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ON WEAKLY Fπ-REGULAR AND WEAKLY wFπ-REGULAR RINGS

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Abstract. The purpose of this paper is to study a new class of regular rings, namely weakly Fπ-regular and weakly wFπ-regular rings. Their key properties and features, as well as the relationship between these rings and other rings, such as CS-rings, FGP-rings, and GMP-rings are investigated. Moreover, a new class of sets is introduced, namely weakly motivating and weakly w-motivating sets, after which their connections with the weakly Fπ-regular and the weakly wFπ-regular rings are also studied. Weakly F*π-regular and weakly wF*π-regular rings are discussed, alongside their relationships with weakly Fπ-regular and weakly wFπ-regular rings. 2-primal rings are classified, and the conditions required to render the 2-primal weakly Fπ-regular and weakly wFπ-regular rings are also examined.

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1. Introduction

Von Neumann introduced the concept of regularity in ring theory during the 1900s, and it plays a fundamental role in understanding the structural properties of rings and modules. A ring $\Re$ is called von-Neumann regular if for every element $\mathrm{a} \in \Re$, there exists $x \in \Re$ such that $\mathrm{a} = \mathrm{a}x\mathrm{a}$. This notion has been widely studied and generalized in various directions, particularly in the context of non-commutative rings.

Among such generalizations, the concept of weakly regular rings appeared by Ramamurthi in 1973. The study of right (resp. left) weakly regular rings was motivated by the point of view of the generalization of regular rings, which also provided examples for regular rings that are right (resp. left) weakly regular rings. Ramamurthi generalized the property of the regular rings, namely $\mathcal{I}^2 = \mathcal{I}$, for every right (resp. left) ideal of $\Re$. The latter researcher also represented the concept of a right

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(resp. left) weakly regular ring $\Re$ such that for all $\mathrm{a} \in \Re$ we have $\mathrm{a} \in (\mathrm{a}\Re)^2$ ($\mathrm{a} \in (\Re\mathrm{a})^2$) [15]. Later, the concept of weakly regular rings has been utilized and studied by several other authors, such as Camillo and Xiao. In addition, Gupta defined the concept of weakly π-regular rings through generalizations made about weakly regular rings and described weakly π-regular as a broad class of rings that includes π-regular and weakly regular rings. According to Gupta, $\Re$ is a right (left) weakly π-regular ring provided that for all $\mathrm{a} \in \Re$ there is an $m \in \mathbb{Z}^+$ such that $\mathrm{a}^m \in (\mathrm{a}^m\Re)^2$ ($\mathrm{a}^m \in (\Re\mathrm{a}^m)^2$) respectively.

In 1997, Hong, Kim, Kwak, and Lee [12] investigated the relationships between the maximality of prime ideals in 2-primal rings and the weak π-regularity from the right (or left). They extended the π-regularity rings to weakly π-regularity rings. Their findings indicated that the π-regular is a weakly π-regular ring, but the converse is not true.

Throughout this paper, we present new generalizations of regular rings; namely, right (resp. left) weakly Fπ-regular and weakly wFπ-regular rings. The system of weakly Fπ-regular and weakly wFπ-regular rings is indeed a wide class of rings that strictly includes weakly π-regular and Fπ-regular rings. We study their properties by generalizing some results in [3, 6, 7, 9, 10, 14] and also provide some examples. In addition, we establish the relationship between weakly Fπ-regular and weakly wFπ-regular rings with some other rings.

2. Preliminaries

Throughout this paper, $\Re$ denotes an identity-associated ring. A ring $\Re$ is a (Von Neuman) regular ring if for each $\mathrm{a} \in \Re$, there exists $b \in \Re$ such that $\mathrm{a} = \mathrm{a}b\mathrm{a}$ [8]. Let $\mathcal{S}$ be a subset of a ring $\Re$, then the annihilator of $\mathcal{S}$ is $\mathrm{a}nn_l(\mathcal{S}) = \{r \in \Re | rs = 0, \text{ for all } s \in \mathcal{S}\}$ .This is a left annihilator, (dually the right annihilator $\mathrm{a}nn_r(\mathcal{S}) = \{r \in \Re | sr = 0, \text{ for all } s \in \mathcal{S}\}$). Let $\mathcal{U}(\Re)$, $\mathcal{Z}(\Re)$, $\mathcal{P}(\Re)$, $\mathcal{N}(\Re)$, and $\mathcal{C}(\Re)$ denote the set of all unit elements, the set of all zero divisors, the prime radical, the nilpotent elements, and the center of $\Re$, respectively.

Definition 2.1. *Let $\Re$ be a ring then:*

(*a*) *$\Re$ is a right (resp. left) Gw-R if for every $\mathrm{a} \in \Re$, there is $x \in \Re\mathrm{a}\Re$ such that $\mathrm{a} = \mathrm{a}x$ (resp. $\mathrm{a} = x\mathrm{a}$) [10].*

(*b*) *$\Re$ is a right (resp. left) w-R if for all $\mathrm{a} \in \Re$ we have $\mathrm{a} = \mathrm{a}b_1\mathrm{a}b_2$ (resp. $\mathrm{a} = b_1\mathrm{a}b_2\mathrm{a}$) where $b_1, b_2 \in \Re$ [4].*

(*c*) *$\Re$ is a right (resp. left) $w\pi$-R if for all $\mathrm{a} \in \Re$, there is an $m \in \mathbb{Z}^+$ such that $\mathrm{a}^m = \mathrm{a}^m d$ (resp. $\mathrm{a}^m = d\mathrm{a}^m$) where $d \in \Re\mathrm{a}^m\Re$ [10].*

(*d*) *$\Re$ is a right (resp. left) $w\pi'$-R if for all $\mathrm{a} \in \Re$, there is an $m \in \mathbb{Z}^+$ such that $\mathrm{a}^m = \mathrm{a}^m b_1 \mathrm{a}^m b_2$ (resp. $\mathrm{a}^m = b_1\mathrm{a}^m b_2\mathrm{a}^m$) where $b_1, b_2 \in \Re$ [4].*

(*e*) *$\Re$ is an Fπ-R if for every $\mathrm{a} \in \Re$, there exist $0 \neq \mathrm{c} \in \Re$ and $b \in \Re$ such that $\mathrm{ac} = \mathrm{ac}b\mathrm{ac}$ [3].*

Definition 2.2. [13] *An ideal $\mathcal{I}$ of a ring $\Re$ is said to be regular if for each $x \in \mathcal{I}$, there exists $y \in \mathcal{I}$ such that $x = xyx$.*

Proposition 2.1. [13] *Let $\Re$ be a regular ring, and $\mathcal{I}$ be an ideal of $\Re$. Then, $\mathcal{I}$ is regular.*

