$\begin{aligned} u(x,t) &= u_0(x) + (1-\alpha)g(x,t,u(x,t)) \\ &\quad + \alpha\int_0^t g(x,s,u(x,s))ds \ \ ;t>0 \end{aligned} \tag{4.3}$$

Proof. We verify that functiondefined in (4.3) is the solution of theproblem (4.2), by (4.3), we get

$$\frac{\partial}{\partial x}u(x,t) = (1-\alpha)\frac{\partial}{\partial x}g(x,t,u(x,t)) + \alpha g(x,t,u(x,t)) \tag{4.4}$$

Hence, multipelied by functon $exp\left(\frac{\alpha}{1-\alpha}t\right)$ on bothsides of (4.4) , for $t \geq 0$:

$$\begin{aligned} &exp\left(\frac{\alpha}{1-\alpha}t\right)\frac{\partial}{\partial x}u(x,t) = \\ &exp\left(\frac{\alpha}{1-\alpha}t\right)\left((1-\alpha)\frac{\partial}{\partial x}g(x,t,u(x,t)) + \alpha g(x,t,u(x,t))\right) \\ &= (1-\alpha)\frac{\partial}{\partial t}\left(exp\left(\frac{\alpha}{1-\alpha}t\right)g(x,t,u(x,t))\right) \end{aligned} \tag{4.5}$$

Integreting from 0 to t on bothsides of (4.5), for $t \geq 0$, we get

$$\begin{aligned} &\int_0^t exp\left(\frac{\alpha}{1-\alpha}s\right)\frac{\partial}{\partial x}u(x,s)ds \\ &= (1-\alpha)\left(exp\left(\frac{\alpha}{1-\alpha}t\right)g(x,t,u(x,t)) - g(x,0,u(x,0))\right) \end{aligned}$$

From (4.6) , for $t \geq 0$, we get

$$ {}^{CF}D_t^{\alpha}u(x,t) = g(x,t,u(x,t)) - g(x,0,u(x,0))\left(exp\left(-\frac{\alpha}{1-\alpha}t\right)\right) \tag{4.6}$$

As well, from (4.3), we know that $u(x,0) = u_0(x)$. With (4.6) combined, we get that $u(x,t)$ is the solution of (4.2). □

Theorem 4.2. Let $0 < \alpha < 1$, $g : [a, b] \times [0, \infty) \times H^1([a, b] \times [0, \infty)) \longrightarrow R$ be a continuous map, then the solution of theIVP:

$$
\begin{aligned}
&{}^{CF}D_t^{\alpha} u(x,t) = g(x,t,u(x,t)) - g(x,0,u(x,0)) \\
&u(x,0) = u_0(x)
\end{aligned}
\tag{4.7}
$$

is given by

$$
\begin{aligned}
u(x,t) &= u_0(x) + (1-\alpha)\left[g(x,t,u(x,t)) - g(x,0,u(x,0))\right] \\
&\quad + \alpha \int_0^t g(x,s,u(x,s))ds - \alpha g(x,0,u(x,0))t \ \ ; t \geq 0
\end{aligned}
\tag{4.8}
$$

Proof. Here, we test that function defined in (4.8) is the solution of (4.7). By (4.8), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= (1-\alpha) \frac{\partial}{\partial t} g(x,t,u(x,t)) \\
&\quad + \alpha g(x,t,u(x,t)) - \alpha g(x,0,u(x,0)) \ \ ; t \geq 0
\end{aligned}
\tag{4.9}
$$

Multiplying by $exp\left(\frac{\alpha}{1-\alpha}t\right)$ on (4.9), we obtain

$$
\begin{aligned}
exp\left(\frac{\alpha}{1-\alpha}t\right)\frac{\partial}{\partial t} u(x,t) &= exp\left(\frac{\alpha}{1-\alpha}t\right)(1-\alpha)\frac{\partial}{\partial t} g(x,t,u(x,t)) \\
&\quad + exp\left(\frac{\alpha}{1-\alpha}t\right)\alpha g(x,t,u(x,t)) \\
&\quad - exp\left(\frac{\alpha}{1-\alpha}t\right)\alpha g(x,0,u(x,0)) \\
&= (1-\alpha)\frac{\partial}{\partial t}\left(exp\left(\frac{\alpha}{1-\alpha}t\right) g(x,t,u(x,t))\right) \\
&\quad - exp\left(\frac{\alpha}{1-\alpha}t\right)\alpha g(x,0,u(x,0)).
\end{aligned}
\tag{4.10}
$$

Now Integreting from 0 to t on the equality above, we obtain

$$
\begin{aligned}
&\int_0^t exp\left(\frac{\alpha}{1-\alpha}s\right)\frac{\partial}{\partial s} u(x,s)ds \\
&= (1-\alpha)\left(exp\left(\frac{\alpha}{1-\alpha}t\right) g(x,t,u(x,t) - g(x,0,u(x,0))\right) \\
&\quad -(1-\alpha)g(x,0,u(x,0))\left(exp\left(\frac{\alpha}{1-\alpha}t\right) - 1\right) \\
&= (1-\alpha)exp\left(\frac{\alpha}{1-\alpha}t\right)\left[g(x,t,u(x,t)) - g(x,0,u(x,0))\right]
\end{aligned}
\tag{4.11}
$$

According to the equality above, we get

$$
{}^{CF}D_t^{\alpha} u(x,t) = g(x,t,u) - g(x,0,u(x,0)) \tag{4.12}
$$

Hence, u defined in (4.8) is the solution of (4.7). □

Theorem 4.3. Consider the NFPIDE (4.1) over the region M and suppose that g satisfies LC with respect to u with constant c such that $c < \dfrac{1}{1-\alpha+\alpha t}$, then (4.1)

$$T_u = u_0(x) +^{CF} I_t^{\alpha}(g(x,t,u)) \tag{4.13}$$

has a unique solution.

