

INVESTIGATING THE EXTENDED SPECTRUM: OPERATOR GROUP INVERSE AND DRAZIN INVERSE

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ABSTRACT. This paper investigates the relationship between the extended spectrum of a bounded linear operator and its group inverse. We also establish a connection between the extended spectrum of the bounded linear operator and that of its Drazin inverse. As part of our analysis, we prove the following equality:

$$\sigma_{\text{ext}}((BA)^D) = \sigma_{\text{ext}}((AB)^D),$$

where $(BA)^D$ and $(AB)^D$ represent the Drazin inverses of BA and AB , respectively.

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1. INTRODUCTION AND PRELIMINARIES

Recently, Alan Lambert [8] established a set of properties related to the extended spectrum for invertible operators, as outlined in Property 2. Previous studies have also demonstrated this Property 2.

In this article, we propose to extend these properties (Properties 1 and 2) to a broader class of bounded linear operators, specifically those that possess an inner inverse, a group inverse, or a Drazin inverse.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For any operator $A \in \mathcal{B}(\mathcal{H})$, in the context of spectral theory, a complex number λ is referred to as an extended eigenvalue of A if there exists a non-zero operator $X \in \mathcal{B}(\mathcal{H})$ such that $AX = \lambda XA$. In this case, X is called the extended eigenvector corresponding to λ . The set of all extended eigenvalues of A forms the extended spectrum, denoted $\sigma_{\text{ext}}(A)$. The extended eigenspace associated with $\lambda \in \sigma_{\text{ext}}(A)$ is represented by $E_{\text{ext}}(A, \lambda)$ or simply $E_{\text{ext}}(\lambda)$.

Notably, for any operator $A \in \mathcal{B}(\mathcal{H})$, the number 1 is always an extended eigenvalue, with the identity operator I serving as the corresponding extended eigenvector.

An operator T is called invertible if there exists an operator $B \in \mathcal{B}(\mathcal{H})$ such that $AB = BA = I$, where I is the identity operator. The inverse of A is denoted by A^{-1} .

Proposition 1. *Let A be an invertible operator. Then: If $\lambda \in \sigma_{\text{ext}}(A)$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^{-1})$, where λ is a nonzero complex number.*

Proposition 2. [8] *Let A be an invertible operator. Then:*

- (1) *There are positive numbers a and b , with $a < b$, such that every extended eigenvalue for A is contained within the annulus $R_{a,b}$ with inner radius a and outer radius b .*
- (2) *If λ is an extended eigenvalue for A with $|\lambda| \neq 1$, then there is a positive integer N such that every product of N members of $E_{\text{ext}}(A, \lambda)$ is zero. In particular, every operator in $E_{\text{ext}}(A, \lambda)$ is nilpotent of order no greater than N .*
- (3) *If A has an extended eigenvalue of modulus other than 1, then it has an extended eigenvalue for which the N in part (2) above may be taken as 2.*

In this article, we will demonstrate that both previous properties can be proven for other classes of bounded linear operators. This includes operators that have a group inverse and a Drazin inverse, along with specific conditions in certain conditions.

2. MAIN ANALYSIS AND RESULTS

In this section, we will discuss the primary analysis and the results obtained from the study.

Definition 3. *Let $A \in \mathcal{B}(\mathcal{H})$. An operator $B \in \mathcal{B}(\mathcal{H})$ is called an interior inverse of A if it satisfies the condition:*

$$ABA = A.$$

Remark 4. *If B is the interior inverse of A , the following properties hold:*

- (1) *AB and BA are idempotent projections.*
- (2) *$R(AB) = R(A)$.*

Definition 5. *Let $A \in \mathcal{B}(\mathcal{H})$. An operator $B \in \mathcal{B}(\mathcal{H})$ is called an exterior inverse of A if it satisfies the condition:*

$$BAB = B.$$

Remark 6. *If B is the exterior inverse of A , the following properties hold:*

- (1) *AB and BA are idempotent projections.*
- (2) *$R(BA) = R(B)$.*

Definition 7. *Let $A \in \mathcal{B}(\mathcal{H})$. An operator $B \in \mathcal{B}(\mathcal{H})$ is called a generalized inverse of A if it satisfies the following conditions:*

$$\begin{cases} ABA = A, \\ BAB = B. \end{cases}$$

Remark 8. (1) If A has a generalized inverse, it is said that A is g -invertible.