Theorem 2.1. [13] *The statements below are equivalent:*

(a) *$\Re$ is a right w-R regular.*
(b) *A right ideal $\mathcal{I}$ of $\Re$, is an idempotent.*
(c) *All right ideal of $\Re$, is the intersection of prime right ideal.*
(d) *For every $\mathrm{a} \in \Re$, $\mathrm{a} \in (\mathrm{a}\Re)^2$.*
(e) *For any two right ideals $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{I} \subseteq \mathcal{J}$, $\mathcal{I}\mathcal{J} = \mathcal{I}$.*

Definition 2.3. [13] *Let $\mathcal{L}$ be a right ideal of $\Re$. Then, $\mathcal{L}$ is called a prime (resp. semi-prime) if $\mathbf{x}\Re\mathbf{y} \subseteq \mathcal{L}$ (resp. $\mathbf{x}\Re\mathbf{x} \subseteq \mathcal{L}$) implies $\mathbf{x} \in \mathcal{L}$ or $\mathbf{y} \in \mathcal{L}$ (resp. $\mathbf{x} \in \mathcal{L}$) for all $\mathbf{x}, \mathbf{y} \in \Re$. Moreover, $\Re$ contains no non-zero nilpotent ideals if and only if $\Re$ is a semiprime ring. If $\Re$ is a reduced ring, then $\Re$ is a semiprime.*

Definition 2.4. [13] *If $\Re$ satisfies the ascending chain condition on both complement and annihilator right ideals, then $\Re$ is a right Goldie ring.*

3. Weakly Fπ-Regular and Weakly wFπ-Regular Rings

In this section, we define wFπ-R and wwFπ-R rings and outline some of their properties.

Definition 3.1. *A ring $\Re$ is called right (resp. left) weakly Fπ-regular (wFπ-R) if for every element $\mathrm{a} \in \Re$, there exists $0 \neq \mathrm{c} \in \Re$ such that $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$ (resp. $\mathrm{ac} = b_1\mathrm{ac}b_2\mathrm{ac}$), where $b_1, b_2 \in \Re$.*

Moreover, we define a right weakly motivating set for an element $\mathrm{a} \in \Re$ as $\mathbf{M}^w(\mathrm{a}) = \{0 \neq \mathrm{c} \in \Re : \mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$, for some $b_1, b_2 \in \Re\}$, (dually, a left weakly motivating set as ${}^w\mathbf{M}(\mathrm{a}) = \{0 \neq \mathrm{c} \in \Re : \mathrm{ac} = b_1\mathrm{ac}b_2\mathrm{ac}$, for some $b_1, b_2 \in \Re\}$).

Definition 3.2. *A ring $\Re$ is called a right (resp. left) weakly wFπ-regular (wwFπ-R), if for every element $\mathrm{a} \in \Re$ and $r \in \Re$, there exists an element $d \in \Re\mathrm{ac}\Re$ such that $\mathrm{a}r = \mathrm{a}rd$ (resp. $\mathrm{a}r = d\mathrm{a}r$).*

Moreover, we define right (dually, left) weakly w-motivating sets for an element $\mathrm{a} \in \Re$ as $\mathbf{M}^{ww}(\mathrm{a}) = \{0 \neq \mathrm{c} \in \Re : \mathrm{ac} = \mathrm{ac}x, x \in \Re\mathrm{ac}\Re\}$ (dually, ${}^{ww}\mathbf{M}(\mathrm{a}) = \{0 \neq \mathrm{c} \in \Re : \mathrm{ac} = x\mathrm{ac}, x \in \Re\mathrm{ac}\Re\}$).

Clearly, every right $w\pi$-R is a right wFπ-R ring. In addition, every right wFπ-R and right $w\pi$-R are right wwFπ-R rings.

Example 3.1. *A ring $\Re = \left\{ \begin{bmatrix} u & v \\ 0 & u \end{bmatrix} |u, v \in D, \text{ where } D \text{ is simple domain} \right\}$ is a right wwFπ-R ring by [11, Example 4.2]. Thus, the ring $\Re$ shows that it is not necessarily a right wπ-R or right wwFπ-R is an Fπ-R ring. Indeed; for $\mathrm{A} = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ such that $0 \neq u \in D$ and is not invertible, there is no $0 \neq C \in \Re$ such that $AC \notin AC\Re AC$.*

Subsequently, we have the assertion below:

Proposition 3.1. *(a) Let $\Re$ be a right wwFπ-R where* $\mathbf{r}(\mathrm{ac}) \subset \mathbf{l}(\mathrm{ac})$ *for some* $\mathrm{c} \in \mathbf{M}^{ww}(\mathrm{a})$*, then $\Re$ is a left wwFπ-R ring.*

(b) If $\Re$ is a left wwFπ-R such that $\mathbf{l}(\mathrm{ac}) \subset \mathbf{r}(\mathrm{ac})$ *for some* $\mathrm{c} \in {}^{ww}\mathbf{M}(\mathrm{a})$*, then $\Re$ is a right wwFπ-regular ring.*

Proof. (a) Let $\Re$ be a right wwFπ-R, where $a \in \Re$ and $c \in M^{ww}(a)$ with $\mathbf{r}(\mathrm{ac}) \subset \mathbf{l}(\mathrm{ac})$. Then, there exists $0 \neq c \in \Re$ such that $ac = act$ for some $t \in RacR$. Therefore, $ac(1-t) = 0$ implies $1-t \in \mathbf{r}(\mathrm{ac})$. Since $\mathbf{r}(\mathrm{ac}) \subset \mathbf{l}(\mathrm{ac})$ then $1-t \in \mathbf{l}(\mathrm{ac})$ and $(1-t)ac = 0$, and hence $ac - tac = 0$. Thus, $ac = tac$ and then, $\Re$ is a left wwFπ-R ring.

(b) Straightforward using a similar argument in (a).

□

Lemma 3.1. [13] *Let $\Re$ be a reduced ring. Then,* $\mathbf{r}(\mathrm{a}) = \mathbf{l}(\mathrm{a})$ *for all* $0 \neq \mathrm{a} \in \Re$.

Corollary 3.1. *A reduced ring $\Re$ is a right wwFπ-R if and only if $\Re$ is a left wwFπ-R.*

Proposition 3.2. *Let $\Re$ be a domain. Then:*

(a) If $\Re$ is an Fπ-R ring, then it is a division ring.

(b) If $\Re$ is a right wwFπ-R ring, then it is a simple.

(c) If $\Re$ is a right wFπ-R ring, then $\Re$ is a division ring.

Proof. (a) Strightforword by [3, Corollary 3.7].

(b) Since $\Re$ is a domain, then it does not have zero divisors. By the definition of wwFπ-R, we obtain that for any nonzero element a in a proper ideal $\mathcal{I} \subset \Re$, we have $1 - b = 0$ for some $b \in \Re\mathrm{ac}\Re$ where $\mathrm{c} \in \mathbf{M}^{ww}(\mathrm{a})$. Hence, $1 = b \in \Re\mathrm{ac}\Re \subset \mathcal{I}$, and then $\mathcal{I} = \Re$. Therefore, $\Re$ has no proper ideal; thus $\Re$ is simple.