Proof. The set of all contnuously differentiable functions definedon M form a Banach space.Therefore it remains to show that T is a contractive mapping and for this purpose take $u_1, u_2 \in H^1([a,b]\times[0,\infty))$ and then:

$$\begin{aligned}
\|T_{u_1} - T_{u_2}\| &= \|u_0(x) +^{CF} I_t^{\alpha}(g(x,t,u_1(x,t))) \\
&\quad -u_0(x) -^{CF} I_t^{\alpha}(g(x,t,u_2(x,t)))\| \\
&= \|^{CF} I_t^{\alpha}(g(x,t,u_1(x,t))) -^{CF} I_t^{\alpha}(g(x,t,u_2(x,t)))\| \\
&= \|^{CF} I_t^{\alpha}(g(x,t,u_1(x,t)) - g(x,t,u_2(x,t)))\| \\
&= \|(1-\alpha)[g(x,t,u_1(x,t)) - g(x,t,u_2(x,t))] \\
&\quad +\alpha \int_0^t [g(x,s,u_1(x,s)) - g(x,s,u_2(x,s))ds]\| \\
&\leq (1-\alpha)\|g(x,t,u_1(x,t)) - g(x,t,u_2(x,t))\| \\
&\quad +\alpha\| \int_0^t [g(x,s,u_1(x,s)) - g(x,s,u_2(x,s))ds]\| \\
&\quad (Since\ g\ satisfies\ Lipschitz\ condition) \\
&\leq (1-\alpha)c\|u_1(x,t) - u_2(x,t)\| + \alpha c\|u_1(x,t) - u_2(x,t)\| \int_0^t ds \\
&= (1-\alpha)c\|u_1(x,t) - u_2(x,t)\| + \alpha c\|u_1(x,t) - u_2(x,t)\| t \\
&= \|u_1(x,t) - u_2(x,t)\|(1-\alpha+\alpha t)c
\end{aligned}$$

Since $c < \dfrac{1}{1-\alpha+\alpha t}$ which implies $(1-\alpha+\alpha t)c < 1$, T is a contrective mapping andtherefore T has a unique fixedpoint which means that (4.1) has a unique solution □

5. Illustrative Examples

Example 5.1. We consider the following nonlinear Burger equation with CFFD:

$$^{CF}D_t^{\alpha} u + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \tag{5.1}$$

subject to the initial condition

$$u(x,0) = x \tag{5.2}$$

Taking ST of (5.1):

$$S\{^{CF}D_t^{\alpha}u\} = S\left\{\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right\} \tag{5.3}$$

By using ST inverse of (5.3):

$$v = x + S^{-1}\left((1-\alpha+\alpha u)\left(S\left\{\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} v_n\right\} - S\left\{\sum_{n=0}^{\infty} A_n\right\}\right)\right). \tag{5.4}$$

Now

$$\begin{aligned} A_0 &= u_0\frac{\partial u_0}{\partial x} \\ A_1 &= u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} \\ A_2 &= u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x} \end{aligned} \tag{5.5}$$

From relation (3.10) we get

$$\begin{aligned} u_0 &= x \\ u_1 &= S^{-1}\left((1-\alpha+\alpha z)\left(S\left\{\frac{\partial^2 u_0}{\partial x^2}\right\} - S\{A_0\}\right)\right) \\ &= S^{-1}\left((1-\alpha+\alpha z)S\{0\} - S\{x\}\right) \\ &= -S^{-1}(1-\alpha+\alpha z) \\ &= -x(1-\alpha+\alpha t) \\ u_2 &= S^{-1}\left((1-\alpha+\alpha z)\left(S\left\{\frac{\partial^2 u_1}{\partial x^2}\right\} - S\{A_1\}\right)\right) \\ &= S^{-1}\left((1-\alpha+\alpha z)S\{2x(1-\alpha+\alpha t)\}\right) \\ &= S^{-1}\left(2x(1-\alpha+\alpha z)^2\right) \\ &= 2x\left((1+\alpha^2-2\alpha) + 2(\alpha-\alpha^2)t + \frac{1}{2}\alpha^2 t^2\right) \end{aligned} \tag{5.6}$$

Therefore, the solution of (5.1) is

$$u(t,x) = x - x(1-\alpha+\alpha t) + 2x\left((1+\alpha^2-2\alpha) + 2(\alpha-\alpha^2)t + \frac{1}{2}\alpha^2 t^2\right) + \cdots \tag{5.7}$$

Therefore, the approximate solution of (5.7) is

$$u(x,t) = \frac{x}{1+t},$$

for $\alpha = 1$, which is the exact solution of (5.1).

Table 1 shows the values of the exact and approximate solution of $u(x,t)$ among the different values of x, t and α, as well as the values of the absolute error between the exact solution and the approximate solution when $\alpha = 1$ of $u(x,t)$ for equation (5.1). In Figure 1 , we plotted the graphs of the approximate and exact solutions among different values of t and α when x is fixed and the Figures 2,3,4,5 show the graphs of the approximate and exact solutions among different values of x and t when $\alpha = 0.8, 0.9, 1$ respectively for equation (5.1).