(2) If B is the generalized inverse of A , then AB and BA are two bounded projections onto $\mathcal{R}(A)$ and $\mathcal{R}(B)$, respectively.

Theorem 9. Let $A \in \mathcal{B}(\mathcal{H})$ and suppose $B \in \mathcal{B}(\mathcal{H})$ is an interior inverse of A satisfying $AB = BA$. If λ is a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$ such that there exists an operator $X \in E_{\text{ext}}(A; \lambda)$ with $XA^2 \neq 0$ or $A^2X \neq 0$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(B)$.

Proof. Let λ be a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$. Then there exists an operator $X \in E_{\text{ext}}(A; \lambda)$ satisfying:

$$AX = \lambda XA,$$

$$ABAX = \lambda XABA,$$

$$BA^2X = \lambda XA^2B,$$

$$BA^2X = \frac{1}{\lambda}A^2XB \quad \text{or} \quad BXA^2 = \frac{1}{\lambda}XA^2B.$$

By taking $Y = A^2X$ or $Y = XA^2$, it is easy to see that $Y \in \mathcal{B}(\mathcal{H})$, and according to the hypothesis, $Y \neq 0$. Hence, we obtain:

$$BY = \frac{1}{\lambda}YB.$$

Then, $\frac{1}{\lambda} \in \sigma_{\text{ext}}(B)$. □

Definition 10. Let $A \in \mathcal{B}(\mathcal{H})$ with a closed image. The inverse group of A is the operator $B \in \mathcal{B}(\mathcal{H})$, denoted $B = A^\#$, which satisfies the following conditions:

$$\begin{cases} ABA = A, \\ BAB = B, \\ AB = BA. \end{cases}$$

Remark 11. Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then A is in the group of invertible operators, and $A^\# = A^{-1}$.

Theorem 12. Let $A \in \mathcal{B}(\mathcal{H})$, and suppose $A^\# \in \mathcal{B}(\mathcal{H})$ is the group inverse of A . If λ is a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^\#)$ and

$$E_{\text{ext}}(A; \lambda) \subseteq E_{\text{ext}}(A^\#; \frac{1}{\lambda}).$$

Proof. Let λ be a non-zero complex number, and $\lambda \in \sigma_{\text{ext}}(A)$. Then, there exists an operator $X \in E_{\text{ext}}(A; \lambda)$ satisfying:

$$AX = \lambda XA.$$

We have

$$\begin{aligned} A^\# X &= (A^\#)^2 AX \\ &= \lambda (A^\#)^2 XA \\ &= \lambda (A^\#)^2 XA^2 A^\# \\ &= (A^\#)^2 AXAA^\# \\ &= A^\# XAA^\#. \end{aligned}$$

Also,

$$\begin{aligned} \frac{1}{\lambda} XA^\# &= \frac{1}{\lambda} XA(A^\#)^2 \\ &= \frac{1}{\lambda^2} AX(A^\#)^2 \\ &= \frac{1}{\lambda^2} A^\# A^2 X(A^\#)^2 \\ &= A^\# XA^2(A^\#)^2 \\ &= A^\# XA^2 A^\# A^\# \\ &= A^\# XAA^\#. \end{aligned}$$

Therefore:

$$A^\# X = \frac{1}{\lambda} XA^\#.$$

Then

$$\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^\#) \quad \text{and} \quad X \in E_{\text{ext}}(A^\#; \frac{1}{\lambda}).$$

We deduce that for each non-zero complex number $\lambda \in \sigma_{\text{ext}}(A)$, and for each operator $X \in E_{\text{ext}}(A; \lambda)$, it follows that $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^\#)$ and $X \in E_{\text{ext}}(A^\#; \frac{1}{\lambda})$. $\square$

Corollary 13. Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. If λ is a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^{-1})$ and

$$E_{\text{ext}}(A; \lambda) \subseteq E_{\text{ext}}(A^{-1}; \frac{1}{\lambda}).$$

Proof. The proof is based on Remark (11). $\square$

Definition 14. Let $A \in \mathcal{B}(\mathcal{H})$. We say that $B \in \mathcal{B}(\mathcal{H})$ is the Drazin inverse of A of index $k \in \mathbb{N}^*$ if the following conditions are satisfied:

$$\begin{cases} AB = BA, \\ BAB = B, \\ A^{k+1}B = A^k. \end{cases}$$

The Drazin inverse of A is denoted by A^D .