(c) Let $\Re$ be a domain and wFπ-R ring. Then, for any $0 \neq \mathrm{a} \in \Re$, there exists $0 \neq \mathrm{c} \in \Re$ such that $\mathrm{ac} = \mathrm{ac}r\mathrm{ac}s$ for some $r, s \in \Re$, then $\mathrm{ac}(1 - r\mathrm{ac}s) = 0$. Since $\Re$ is domain, then it has no zero divisor, so $1 = r\mathrm{ac}s$ implies that a is invertible. Hence, $\Re$ is a division ring.

□

Theorem 3.2. *Let $\Re_i$ be rings with identities. Then, the direct product of $\Re_i$ is a right wFπ-R if at least one of them is a right wFπ-R ring.*

Proof. Let $\Re_1$ and $\Re_2$ be rings where $\Re_1$ is a right wFπ-R ring. It is sufficient to show that $\Re = \Re_1 \times \Re_2$ is a right wFπ-regular ring. Let $(\mathrm{a}_1, \mathrm{a}_2) \in \Re$ where $\mathrm{a}_1 \in \Re_1$ and $\mathrm{a}_2 \in \Re_2$. Since $\mathrm{a}_1 \in \Re_1$ and $\Re_1$ is a right wFπ-regular, there exist $0 \neq \mathrm{c}_1 \in \Re_1$ and $b_1, d_1 \in \Re_1$ such that $\mathrm{a}_1\mathrm{c}_1 b_1 \mathrm{a}_1\mathrm{c}_1 d_1 = \mathrm{a}_1\mathrm{c}_1$. Take

$(0,0) \neq \mathrm{c} = (\mathrm{c}_1, 0) \in \Re$ with $\mathrm{c}_1 \neq 0$ and $b = (b_1, b_2), d = (d_1, d_2) \in \Re$ where $b_2, d_2 \in \Re_2$. Now, $(\mathrm{a}_1, \mathrm{a}_2)(\mathrm{c}_1, 0)(b_1, b_2)(\mathrm{a}_1, \mathrm{a}_2)(\mathrm{c}_1, 0)(d_1, d_2) = (\mathrm{a}_1\mathrm{c}_1 b_1 \mathrm{a}_1 \mathrm{c}_1 d_1, 0) = (\mathrm{a}_1\mathrm{c}_1, 0) = (\mathrm{a}_1, \mathrm{a}_2)(\mathrm{c}_1, 0)$. Therefore, $\Re$ is a right wFπ-R ring. □

Similar argument can be utilized to show the assertion below:

Theorem 3.3. *Let $\Re_i$ be rings with identities. Then, the direct product of $\Re_i$ is a right wwFπ-R if at least one of them is a right wwFπ-R ring.*

The next example shows that not necessarily every right wwFπ-R is a right $w\pi$-R ring.

Example 3.4. *Let $\mathcal{A} = \mathbb{Z} \times \Re$, where $\Re = \left\{ \begin{bmatrix} u & v \\ 0 & u \end{bmatrix} | u, v \in D, \text{ where } D \text{ is simple domain} \right\}$. From Example 3.1, $\Re$ is a right wwFπ-R. Then, $\mathcal{A}$ is a right wwFπ-R ring by Theorem 3.3. Indeed; $(2,0)^m \notin (2,0)^m \mathcal{A} (2,0)^m \mathcal{A}$, for any positive integer m implies that $\Re$ is not $w\pi$-R ring.*

Theorem 3.5. *Let $\Re/\mathcal{I}$ be a right wwFπ-R ring where $\mathcal{I}$ is the right ideal Gw-R of $\Re$. Then, $\Re$ is a wwFπ-R ring.*

Proof. Let $\Re/\mathcal{I} = \bar{\Re}$ where $\bar{\Re}$ is a right wwFπ-R ring. Then for every $\bar{\mathrm{a}} \in \bar{\Re}$, there exist $0 \neq \bar{\mathrm{c}} \in \bar{\Re}$ and $\bar{x} \in \bar{\Re}\bar{\mathrm{a}}\bar{\mathrm{c}}\bar{\Re}$ such that, $\mathrm{ac} + \mathcal{I} = (\mathrm{ac} + \mathcal{I})(x + \mathcal{I}) = (\mathrm{ac}x + \mathcal{I})$. This implies that $\mathrm{ac} - \mathrm{ac}x \in \mathcal{I}$. Since $\mathcal{I}$ is a right Gw-R ideal, then there exists $t \in \mathcal{I}(\mathrm{ac} - \mathrm{ac}x)\mathcal{I} \subset \Re\mathrm{ac}(1 - x)\Re \subset \Re\mathrm{ac}\Re$ such that $\mathrm{ac} - \mathrm{ac}x = (\mathrm{ac} - \mathrm{ac}x)t = \mathrm{ac}t - \mathrm{ac}xt$ then $\mathrm{ac} = \mathrm{ac}(x + t - xt)$. Since $x + t - xt \in \Re\mathrm{ac}\Re$. Then, $\mathrm{ac} = \mathrm{ac}(x + t - xt)$, and therefore $\Re$ is a right wwFπ-R ring. □

Proposition 3.3. *Let $\Re$ be a right wwFπ-R ring. Then:*

(*a*) *If $\mathcal{I}$ is a proper ideal of $\Re$, then $\mathbf{M}^{ww}(\mathrm{a}) \subset \mathcal{I}$ for any right regular element (i.e. not a non-zero left zero divisor) $\mathrm{a} \in \mathcal{I}$.*

(*b*) *If $\Re$ has no nonzero left zero divisor, then $\Re$ is a simple.*

Proof. (a) Let $\mathrm{c} \in \mathbf{M}^{ww}(\mathrm{a})$, then $\mathrm{ac} = \mathrm{ac}x$ for some $x \in (\mathrm{ac})$. This implies that $\mathrm{a}(\mathrm{c} - \mathrm{c}x) = 0$, and if a is a right regular element, then $\mathrm{c} = \mathrm{c}x \in \mathrm{c}(\mathrm{ac}) \subset \mathcal{I}$ implies $\mathbf{M}^{ww}(\mathrm{a}) \subset \mathcal{I}$.

(b) If $\Re$ is a right wwFπ-R ring without nonzero left zero divisor and $\mathcal{I}$ is a proper ideal, then $0 \neq \mathrm{a} \in \mathcal{I}$, there exists $0 \neq \mathrm{c} \in \Re$ such that $\mathrm{ac} = \mathrm{ac}x$ and $x \in (\mathrm{ac})$ implies $1 = x \in (\mathrm{ac}) \subset \mathcal{I}$. Hence, $\Re$ is a simple.