Example 5.2. We consider the following linear heat-like equation in the Caputo-Fabrizio sense

$$ {}^{CF}D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{5.8} $$

where $0 \leq x, y \leq 2\pi$, $0 < \alpha \leq 1$, $t > 0$, with the initial condition

$$ u(x, y, 0) = \sin(x)\sin(y) \tag{5.9} $$

By using ST $G(z)$ to both sides of (5.8) we get

$$ S\left\{{}^{CF}D_t^{\alpha} u\right\} = S\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} \tag{5.10} $$

Taking ST inverse $G^{-1}(z)$to both sides of (5.10) we get

$$ u = \sin(x)\sin(y) + S^{-1}\left((1-\alpha+\alpha z)S\left\{\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n + \frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty} u_n\right\}\right) \tag{5.11} $$

From relation (3.10) we get

$$ \begin{aligned} u_0 &= \sin(x)\sin(y) \\ u_1 &= S^{-1}\left((1-\alpha+\alpha z)S\left\{\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right\}\right) \\ &= S^{-1}\left((1-\alpha+\alpha z)S\left\{-2\sin(x)\sin(y)\right\}\right) \\ &= -S^{-1}\left((1-\alpha+\alpha z)(2\sin(x)\sin(y))\right) \\ &= -2\sin(x)\sin(y)(1-\alpha+\alpha t) \end{aligned} $$

$$ \begin{aligned} u_2 &= S^{-1}\left((1-\alpha+\alpha z)S\left\{\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right\}\right) \\ &= S^{-1}\left((1-\alpha+\alpha z)S\left\{4\sin(x)\sin(y)(1-\alpha+\alpha t)\right\}\right) \\ &= 4\sin(x)\sin(y)S^{-1}\left((1-\alpha+\alpha z)^2\right) \\ &= 4\sin(x)\sin(y)\left((1+\alpha^2-2\alpha) + (2\alpha-2\alpha^2)t + \frac{1}{2}\alpha^2 t^2\right) \end{aligned} \tag{5.12} $$

Then, the approximate solution of (5.8) is

$$ \begin{aligned} u(t,x) &= \sin(x)\sin(y) - 2\sin(x)\sin(y)(1-\alpha+\alpha t) \\ &\quad + 4\sin(x)\sin(y)\left((1+\alpha^2-2\alpha) + (2\alpha-2\alpha^2)t + \frac{1}{2}\alpha^2 t^2\right) - \cdots \end{aligned} \tag{5.13} $$

The Eq. (5.13) is approximate solution to the form

$$ u(x,t) = \sin(x)\sin(y)e^{-2t}, $$

for $\alpha = 1$, which is the exact solution of (5.8).

Table 2 shows the values of the exact and approximate solution of $u(x, y, t)$ among the different values of x, y, t and α , as well as the values of the absolute error between the exact solution and the approximate solution when $\alpha = 1$ of u(x,y,t) for equation (5.8). In Figure 6 , we plotted the graphs of the approximate and exact solutions among different values of t and α whenx, y are fixed and the Figures 7,8,9,10 show the graphs of the approximate and exact solutions among different values of x and t when y is fixed and $\alpha = 0.8, 0.9, 1$ respectively for equation (5.8).

Table 1. The results obtained by the SDM for different values of α for equation (5.1)

x	t	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\|u_{\text{exact}} - u_{\alpha=1}\|$
0.25	0.2	0.2184	0.2111	0.2100	0.2083	0.0017
	0.4	0.2296	0.2084	0.1900	0.1786	0.0114
	0.6	0.2536	0.2219	0.1900	0.1563	0.0338
0.50	0.2	0.4368	0.4222	0.4200	0.4167	0.0033
	0.4	0.4592	0.4168	0.3800	0.3571	0.0229
	0.6	0.5072	0.4438	0.3800	0.3125	0.0675
0.75	0.2	0.6552	0.6333	0.6300	0.6250	0.0050
	0.4	0.6888	0.6252	0.5700	0.5357	0.0343
	0.6	0.7608	0.6657	0.5700	0.4688	0.1013

Table 2. The results obtained by the DJM and SDM for different values of α for equation (5.8)

x	y	t	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\|u_{\text{exact}} - u_{\alpha=1}\|$
0.25	0.2000	0.1000	0.0364	0.0368	0.0403	0.0402	0.0001
	0.4000	0.2000	0.0720	0.0664	0.0655	0.0646	0.0009
	0.6000	0.3000	0.1089	0.0925	0.0810	0.0767	0.0044
0.50	0.2000	0.1000	0.0706	0.0713	0.0781	0.0780	0.0001
	0.4000	0.2000	0.1395	0.1286	0.1270	0.1251	0.0018
	0.6000	0.3000	0.2109	0.1792	0.1570	0.1486	0.0084
0.75	0.2000	0.1000	0.1003	0.1013	0.1110	0.1109	0.0002
	0.4000	0.2000	0.1983	0.1828	0.1805	0.1779	0.0026
	0.6000	0.3000	0.2999	0.2547	0.2232	0.2112	0.0120