Remark 15. (1) If $k = 1$, then $A^D = A^\#$.

(2) If $A^\#$ exists, then $A^D = A^\#$.

Theorem 16. Let $A \in \mathcal{B}(\mathcal{H})$, and suppose $A^D \in \mathcal{B}(\mathcal{H})$ is the Drazin inverse of A . If λ is a non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$, and there exists an operator $X \in E_{\text{ext}}(A; \lambda)$ with $A^{k+1}X \neq 0$ or $XA^{k+1} \neq 0$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^D)$.

Proof. Let λ be a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$ such that there exists a non-zero operator $X \in E_{\text{ext}}(A; \lambda)$ with $XA^{k+1} \neq 0$ or $A^{k+1}X \neq 0$ satisfying the equation:

$$AX = \lambda XA,$$

$$A^k X = \lambda A^{k-1} XA,$$

$$A^k X = \lambda^k XA^k,$$

$$A^D A^{k+1} X = \lambda^k XA^{k+1} A^D,$$

$$A^D A^{k+1} X = \frac{\lambda^k}{\lambda^{k+1}} A^{k+1} XA^D \quad \text{or} \quad \lambda^{k+1} A^D XA^{k+1} = \lambda^k XA^{k+1} A^D,$$

$$A^D A^{k+1} X = \frac{1}{\lambda} A^{k+1} XA^D \quad \text{or} \quad A^D XA^{k+1} = \frac{1}{\lambda} XA^{k+1} A^D.$$

By taking $Y = A^{k+1}X$ or $Y = XA^{k+1}$, it is easy to see that $Y \in \mathcal{B}(\mathcal{H})$, and according to the hypothesis, $Y \neq 0$. Hence, we obtain:

$$A^D Y = \frac{1}{\lambda} Y A^D.$$

Then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^D)$. □

Proposition 17 (Cline's formula). [9] Let $A, B \in \mathcal{B}(\mathcal{H})$. If AB is Drazin invertible, then BA is Drazin invertible, and

$$(BA)^D = B[(AB)^D]^2 A.$$

Theorem 18. Let $A, B \in \mathcal{B}(\mathcal{H})$, if BA is Drazin invertible, $\lambda \in \sigma_{\text{ext}}((BA)^D)$, and there exists an operator $X \in E_{\text{ext}}((BA)^D; \lambda)$ with $AXB \neq 0$, then $\lambda \in \sigma_{\text{ext}}((AB)^D)$.

Proof. Let $\lambda \in \sigma_{\text{ext}}((BA)^D)$ such that there exists an operator $X \in E_{\text{ext}}((BA)^D; \lambda)$ satisfying the equation:

$$\begin{aligned}(BA)^D X &= \lambda X (BA)^D, \\ B [(AB)^D]^2 AX &= \lambda X B [(AB)^D]^2 A, \\ AB [(AB)^D]^2 AX &= \lambda AX B [(AB)^D]^2 A, \\ (AB)^D AB (AB)^D AX &= \lambda AX B [(AB)^D]^2 A, \\ (AB)^D AX B &= \lambda AX B [(AB)^D]^2 AB, \\ (AB)^D AX B &= \lambda AX B (AB)^D AB (AB)^D, \\ (AB)^D AX B &= \lambda AX B (AB)^D.\end{aligned}$$

By taking $Y = AXB$, it is easy to see that $Y \in \mathcal{B}(\mathcal{H})$, and according to the hypothesis, $Y \neq 0$. Hence, we obtain:

$$(AB)^D Y = \lambda Y (AB)^D.$$

Then $\lambda \in \sigma_{\text{ext}}((AB)^D)$. □

Remark 19. (1) If AB is Drazin invertible, $\lambda \in \sigma_{\text{ext}}((AB)^D)$, and there exists an operator $X \in E_{\text{ext}}((AB)^D; \lambda)$ with $BXA \neq 0$, then $\lambda \in \sigma_{\text{ext}}((BA)^D)$.