□

Proposition 3.4. [3] *Let $\Re$ be a ring. Then the following are satisfied:*

(*a*) *If $\mathrm{a} \in \mathcal{U}(\Re)$, then a is a regular.*

(*b*) *If $\mathrm{a} \in \mathcal{N}(\Re) \cup \mathcal{U}(\Re)$, then a is a π-regular.*

Proposition 3.5. [3] *Let $\Re$ be a ring. Then the following are satisfied:*

(*a*) *If* $\mathrm{a} \in \mathcal{Z}(\Re)$*, then* a *is an* $F\pi$*-regular.*

(*b*) *If* $\mathrm{a} \in \mathcal{U}(\Re)$*, then* $\mathcal{U}(\Re) \subset \mathbf{M}(\mathrm{a})$*.*

According to propositions 3.4 and 3.5, we have the proposition below.

Proposition 3.6. (*a*) *$\Re$ is a right $w\pi$-R if and only if for all* $\mathrm{a} \notin \mathcal{N}(\Re) \cup \mathcal{U}(\Re)$*, there is a positive integer* m *where* $\mathrm{a}^m \in \mathbf{M}^{ww}(\mathrm{a})$*.*

(*b*) *$\Re$ is a right $wwF\pi$-R if and only if for all* $\mathrm{a} \notin \mathcal{Z}(\Re) \cup \mathcal{U}(\Re)$*,* $\mathbf{M}^{ww}(\mathrm{a}) \neq \emptyset$*.*

(*c*) *$\Re$ is a right $w\pi$-R if and only if for all* $\mathrm{a} \notin \mathcal{U}(\Re)$*, then* $1 \in \mathbf{M}^{w}(\mathrm{a})$*.*

Definition 3.3. *Let $\mathcal{X} \subset \Re$. A set $\mathcal{X}$ is called a weakly regular set if for all $x \in \mathcal{X}$, there exists $y \in \Re x \Re$ such that $x = xy$.*

Corollary 3.2. *Let $\Re$ be a right $wwF\pi$-R and* $\mathrm{a} \notin \mathcal{Z}(\Re)$*, then* $\mathbf{M}^{ww}(\mathrm{a})$ *is a weakly regular set.*

Proof. Let $\Re$ be a right $ww\mathrm{F}\pi$-R and $\mathrm{a} \notin Z(\Re)$. Then, by Proposition (3.6 -2) $\mathbf{M}^{ww}(\mathrm{a}) \neq \emptyset$. Suppose $x \in \mathbf{M}^{ww}(\mathrm{a})$, then $0 \neq x \in \Re$ and $\mathrm{a}x = \mathrm{a}xd$ for some $d \in \Re\mathrm{ac}\Re$. Hence, $\mathrm{a}(x - xd) = 0$. Since $\mathrm{a} \notin \mathcal{Z}(\Re)$, then $x - xd = 0$ and $x = xd$. Therefore, $\mathbf{M}^{ww}(\mathrm{a})$ is a right weakly regular set. $\square$

Proposition 3.7. *Let $\Re$ be a ring such that for every* $\mathrm{a} \in \Re$*, there is* $\mathrm{c} \in \mathbf{M}^{ww}(\mathrm{a})$ *where* $\mathbf{r}(\mathrm{ac}) = \mathbf{r}(\mathrm{a})$*. Then the following statements are equivalent:*

(*a*) *$\Re$ is a right w-R ring.*

(*b*) *$\Re$ is a right $w\pi$-R ring.*

(*c*) *$\Re$ is a right $wwF\pi$-R ring.*

Proof. It is clear that $a \Longrightarrow b \Longrightarrow c$.
For $c \Longrightarrow a$, let $\Re$ be a right $ww\mathrm{F}\pi$-regular ring where $\mathrm{a} \in \Re$ and $\mathrm{c} \in \mathbf{M}^{ww}(\mathrm{a})$ with $\mathbf{r}(\mathrm{ac}) = \mathbf{r}(\mathrm{a})$. Then $\mathrm{ac} = \mathrm{ac}x$ where $x \in \Re\mathrm{ac}\Re$. Hence $1 - x \in \mathbf{r}(\mathrm{a})$. Then $\mathrm{a} = \mathrm{a}x$ and $x \in \Re\mathrm{a}\Re$. Thus, $\Re$ is a right w-R ring. $\square$

Since each prime reduced ring is a domain, the following proposition yields.

Proposition 3.8. *Let $\Re$ be a reduced ring. Then the following statements are equivalent;*

(*a*) *$\Re$ is a prime $w\pi$-R ring.*

(*b*) *$\Re$ is a prime $wwF\pi$-R ring.*

(*c*) *$\Re$ is a simple.*

Theorem 3.6. *Let $\Re$ be a semiprime right Goldie ring such that every essential right ideal of $\Re$ is an ideal. Then, the following are equivalent.*

(*a*) *$\Re$ is a right $w\pi$-R ring.*

(*b*) *$\Re$ is a right wwFπ-R ring, where $\mathbf{M}^{ww}(\mathrm{a})$ contains an element that is not a nonzero left zero divisor whenever a is a regular element.*

(*c*) *$\Re$ is a semi-simple artinian.*

Proof. (a)$\Longleftrightarrow$(c), [10, Theorem 3.4].

For (b) $\Longrightarrow$(c), let $\Re$ be a semiprime right Goldie ring. Then every essential right ideal of $\Re$ contains a regular element. Let $\mathcal{I}$ be a proper essential right ideal of $\Re$. Then $\mathcal{I}$ contains a regular element a. Since $\Re$ is wwFπ-R, then $\mathcal{I}$ contains $\mathbf{M}^{ww}(\mathrm{a})$. Hence, $\mathbf{M}^{ww}(\mathrm{a})$ contains an element that is not a nonzero left-zero divisor. Therefore, $\mathcal{I} = \Re$ according to Proposition 3.3. Suppose $\mathcal{J}$ is a proper right ideal of $\Re$. Then there exists a right ideal $\mathcal{K} \subset \Re$ where $\mathcal{J} + \mathcal{K} = \mathcal{J} \oplus \mathcal{K}$ that is essential. Hence, $\mathcal{J} \oplus \mathcal{K} = \Re$. Thus, $\Re$ is semi-simple artinian.