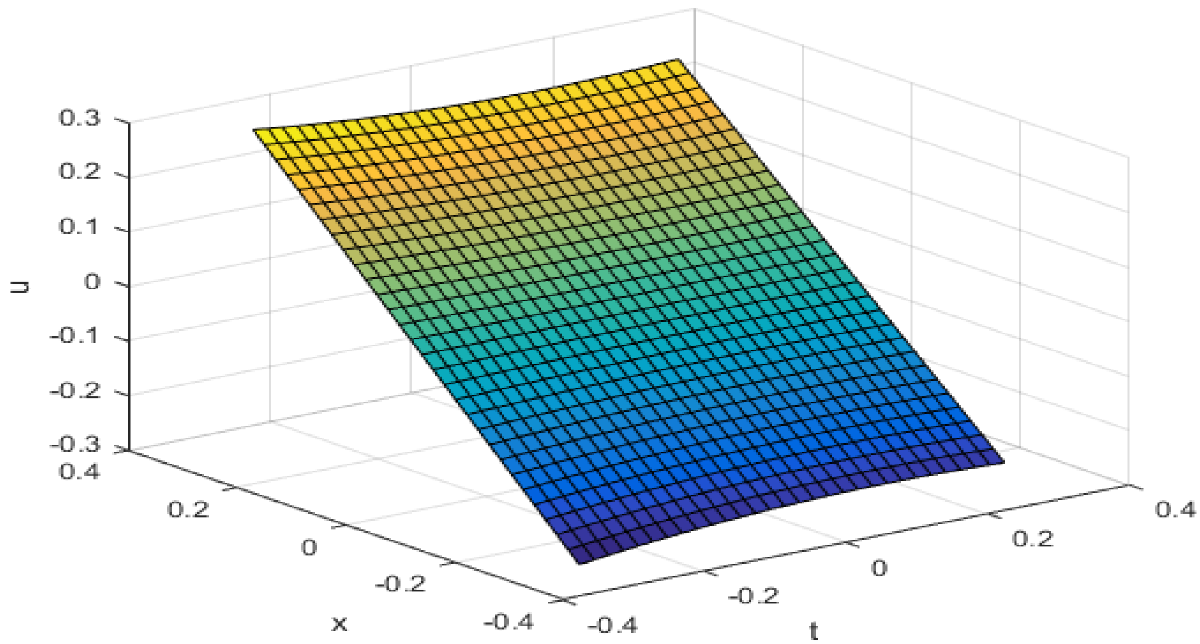

FIGURE 2. The surface generated from the numerical solution for equation (5.1) at $\alpha = 0.8$.

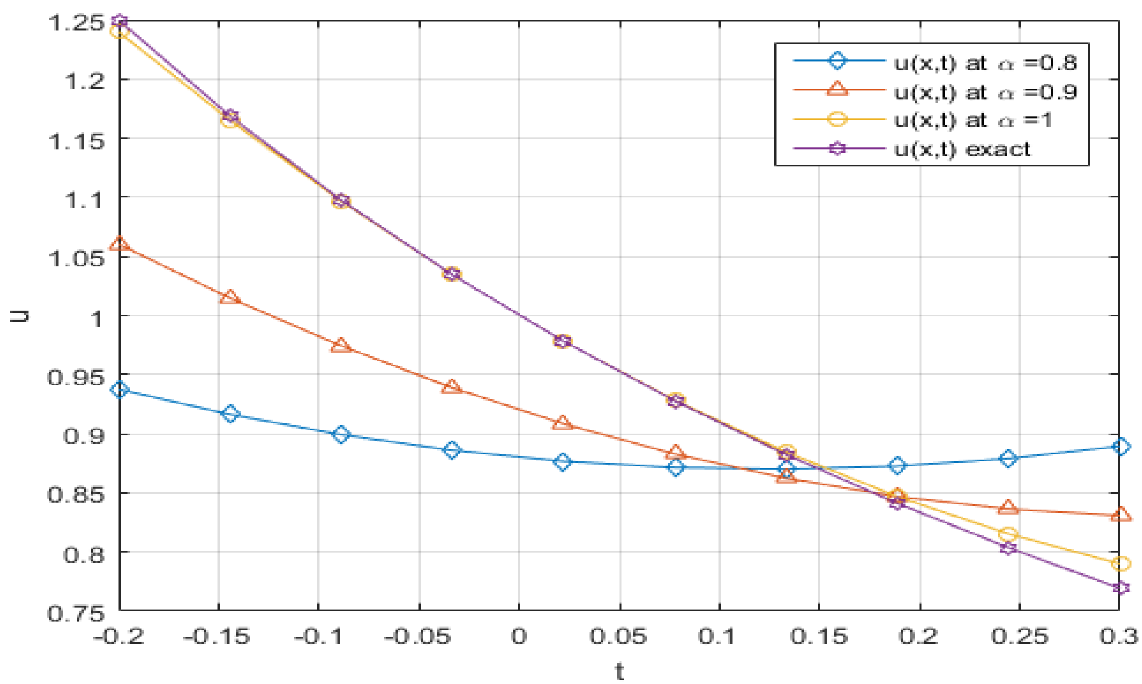

FIGURE 1. The plot for the approximate solution for equation (5.1) at ($\alpha = 0.8, \alpha = 0.9,$ $\alpha = 1$) and exact solution.