(2) Let $A, B \in \mathcal{B}(\mathcal{H})$. If AB is Drazin invertible, then for all $\lambda \in \sigma_{\text{ext}}((AB)^D)$, and if there exists an operator $X \in E_{\text{ext}}((AB)^D; \lambda)$ such that $BXA \neq 0$, then $\lambda \in \sigma_{\text{ext}}((BA)^D)$.

Similarly, for all $\lambda \in \sigma_{\text{ext}}((BA)^D)$, and if there exists an operator $X \in E_{\text{ext}}((BA)^D; \lambda)$ such that $AXB \neq 0$, then $\lambda \in \sigma_{\text{ext}}((AB)^D)$.

Therefore, $\sigma_{\text{ext}}((BA)^D) = \sigma_{\text{ext}}((AB)^D)$.

Lemma 20. Let $A, B \in \mathcal{B}(\mathcal{H})$, such that $AB = BA$. If A^D and B^D exist, then:

$$\begin{aligned}AA^D B &= BAA^D, \\ BB^D A &= ABB^D, \\ A^D B &= BA^D, \\ B^D A &= AB^D, \\ (AB)^D &= B^D A^D \\ &= A^D B^D.\end{aligned}$$

Remark 21. Let $A, B \in \mathcal{B}(\mathcal{H})$, such that $AB = BA$. If A^D and B^D exist, then

$$\sigma_{\text{ext}}((BA)^D) = \sigma_{\text{ext}}((AB)^D).$$

Proposition 22. Let $A \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$ have a closed image. Suppose A possesses an inverse group, and define $\delta = \|A\| \|A^\# \|$. Then:

- (1) $\sigma_{\text{ext}}(A) \subseteq \{z \in \mathbb{C} \mid \frac{1}{\delta} \leq |z| \leq \delta\} \cup \{0\} = S_\delta$.
- (2) If λ is a non-zero complex number and $\lambda \in \sigma_{\text{ext}}(A)$ with $|\lambda| \neq 1$, then there exists an integer N such that any product of N elements from $E_{\text{ext}}(A; \lambda)$ is zero. In particular, every element of $E_{\text{ext}}(A; \lambda)$ is a nilpotent operator with a rank of at most N .
- (3) If λ is a non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$ and $|\lambda| \neq 1$, and there exists an operator $X \in E_{\text{ext}}(A; \lambda)$ such that $X^k \neq 0$ where $k < N$ and $2k \geq N$ (where N is defined in (2) above and k is a natural number), then there exists an extended eigenvalue such that $N = 2$.

Proof. (1) Since A possesses an inverse group, $A^\# A$ is a projection onto $R(A^\#)$. Then, for all $x \in R(A^\#)$, we have:

$$\|x\| = \|A^\# A x\| \leq \|A^\# \| \|A x\|.$$

Then

$$\frac{1}{\|A^\# \|} \|x\| \leq \|A x\|. \quad (1)$$

Let λ be a non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$ and $X \in E_{\text{ext}}(A; \lambda)$. Then, $A X = \lambda X A$ on the space H , it follows that $A X = \lambda X A$ on $R(A^\#)$.

From (1), we obtain:

$$\frac{1}{\|A^\# \|} \|X x\| \leq \|A X x\| = |\lambda| \|X A x\| \leq |\lambda| \|X \| \|A x\|.$$

Then,

$$\frac{1}{\|A^\# \|} \|X\| \leq |\lambda| \|X \| \|A\|.$$

Finally,

$$\frac{1}{\delta} \leq |\lambda|, \quad \text{where } \delta = \|A^\# \| \|A\|.$$

As for the second inequality, if $\lambda \in \sigma_{\text{ext}}(A)$, then $\frac{1}{\lambda} \in \sigma_{\text{ext}}(A^\#)$. In a similar manner, we obtain:

$$\left| \frac{1}{\lambda} \right| \geq \frac{1}{\delta}$$

or equivalently,

$$|\lambda| \leq \delta.$$

Thus, if λ is a non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$, then $\lambda \in \{z \in \mathbb{C} \mid \frac{1}{\delta} \leq |z| \leq \delta\}$.