For (c) $\Longrightarrow$ (b), since $\Re$ is a semi-simple artinian right ring, then it is a von Neumann regular and therefore, it is a right wwFπ-R ring. Also, for each regular a, $\mathbf{M}^{ww}(\mathrm{a})$ contains a non-zero left divisor. □

Definition 3.4. [3] *For an element $\mathrm{a} \in \Re$ is called a right semi-Fπ-regular if there exist $0 \neq \mathrm{c} \in \Re$ and $t \in \Re$ such that $\mathrm{ac} = \mathrm{act}$ and $\mathbf{r}(\mathrm{ac}) = \mathbf{r}(t)$. Moreover, $\Re$ is a right semi-Fπ-regular if and only if all elements of $\Re$ are right semi-Fπ-regular elements.*

Proposition 3.9. *Let $\Re$ be a ring satisfying that for all $0 \neq \mathrm{a} \in \Re$, there exists $0 \neq \mathrm{c} \in \Re$, such that $\mathbf{r}(\mathrm{ac}) \cap \Re\mathrm{ac}\Re = 0$. Then, every right wwFπ-R ring is a right semi-Fπ-regular element.*

Proof. Let $0 \neq \mathrm{a} \in \Re$. Then there exists $0 \neq \mathrm{c} \in \Re$ such that $\mathbf{r}(\mathrm{ac}) \cap \Re\mathrm{ac}\Re = 0$. Since $\Re$ is a right wwFπ-R ring, then $\mathrm{ac} = \mathrm{ac}d$ where $d \in \Re\mathrm{ac}\Re$. We need to show that $\mathbf{r}(d) = \mathbf{r}(\mathrm{ac})$. If $x \in \mathbf{r}(d)$, then $dx = 0$. Multiplying both sides by ac , we obtain that $\mathrm{ac}dx = 0$. Since $\mathrm{ac} = \mathrm{ac}d$, then $\mathrm{ac}x = 0$, hence $x \in \mathbf{r}(\mathrm{ac})$ and $\mathbf{r}(d) \subseteq \mathbf{r}(\mathrm{ac})$. Next, let $x \in \mathbf{r}(\mathrm{ac})$, then $\mathrm{ac}x = 0$. Since $\mathrm{ac} = \mathrm{ac}d$, then $\mathrm{ac}dx = 0$, hence $d \in \Re\mathrm{ac}\Re$ and then $dx \in \Re\mathrm{ac}\Re$. However, by multiplying both sides by x, we obtain $(\mathrm{ac})x = (\mathrm{ac}d)x$. Thus, $\mathrm{ac}dx = 0$ and $dx \in \mathbf{r}(\mathrm{ac})$. Hence, $dx \in \mathbf{r}(\mathrm{ac}) \cap \Re\mathrm{ac}\Re = 0$ and $dx = 0$ implies that $x \in \mathbf{r}(d)$. Then, $\mathbf{r}(\mathrm{ac}) \subseteq \mathbf{r}(d)$. Therefore, $\mathbf{r}(\mathrm{ac}) = \mathbf{r}(d)$ and so $\Re$ is a right semi-Fπ-regular ring. □

4. Relation between Weakly wFπ-Regular Ring and Other Rings

In this section, we examine the relation between a wwFπ-R ring and other rings such as CS-rings, GMP-rings, FGP-rings, and quasi-duo rings.

Definition 4.1. [16] *An $\Re$ is a right (resp. left) CS-ring if every nonzero right (resp. left) ideal is essential in a direct summand. Equivalently, every right (resp. left) closed ideal is a direct summand.*

Clearly, every maximal right ideal is indeed a right-closed ideal.

Theorem 4.1. *Let $\Re$ be a right CS-ring. Then $\Re$ is a left wwFπ-R ring, if for all elements $0 \neq \mathrm{a} \in \Re$, $\mathbf{l}(\mathrm{ac})$ is a two-sided ideal with $0 \neq \mathrm{c} \in \Re$.*

Proof. Suppose $\Re$ is a right CS-ring and for all $0 \neq \mathrm{a} \in \Re$, there exists $0 \neq \mathrm{c} \in \Re$ such that $\mathbf{l}(\mathrm{ac})$ is a two-sided ideal. Claim, $\Re\mathrm{ac}\Re + \mathbf{l}(\mathrm{ac}) = \Re$, if not then there exists a maximal right ideal $\mathcal{M}$ of $\Re$ such that $\Re\mathrm{ac}\Re + \mathbf{l}(\mathrm{ac}) \subseteq \mathcal{M}$. Since $\Re$ is CS-ring and $\mathcal{M}$ is maximal right ideal of $\Re$, then $\mathcal{M}$ is a direct summand, such that $\mathcal{M} + \mathcal{K} = \Re$ and $\mathcal{M} \cap \mathcal{K} = 0$, where $\mathcal{K}$ is a right ideal of $\Re$. Hence, $(\Re\mathrm{ac}\Re + \mathbf{l}(\mathrm{ac})) \cap \mathcal{K} \subseteq \mathcal{M} \cap \mathcal{K} = 0$. Then, $(\Re\mathrm{ac}\Re + \mathbf{l}(\mathrm{ac})) \cap \mathcal{K} = 0$. Therefore, $\Re\mathrm{ac}\Re \cap \mathcal{K} = 0$ and $\mathbf{l}(\mathrm{ac}) \cap \mathcal{K} = 0$ so $k\mathrm{ac} = 0$. Thus, $\mathcal{K} \subseteq \mathbf{l}(\mathrm{ac})$ which is a contradiction. Hence, $\Re\mathrm{ac}\Re + \mathbf{l}(\mathrm{ac}) = \Re$ and so $x + d = 1$ where $x \in \Re\mathrm{ac}\Re$ and $d \in \mathbf{l}(\mathrm{ac})$. Thus, $\mathrm{ac} = x\mathrm{ac}$ and $\Re$ is a left wwFπ-R ring. □

Definition 4.2. *Let $\Re$ be a ring and $\mathcal{I} \subseteq \Re$ be an ideal. Then $\mathcal{I}$ is a right (resp. left) Fπ-pure ideal, if for all $\mathrm{a} \in \mathcal{I}$, there exist some $\mathrm{c} \in \mathcal{I}$ and $d \in \mathcal{I}$ where $\mathrm{ac} = \mathrm{ac}d$ (resp. $\mathrm{ac} = d\mathrm{ac}$).*

Definition 4.3. *A right (resp. left) GMP-ring is a ring in which every maximal right (resp. left) ideal is a left (resp. right) Fπ-Pure ideal.*

Theorem 4.2. *Suppose $\Re$ is a right GMP-ring with $\mathrm{a} \in \Re$. If $x \in \mathbf{l}(\mathrm{ac})$ for some $0 \neq \mathrm{c} \in \Re$, then $x \in \mathbf{r}(\mathrm{ac})$. Then $\Re$ is a right wwFπ-R ring.*

Proof. Let $\Re$ be a right GMP-ring, and suppose with aim of contradiction, that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) \neq \Re$. Then, a maximal right ideal $\mathcal{M}$ of $\Re$ exists such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) \subseteq \mathcal{M}$. Since $\Re$ is a right GMP-ring, then $\mathcal{M}$ is left Fπ-pure ideal. Then, for all $\mathrm{a} \in \mathcal{M}$, there exist some $0 \neq \mathrm{c}, t \in \mathcal{M}$ such that $\mathrm{ac} = t\mathrm{ac}$. Then $\mathrm{ac} - t\mathrm{ac} = 0$ and $1 - t \in \mathbf{r}(\mathrm{ac})$ then $1 - t \in \mathbf{l}(\mathrm{ac})$. Therefore, $1 - t \in \mathcal{M}$, which is a contradiction. Therefore, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$. Let $x \in \Re\mathrm{ac}\Re$ and $d \in \mathbf{r}(ac)$ such that $x + d = 1$. By multiplying both sides by ac we obtain that $x\mathrm{ac} = \mathrm{ac}$. Therefore, $\Re$ is a right wwFπ-R ring. □