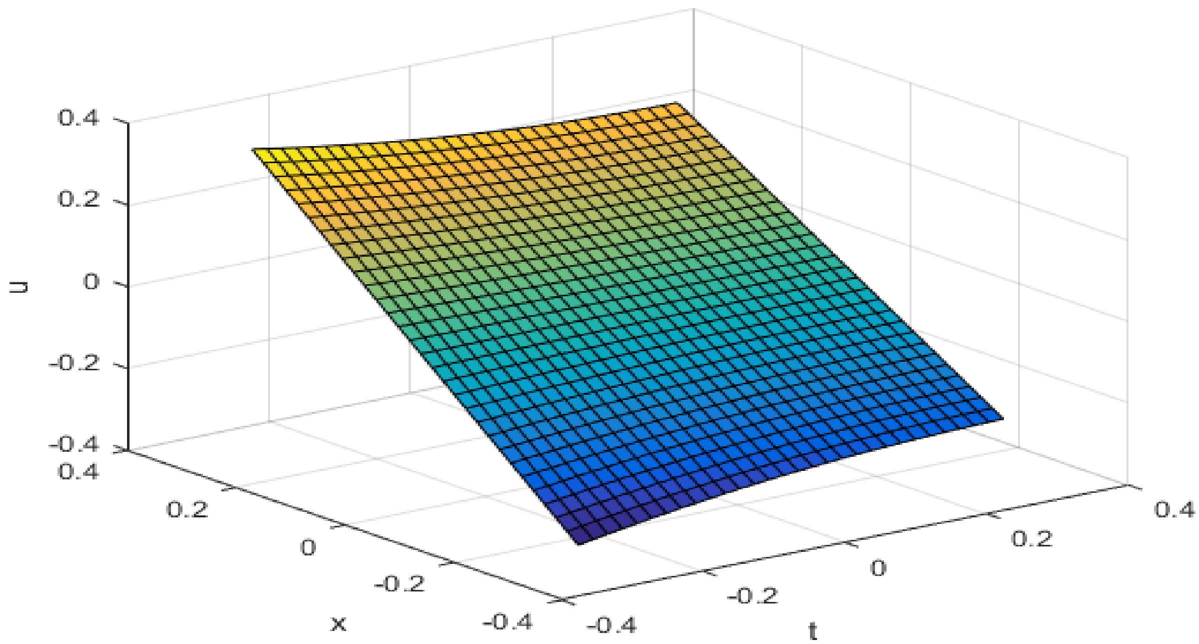

FIGURE 3. The surface generated from the numerical solution for equation (5.1) at $\alpha = 0.9$.

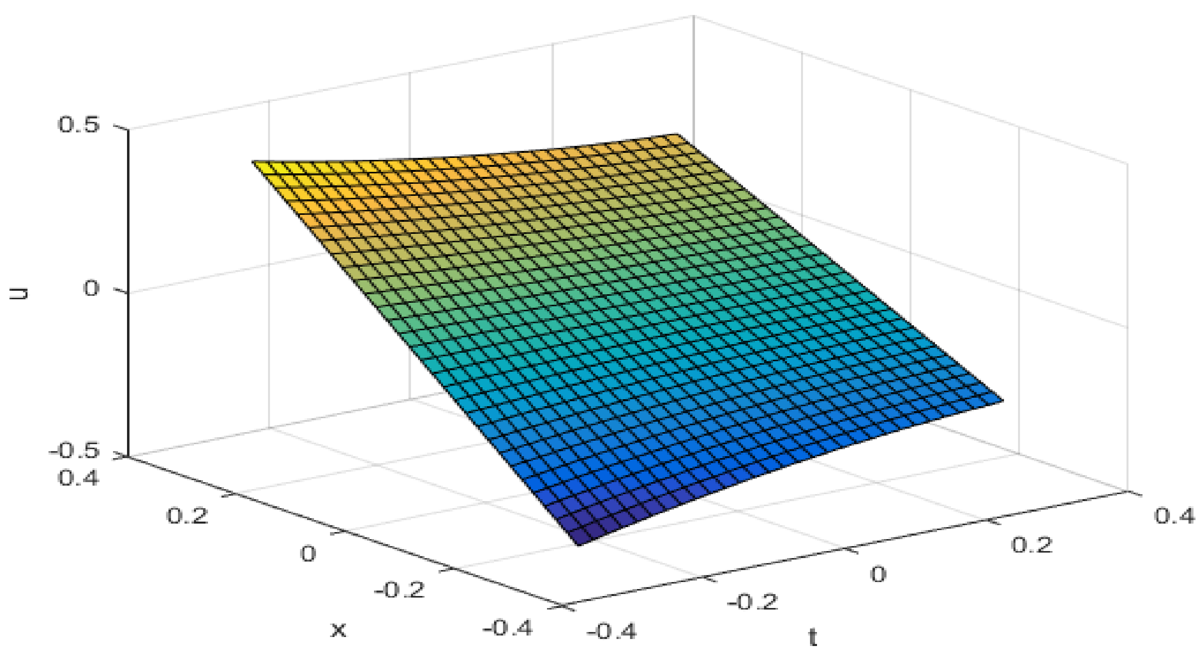

FIGURE 4. The surface generated from the numerical solution for equation (5.1) at $\alpha = 1$.