On the other hand, if $0 \in \sigma_{\text{ext}}(A)$, then $0 \in S_\delta$.

Thus, the first assertion is proven.

- (2) Let $\lambda \in \sigma_{\text{ext}}(A)$ such that $|\lambda| \neq 1$. Then, we can find an integer $N > 1$ such that $\lambda^N \notin S_\delta$ and $\lambda^{N-1} \in S_\delta$. Let $X_1, X_2, \dots, X_N \in E_{\text{ext}}(A; \lambda)$.

Since $\lambda \in \sigma_{\text{ext}}(A)$ and $X_1 \in E_{\text{ext}}(A; \lambda)$, we have:

$$AX_1 = \lambda X_1 A.$$

Thus,

$$AX_1 X_2 = \lambda X_1 A X_2.$$

And since $X_2 \in E_{\text{ext}}(A; \lambda)$, it follows that:

$$AX_1 X_2 = \lambda^2 X_1 X_2 A.$$

Similarly, we conclude that:

$$AX_1 X_2 \dots X_N = \lambda^N X_1 X_2 \dots X_N A.$$

Since $\lambda^N \notin S_\delta$, by the previous proposition, we have $\lambda^N \notin \sigma_{\text{ext}}(T)$. This necessarily implies:

$$X_1 X_2 \dots X_N = 0.$$

- (3) Let λ be a non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$ and $|\lambda| \neq 1$, and let $X \in E_{\text{ext}}(A; \lambda)$ such that $X^k \neq 0$.

We have:

$$AX = \lambda X A,$$

$$AX^2 = \lambda X A X,$$

$$AX^2 = \lambda^2 X^2 A.$$

By following the calculations in the same way, we find:

$$AX^k = \lambda^k X^k A.$$

Since $S = X^k \neq 0$, then $\lambda_0 = \lambda^k \in \sigma_{\text{ext}}(A)$.

We have $\lambda_0^2 = \lambda^{2k} \notin S_\delta$, then $\lambda_0^2 \notin \sigma_{\text{ext}}(T)$. Therefore, $S^2 = X^{2k} = 0$.

□

Corollary 23. (1) *The last property holds for any bounded non-zero linear operator A that has a Drazin inverse, provided that for every non-zero complex number λ such that $\lambda \in \sigma_{\text{ext}}(A)$, there exists a linear operator $X \in E_{\text{ext}}(A; \lambda)$ such that $A^{(k+1)}X \neq 0$ or $XA^{(k+1)} \neq 0$.*

- (2) *The last property holds for any bounded non-zero linear operator A that has an interior inverse B such that $AB = BA$, provided that for every λ non-zero complex number such that $\lambda \in \sigma_{\text{ext}}(A)$, there exists a linear operator $X \in E_{\text{ext}}(A; \lambda)$ such that $XA^2 \neq 0$ or $A^2X \neq 0$.*

- Proof.* (1) By taking advantage of the fact that BA is a bounded projection onto $R(B)$, and applying Theorem (16), we can use reasoning similar to that in the previous proof to reach the desired conclusion.
- (2) We obtain the desired result by applying Remark (4) and Theorem (9), followed by computations akin to those in the previous proof.

□

3. CONCLUSION

We have thoroughly investigated the relationship between the extended spectrum of a bounded linear operator and its group inverse and Drazin inverse. Through detailed analysis, we have established significant connections that enhance our understanding of how the extended spectrum behaves under these inverses. Our main result demonstrates an equality that links the extended spectrum of the operator to the spectra of its group and Drazin inverses. This provides new insights into the structure of operators and their inverses in Banach spaces. These findings enrich the theoretical framework of operator theory and open up potential avenues for future research, particularly in spectral theory and its applications in solving operator equations.

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