Definition 4.4. [16] *An $\Re$ is a right (resp. left) quasi-duo ring whenever every maximal right (resp. left) ideal of $\Re$ is a two-sided ideal.*

Lemma 4.1. *Let $\Re$ be a quasi-duo ring and a left GMP-ring. Then $\Re$ is a right wwFπ-R ring.*

Proof. Let $\Re$ be a quasi-duo ring and a left GMP-ring. Then there is a maximal right ideal $\mathcal{M} \subset \Re$ such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) \subseteq \mathcal{M}$. $\mathcal{M}$ is a right Fπ-pure ideal since $\Re$ is a left GMP-ring. Then, for all $\mathrm{a} \in \mathcal{M}$ and $0 \neq \mathrm{c} \in \mathcal{M}$, there exists $t \in \mathcal{M}$ such that $\mathrm{ac} = \mathrm{ac}t$, then $\mathrm{ac}(1 - t) = 0$. Therefore, $1 - t \in \mathbf{r}(\mathrm{ac}) \subseteq \mathcal{M}$, and so $1 \in \mathcal{M}$ which is a contradiction. Therefore, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$. Let $x \in \Re\mathrm{ac}\Re$ and $d \in \mathbf{r}(\mathrm{ac})$ such that $x + d = 1$. Then by multiplying both sides by ac, we obtain that $\mathrm{ac} = \mathrm{ac}x$, and therefore $\Re$ is a right wwFπ-R ring. □

Definition 4.5. [1] *A right (resp. left) FGP-injective is a right (resp. left) $\Re$-module $\mathcal{B}$ where for any* $\mathrm{a} \in \Re$ *there exists* $0 \neq \mathrm{c} \in \Re$ *such that* $0 \neq \mathrm{ac} = \mathrm{ca}$, *and any right (resp. left) $\Re$-homomorphism of* $\mathrm{ac}\Re$ *(resp.* $\Re\mathrm{ac}$*) into $\mathcal{B}$ can be extended to one of $\Re_r$ (resp. $\Re_l$) into $\mathcal{B}$.*

Theorem 4.3. *Let $\Re$ be a ring with every simple right $\Re$-module is an FGP-injective module. Then $\Re$ is a right wwFπ-R ring.*

Proof. Let $\Re$ be a ring with every simple right $\Re$-module is an FGP-injective module. Claim for any $\mathrm{a} \in \Re$ there exists $0 \neq \mathrm{c} \in \Re$ where $0 \neq \mathrm{ac} = \mathrm{ca}$ such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$. If not, then there is a maximal right ideal $\mathcal{M}$ of $\Re$ such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) \subseteq \mathcal{M}$. Define a map $f : \mathrm{ac}\Re \to \Re/\mathcal{M}$ by $f(\mathrm{ac}x) = x + \mathcal{M}$ for all $x \in \Re$. Note that f is well defined right homomorphism. Since $\Re/\mathcal{M}$ is an FGP-injective, then for $\bar{y} \in \Re/\mathcal{M}$ we have, $f(\mathrm{ac}x) = (y+\mathcal{M})(\mathrm{ac}x) = y\mathrm{ac}x+\mathcal{M}$. Since $f(\mathrm{ac}x) = x+\mathcal{M}$, then $x + \mathcal{M} = y\mathrm{ac}x + \mathcal{M}$. In particular, let $x = 1$ and so $1 + \mathcal{M} = y\mathrm{ac} + \mathcal{M}$. Hence, $1 - y\mathrm{ac} \in \mathcal{M}$ implies $1 \in \mathcal{M}$ which is a contradiction. Hence, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$. Then, for $x \in \Re\mathrm{ac}\Re$ and $d \in \mathbf{r}(\mathrm{ac})$ we have $x + d = 1$. Hence, $\mathrm{ac} = \mathrm{ac}x$ and $\Re$ is a right wwFπ-regular ring. □

Lemma 4.2. *Let $\Re$ be a ring. If for* $\mathrm{a} \in \Re$, *there exists* $0 \neq \mathrm{c} \in \Re$ *such that* $\mathbf{l}(\mathrm{ac}) \subseteq \mathbf{r}(\mathrm{ac})$, *then* $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac})$ *is an essential right ideal of $\Re$.*

Proof. Assume that $(\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac})) \cap \mathcal{I} = 0$, where $\mathcal{I}$ is a non-zero right ideal of $\Re$. Then $\Re\mathrm{ac}\Re \cap \mathcal{I} = 0$ and $\mathbf{r}(\mathrm{ac}) \cap \mathcal{I} = 0$. Since $\mathcal{I}\mathrm{ac} \subseteq \Re\mathrm{ac}\Re$ and $\mathcal{I}\mathrm{ac} \subseteq \mathcal{I}$, then $\mathcal{I}\mathrm{ac} \subseteq \Re\mathrm{ac}\Re \cap \mathcal{I} = 0$ implies $\mathcal{I} \subseteq \mathbf{l}(\mathrm{ac}) \subseteq \mathbf{r}(\mathrm{ac})$. Therefore, $\mathcal{I} = 0$, which is a contradiction. Hence, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac})$ is an essential right ideal of $\Re$. □

Corollary 4.1. [2] *If $\Re$ is a ring whose simple singular right $\Re$-modules are FGP-injective, then the center $\mathcal{C}(\Re)$ of $\Re$ is a regular ring.*

Theorem 4.4. *If $\Re$ is a ring with every simple singular right module is an FGP-injective, and for any* $\mathrm{a} \in \Re$, *there exists* $0 \neq \mathrm{c} \in \Re$ *such that* $\mathbf{l}(\mathrm{ac}) \subseteq \mathbf{r}(\mathrm{ac})$. *Then $\Re$ is a right wwFπ-R ring, and the $\mathcal{C}(\Re)$ is a regular ring.*