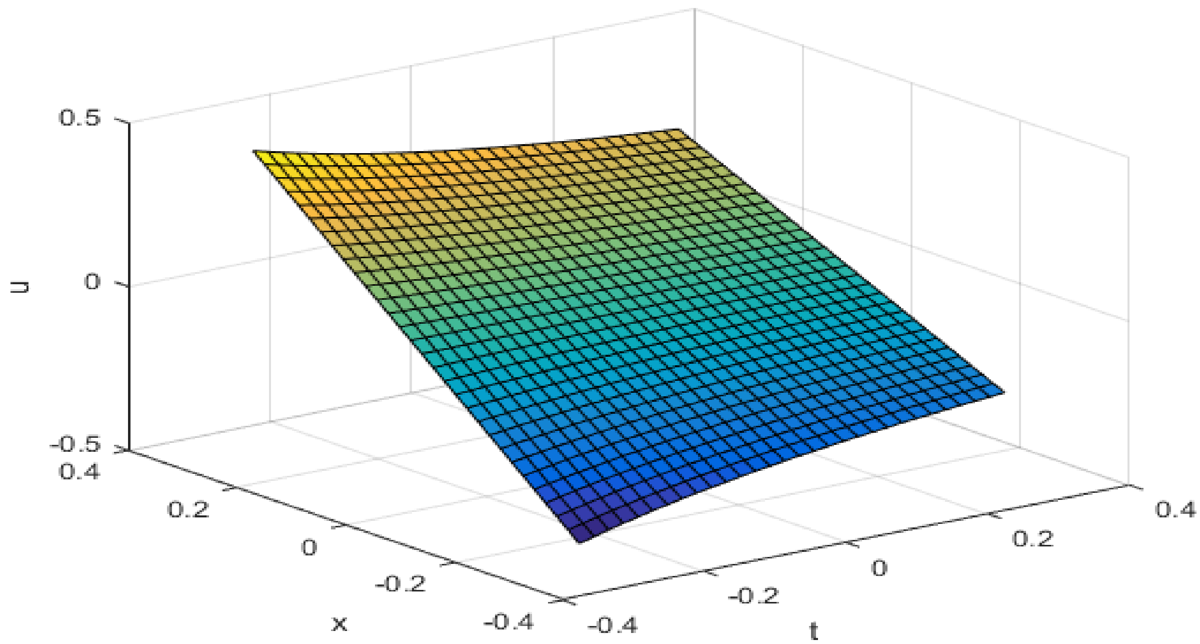

FIGURE 5. The surface generated from the exact solution for equation (5.1).

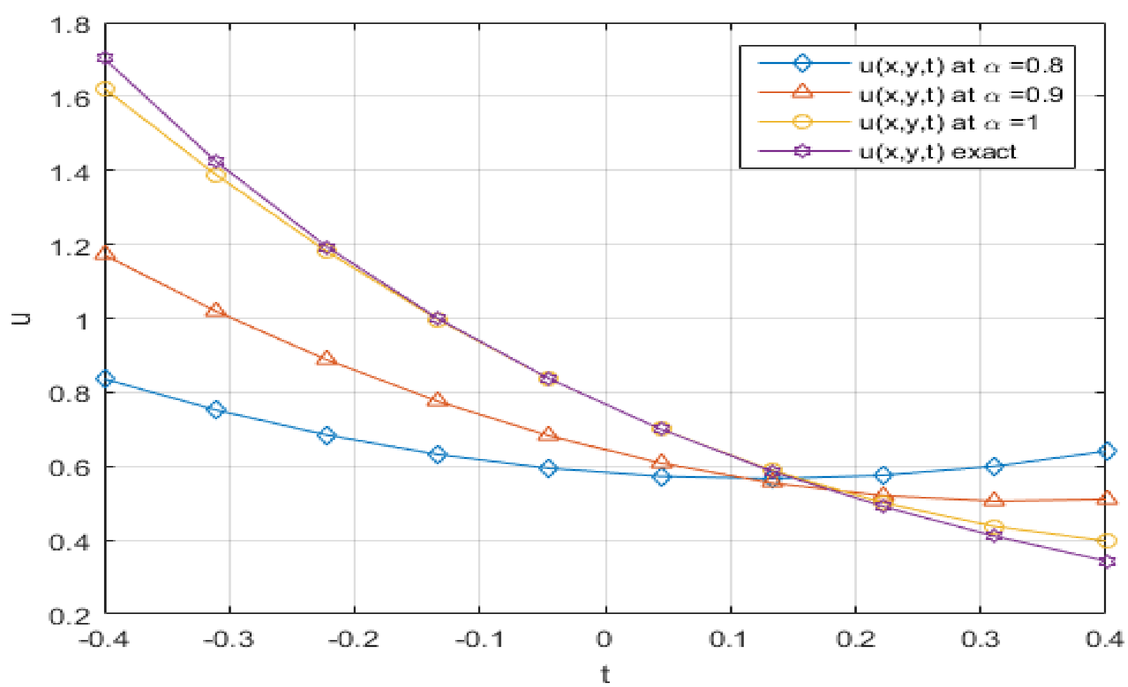

FIGURE 6. The plot for the exact and approximate solutions for equation (5.8) at ($\alpha = 0.8$,$\alpha = 0.9$,$\alpha = 1$).