Proof. Let $\Re$ be a ring with every simple singular right module is an FGP-injective. To prove that for any $\mathrm{a} \in \Re$, there is $0 \neq \mathrm{c} \in \Re$ such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$, suppose with the seek of contradiction that this is not true. Then there exists a maximal right ideal $\mathcal{M}$ of $\Re$ such that $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) \subseteq \mathcal{M}$. Using Lemma 4.2, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac})$ is an essential right ideal of $\Re$, and then $\mathcal{M}$ must be essential in $\Re$. Hence, $\Re/\mathcal{M}$ is an FGP-injective. Let $f : \mathrm{ac}\Re \to \Re/\mathcal{M}$ defined by $f(\mathrm{ac}x) = x + \mathcal{M}$ for all $x \in \Re$. Note that f is a well-defined $\Re$-homomorphism. Since $\Re/\mathcal{M}$ is an FGP-injective, then for $\bar{y} \in \Re/\mathcal{M}$ such that $f(\mathrm{ac}x) = (y + \mathcal{M})(\mathrm{ac}x) = y\mathrm{ac}x + \mathcal{M}$. Since $f(\mathrm{ac}x) = x + \mathcal{M}$, then $x + \mathcal{M} = y\mathrm{ac}x + \mathcal{M}$. Let $x = 1$ and so $1 + \mathcal{M} = y\mathrm{ac} + \mathcal{M}$, then $1 - y\mathrm{ac} \in \mathcal{M}$ implies $1 \in \mathcal{M}$ which is a contradiction. Hence, $\Re\mathrm{ac}\Re + \mathbf{r}(\mathrm{ac}) = \Re$, and $t + d = 1$ where $t \in \Re\mathrm{ac}\Re$ and $d \in \mathbf{r}(\mathrm{ac})$. Therefore, $\mathrm{ac}t = \mathrm{ac}$, and $\Re$ is a right wwFπ-R ring. Moreover, $\mathcal{C}(\Re)$ is a regular ring by Corollary 4.1. □

5. Weakly F*π-Regular and Weakly wF*π-Regular Rings

In this section, we define wwF*π-R and wF*π-R rings. In addition, we introduce the relationship between wwFπ-R and wFπ-R rings with a 2-primary ring.

Definition 5.1. (*a*) *A ring $\Re$ is identified as a right (resp. left) wF*π-R ring if for an ideal $\mathcal{I}$ of $\Re$ and $\mathrm{a} \notin \mathcal{I}$, there exist $0 \neq \mathrm{c} \notin \mathcal{I}$ and $t, s \in \Re$ such that $\mathrm{ac} = \mathrm{ac}t\mathrm{ac}s$ (resp. $\mathrm{ac} = t\mathrm{ac}s\mathrm{ac}$).*

(*b*) *A ring $\Re$ is identified as a right (resp. left) wwF*π-R ring if for an ideal $\mathcal{I}$ of $\Re$ and $\mathrm{a} \notin \mathcal{I}$, there exist $0 \neq \mathrm{c} \notin \mathcal{I}$ and $x \in \Re\mathrm{ac}\Re$ such that $\mathrm{ac} = \mathrm{ac}x$ (resp. $\mathrm{ac} = x\mathrm{ac}$).*

Example 5.1. *Let $\Re = \mathbb{Z}_6$ is a wF*π-R ring. The ideals of $\mathbb{Z}_6$ are $\mathcal{I}_1 = \{0\}$, $\mathcal{I}_2 = \{0, 2, 4\}$, $\mathcal{I}_3 = \{0, 3\}$, and $\mathcal{I}_4 = \{0, 2, 4\}$. Then, for every $\mathrm{a} \in \mathbb{Z}_6$ and $\mathrm{a} \notin \mathcal{I}_j$, where $j \in \{1, 2, 3, 4\}$, there exist $\mathrm{c} \notin \mathcal{I}_j$ and $b_1, b_2 \in \mathbb{Z}_6$ such that $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$.*

Lemma 5.1. (*a*) *Every right wF*π-R ring is a right wFπ-R ring.*

(*b*) *Every right wwF*π-R ring is a right wwFπ-R ring.*

Proof. (a) Let $\Re$ be a wF*π-R ring and $\mathrm{a} \in \Re$. If $0 = \mathrm{a} \in \Re$ done. Suppose $0 \neq \mathrm{a} \in \Re$, and take $\mathcal{I} = \{0\}$. Then, there exist $\mathrm{c} \notin \mathcal{I}$ and $b_1, b_2 \in \Re$ where $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$. Since, $\mathrm{c} \notin \mathcal{I}$ then $0 \neq \mathrm{c} \in \Re$. Hence, for every $\mathrm{a} \in \Re$, there exist $0 \neq \mathrm{c} \in \Re$ and $b_1, b_2 \in \Re$ such that $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$. Hence, $\Re$ is a wFπ-R ring.

(b) Straightforward using a similar argument in (a).

□

However, the converse of the above assertion is not true. For example, let $\Re = \mathbb{Z}_{16}$ and $\mathcal{I} = \{0, 4, 8, 12\}$ be an ideal of $\Re$. In this case, $\Re$ is wFπ-R but not wF*π-R (wwF*π-R). Since for $\mathrm{a} = 2 \notin \mathcal{I}$, there is no such an element $\mathrm{c} \notin \mathcal{I}$ and $b_1, b_2 \in \mathbb{Z}_{16}$ (no $x \in 2c\mathbb{Z}_{16}$) such that $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$ ($\mathrm{ac} = \mathrm{ac}x$).

Theorem 5.2. *If $\Re$ is a right wF*π-R, and $\mathcal{I}$ is an ideal of a ring $\Re$, then $\Re/\mathcal{I}$ is a right wF*π-R.*

Proof. Let $\Re$ be a right wF*π-R ring. We need to show that $\Re/\mathcal{I}$ is a right wF*π-R. Suppose $\bar{\Re} = \Re/\mathcal{I}$, and $\bar{\mathcal{J}}$ be an ideal of $\bar{\Re}$ with $\bar{\mathrm{a}} \notin \bar{\mathcal{J}}$. Since $\Re$ is right wF*π-R, then $\mathrm{a} \notin \mathcal{J}$ when $\mathcal{J}$ is an ideal of $\Re$. Then, there exist $0 \neq \mathrm{c} \notin \mathcal{J}$ and $b_1, b_2 \in \Re$ such that $\mathrm{ac} = \mathrm{ac}b_1\mathrm{ac}b_2$. Hence, $\bar{\mathrm{a}}\bar{\mathrm{c}} = \bar{\mathrm{a}}\bar{\mathrm{c}}\bar{b_1}\bar{\mathrm{a}}\bar{\mathrm{c}}\bar{b_2}$. Since $\mathrm{c} \notin \mathcal{J}$, then $\bar{\mathrm{c}} \notin \bar{\mathcal{J}}$ and $\bar{b_1}, \bar{b_2} \in \bar{\Re}$. Therefore, $\Re$ is a right wF*π-R ring. □

A similar argument can be used to prove the assertion below:

Theorem 5.3. *Let $\mathcal{I}$ be an ideal of a ring $\Re$. If $\Re$ is a right wwF*π-R, then $\Re/\mathcal{I}$ is a right wwF*π-R.*

Theorem 5.4. $\prod \Re_i$ *is a right wF*π-R ring if and only if $\Re_i$ is a right wF*π-R all i.*