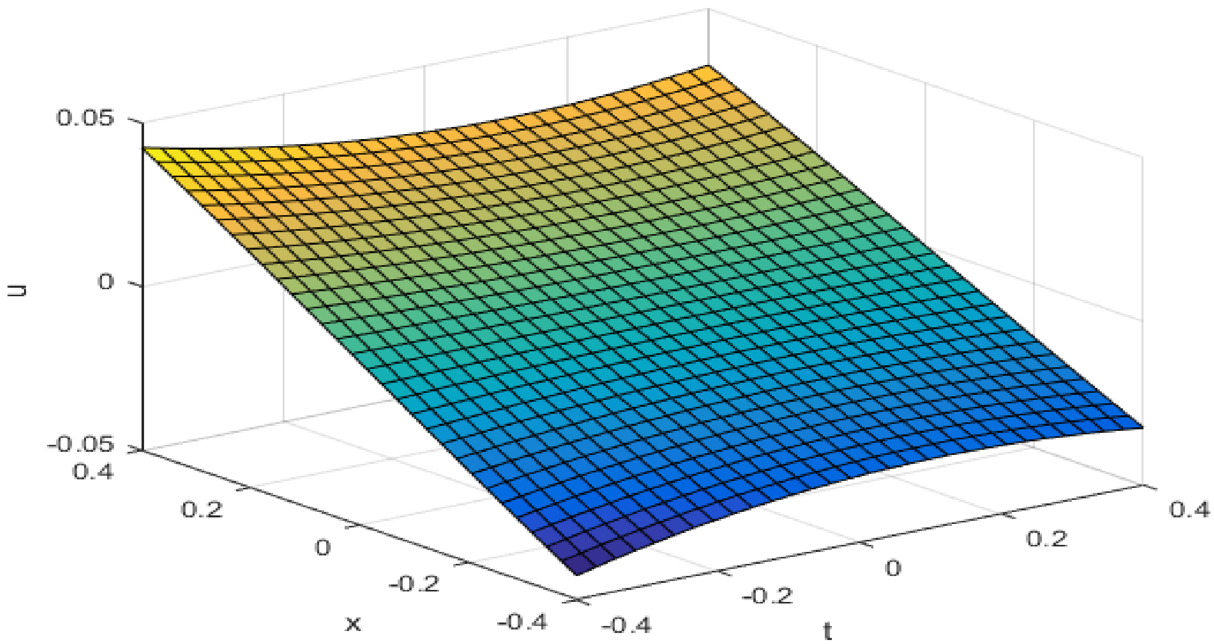

FIGURE 7. The surface generated from the numerical solution for equation (5.8) at $\alpha = 0.8$.

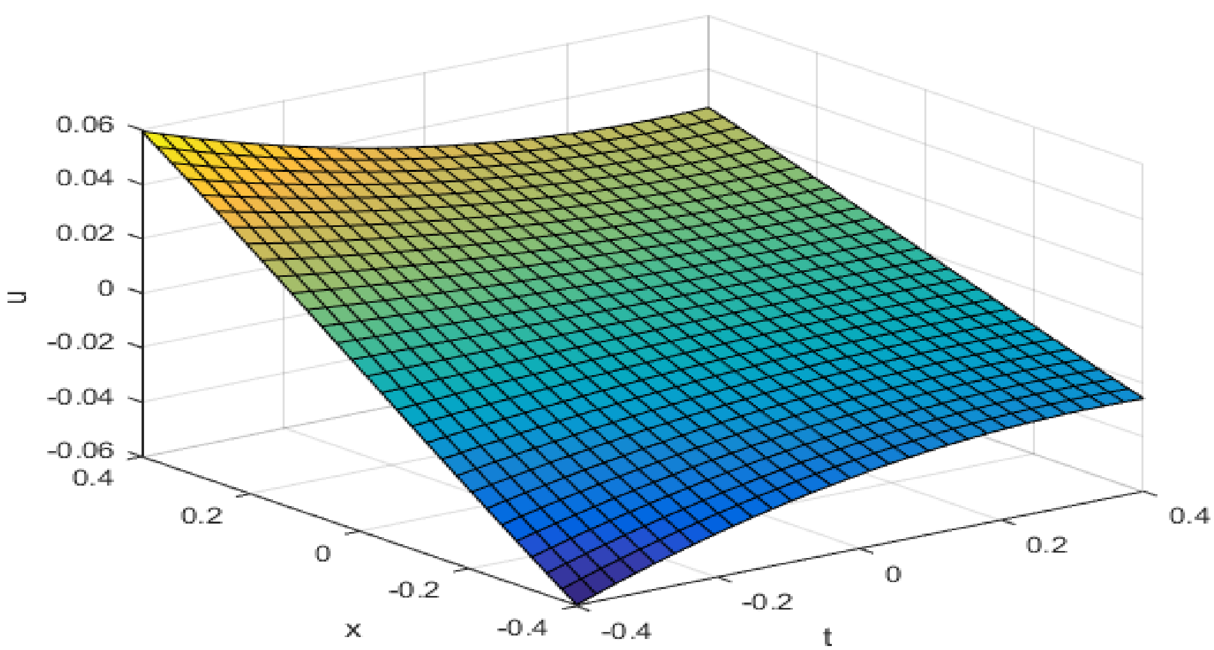

FIGURE 8. The surface generated from the numerical solution for equation (5.8) at $\alpha = 0.9$.