Proof. Let $\prod \Re_i$ be a wF*π-R ring and $\mathrm{a} \notin \mathcal{I}$ where $\mathcal{I}$ is ideal in $\Re_i$. Then, $\bar{\mathcal{I}} = \{(\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_i, \ldots) : \mathrm{a}_j \in \Re_i, \text{for all } j \neq i, \mathrm{a}_i \in \mathcal{I}\}$ is an ideal of $\prod \Re_i$ and $\bar{\mathrm{a}} = (0, \ldots, \mathrm{a}, 0, \ldots) \notin \bar{\mathcal{I}}$. since $\prod \Re_i$ is wF*π-R, then there exist $\bar{\mathrm{c}} \notin \bar{\mathcal{I}}$ with $\bar{\mathrm{c}} = (\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_i, \ldots)$, and $\bar{b}, \bar{d} \in \prod \Re_i$ where $\bar{\mathrm{a}}\bar{\mathrm{c}} = \bar{\mathrm{a}}\bar{\mathrm{c}}\bar{b}\bar{\mathrm{a}}\bar{\mathrm{c}}\bar{d}$. As $\bar{\mathrm{c}} \notin \bar{\mathcal{I}}$, we have $\mathrm{c}_i \notin \mathcal{I}$; otherwise, a contradiction. Thus, $\mathrm{ac_i} = \mathrm{ac_i} b_i \mathrm{ac_i} d_i \in \Re_i$, and hence, $\Re_i$ is wF*π-R.
Conversely, let $\Re_i$ be a right wF*π-R, for all $\mathcal{I}$. Suppose $\mathrm{a} \notin \bar{\mathcal{I}}$ where $\bar{\mathcal{I}}$ is an ideal in $\prod \Re_i$. Since $\mathrm{a} \notin \bar{\mathcal{I}}$, there is at least $\mathrm{a}_i \notin \mathcal{I}_i = \{i_i : (i_1, i_2, \ldots, i_i, \ldots) \in \bar{\mathcal{I}}\}$. As $\Re_i$ is right wF*π-R, there is $\mathrm{c}_i \notin \mathcal{I}_i$ and $b_i, d_i \in \Re_i$ where $\mathrm{a}_i\mathrm{c}_i = \mathrm{a}_i\mathrm{c}_i b_i \mathrm{a}_i \mathrm{c}_i d_i$. Hence, $\mathrm{ac} = \mathrm{ac}b\mathrm{ac}d$ where $\mathrm{c} = (0, \ldots, \mathrm{c}_i, 0, \ldots) \notin \bar{\mathcal{I}}$; otherwise, $\mathrm{c}_i \in \mathcal{I}_i$ a contradiction. Therefore, $\prod \Re_i$ is wF*π-R. □

A similar argument can be used to prove the assertion below:

Theorem 5.5. *$\prod \Re_i$ is a right wwF*π-R ring if and only if $\Re_i$ is a right wwF*π-R all i.*

Definition 5.2. [4] *A 2-primal ring is a ring $\Re$ that satisfied $\mathcal{P}(\Re) = \mathcal{N}(\Re)$. Equivalently, if $\Re/\mathcal{P}(\Re)$ is a reduced ring.*

Lemma 5.2. [4] *A ring $\Re$ is 2-primal if and only if every minimal prime ideal is completely prime.*

Theorem 5.6. [12] *Let $\Re$ be a reduced ring. Then the following statements are equivalent:*

(*a*) *$\Re$ is a right w-R.*
(*b*) *$\Re$ is a right $w\pi$-R.*
(*c*) *Every prime ideal of $\Re$ is a maximal.*
(*d*) *Every prime factor of $\Re$ is a simple domain.*

Lemma 5.3. *Let $\Re$ be a 2-primal ring, and $\Re/\mathcal{P}(\Re)$ is a right wwFπ-R ring. Then every prime ideal of $\Re$ is maximal.*

Proof. Let $\mathbf{P} \subset \Re$ be a prime ideal. Using Lemma 5.2, there is a minimal prime ideal $\mathbf{X}$ of $\Re$ that is completely prime, since $\Re$ is a 2-primal. Let $\bar{\Re} = \Re/\mathbf{X}$, then $\bar{\Re}$ is a wwFπ-R. Let $0 \neq \bar{\mathrm{a}} \in \bar{\Re}$. There exists $0 \neq \bar{\mathrm{c}} \in \bar{\Re}$ such that $\bar{\mathrm{a}}\bar{\mathrm{c}} = \bar{\mathrm{a}}\bar{\mathrm{c}}\bar{b}$, where $\bar{b} \in \bar{\Re}\bar{\mathrm{a}}\bar{\mathrm{c}}\bar{\Re}$. Then, $\bar{\mathrm{a}}\bar{\mathrm{c}}(1 - \bar{b}) = 0$ implies that $1 = \bar{b} \in \bar{\Re}\bar{\mathrm{a}}\bar{\mathrm{c}}\bar{\Re}$. Hence, $\bar{\Re}$ is a simple ring and therefore, $\mathbf{X}$ is a maximal ideal and so is $\mathbf{P}$. □

Corollary 5.1. [4] *Suppose $\Re$ is a 2-primal ring. Then the following statements are equivalent:*

(*a*) *$\Re/\mathcal{P}(\Re)$ is a right w-R ring.*
(*b*) *$\Re/\mathcal{P}(\Re)$ is right $w\pi$-R ring.*
(*c*) *Every prime ideal of $\Re$ is maximal.*

According to Corollary 5.1, Lemma 5.3, and the fact that every right $w\pi$-R is right wwFπ-R, the following assertion yields.

Theorem 5.7. *Let $\Re$ be a 2-primal ring. Then the following statements are equivalent:*

(*a*) $\Re/\mathcal{P}(\Re)$ *is a right w-R ring.*

(*b*) $\Re/\mathcal{P}(\Re)$ *is a right $w\pi$-R ring.*

(*c*) *Every prime ideal of $\Re$ is maximal.*

(*d*) $\Re/\mathcal{P}(\Re)$ *is a right wwFπ-R ring.*

Since every reduced ring is 2-primal, and every right Gw-R is a right wwFπ-R ring. Then, the following theorem yields by applying Theorem 5.2, Lemma 5.3, and Theorem 5.6.

Theorem 5.8. *Let $\Re$ be a reduced ring. Then the following conditions are equivalent:*

(*a*) *$\Re$ is a right w-R.*

(*b*) *$\Re$ is a right $w\pi$-R.*

(*c*) *Every prime ideal of $\Re$ is a maximal.*

(*d*) *Every prime factor of $\Re$ is a simple domain.*

(*e*) *$\Re$ is right wwF*π-R.*

Birkenmeier, Kim, and Park [5, Example 1.7], constructed a ring $\Re$, consisting of upper triangular matrices over a Weyl algebra $\mathbb{W}$. They showed that $\Re$ is a 2-primal and its factor ring $\Re/\mathcal{P}(\Re) \cong \mathbb{W} \times \mathbb{W}$ is a right $w\pi$-R; however, $\Re$ itself failed to be a right $w\pi$-R. This example demonstrates that even when a ring is 2-primal and its prime factor ring is right $w\pi$-R, the ring itself need not be right $w\pi$-R. Hence, they concluded that a 2-primal condition does not guarantee that a ring is $w\pi$-regular, even if its prime factor ring satisfies this property.

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