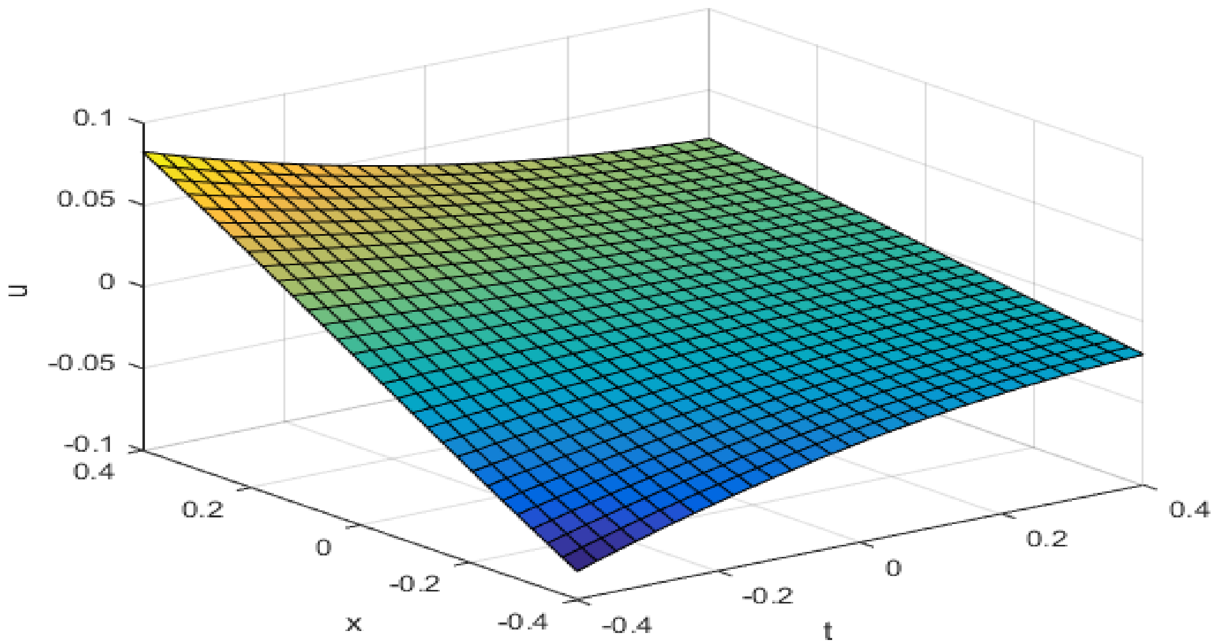

FIGURE 9. The surface generated from the numerical solution for equation (5.8) at $\alpha = 1$.

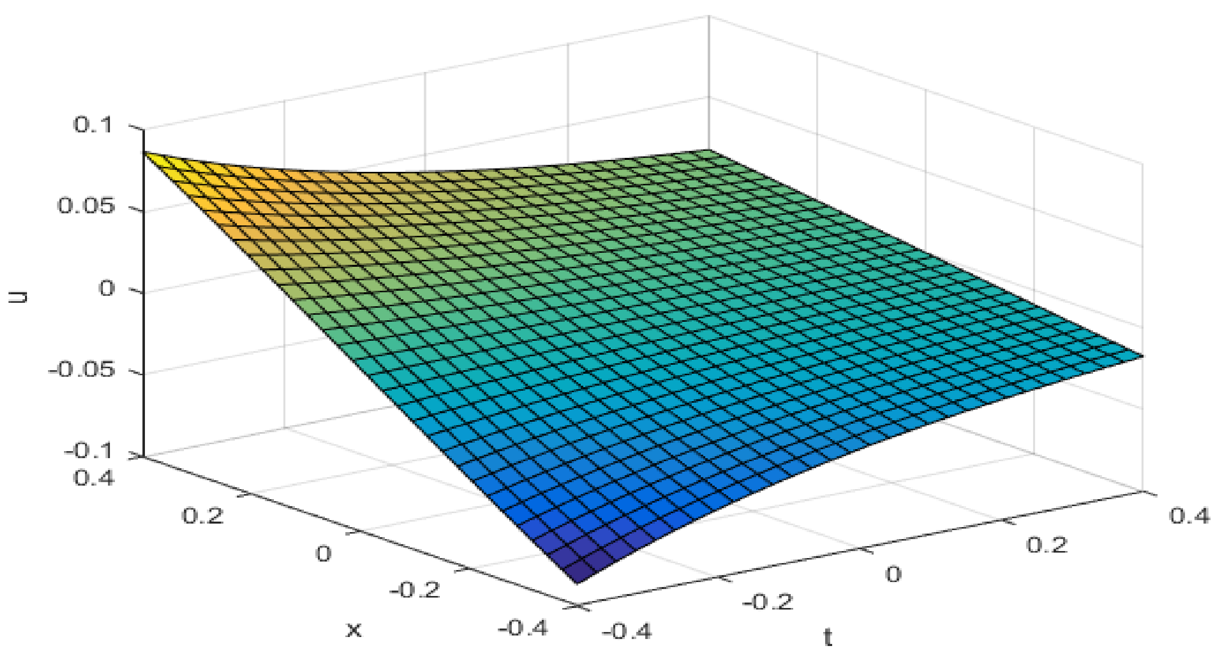

FIGURE 10. The surface generated from the exact solution for equation (5.8).

Conclusions

The MADM is successfully applied to find the approximate solutions of CFPDEs. The reliability of themethod and reduction in the size of the computational work givethis method a wider applicability. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear PDEs. Moreover, we proved the EUR of the solution. The illustrative examples establish the precision and efficiency of the MADM.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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