Asia Pac. J. Math. 2025 **12**:87

A STRUCTURAL DYNAMICS OF NEUTROSOPHIC $\hat{Z}$-IDEALS IN $\hat{Z}$-ALGEBRAS

K. P. SHANMUGAPRIYA[1], P. HEMAVATHI[1], R. VINODKUMAR[2], AIYARED IAMPAN[3,*]

[1]Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Thandalam-602105, India

[2]Department of Mathematics, Rajalakshmi Engineering College (Autonomous), Thandalam-602105, India

[3]Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

[*]Corresponding author: aiyared.ia@up.ac.th

Received Aug. 9, 2025

Abstract. This paper introduces a novel extension of classical $\hat{Z}$-ideals by incorporating the neutrosophic set framework, which models truth, indeterminacy, and falsity as independent components. By unifying concepts from neutrosophic and fuzzy set theories, we define and investigate neutrosophic $\hat{Z}$-ideals within the broader context of $\hat{Z}$-algebras. We establish formal definitions, prove key structural properties, and explore conditions for their stability under algebraic operations such as intersections and homomorphisms. The proposed framework offers a robust and flexible tool for handling uncertainty in algebraic systems and opens pathways for applications in soft computing, approximate reasoning, and decision-making environments where imprecision is inherent.

2020 Mathematics Subject Classification. 03G25; 03E72; 08A72.

Key words and phrases. fuzzy set; neutrosophic set; $\hat{Z}$-algebra; $\hat{Z}$-ideal; neutrosophic $\hat{Z}$-ideal; $\hat{Z}$-homomorphism.

1. Introduction

The evolution of fuzzy logic and algebraic structures has played a transformative role in addressing uncertainty, imprecision, and partial truth in mathematical systems. The foundational work by Zadeh [15] on fuzzy sets initiated a paradigm shift, where traditional binary logic was expanded to accommodate degrees of membership, enabling more flexible modeling of real-world problems. Building on this, Rosenfeld [9] introduced the concept of fuzzy groups, further enhancing algebraic applications under fuzziness and establishing a basis for the development of fuzzy algebraic systems such as fuzzy rings, modules, and ideals.

Subsequent studies extended fuzzy set theory into broader algebraic contexts. Sivaramakrishna Das [4] proposed the theory of fuzzy groups and level subgroups, reinforcing the interplay between

DOI: 10.28924/APJM/12-87

group theory and fuzzy sets. Likewise, Bhattacharya and Mukherjee [1] explored fuzzy relations and fuzzy groups, enriching the structural understanding of fuzzy algebraic systems by linking relational models with group-theoretic properties. These foundational studies provided the impetus for more complex structures involving intuitionistic and type-2 fuzzy sets.

In parallel, researchers like Iséki and Tanaka [7], and Imai and Iséki [5] investigated BCI- and BCK-algebras, which are generalizations of logical algebras instrumental in non-classical logic. These structures, as discussed in [6] by Iséki and in [14] by Xi, became significant in fuzzy settings due to their ability to generalize implications and other logical operations algebraically. The work of Jun et al. [8], who analyzed fuzzy ideals and fuzzy subalgebras in BCK-algebras, marked a key point where abstract algebra merged with fuzzy logic to model uncertainty in non-associative systems, laying the groundwork for algebraic reasoning in uncertain environments.

The concept of $\hat{Z}$-algebras, a structure that generalizes classical rings and fields, has also seen significant developments. These algebras have versatile applications in automata theory, coding, and systems design. Chandramouleeswaran et al. [3] contributed foundational insights into $\hat{Z}$-algebra theory, focusing on its intrinsic properties and structural behaviors. These ideas were further extended into fuzzy contexts by Sowmiya and Jeyalakshmi [11–13], who explored fuzzy algebraic structures and fuzzy $\hat{Z}$-ideals, showing how such ideals operate under graded membership and how fuzziness influences algebraic closure and stability. These efforts reflect the growing relevance of $\hat{Z}$-algebras in soft computing, where algebraic precision must coexist with data uncertainty.

In recent years, novel ideas have emerged to address limitations in classical fuzzy logic. Shanmugapriya and Hemavathi [10] proposed an innovative approach by introducing neutrosophic sets in $\hat{Z}$-algebra, offering a more nuanced framework that accommodates truth, indeterminacy, and falsity simultaneously. This triple-component structure allows for richer representation of uncertainty, especially in scenarios involving contradictory or incomplete information. Similarly, Borumand Saeid et al. [2] investigated bi-normed intuitionistic fuzzy β-ideals in β-algebras, adding depth to the interplay between fuzzy logic and algebraic systems, and demonstrating the potential of intuitionistic and bi-normed frameworks in handling multilayered uncertainty.

These developments reflect a collective effort by the mathematical community to construct algebraic systems that are more adaptable and expressive. With increasing complexity in computational and decision-making environments, the integration of fuzzy, intuitionistic, and neutrosophic frameworks into algebraic structures becomes both essential and promising. These enriched models are now being explored for applications in artificial intelligence, data classification, optimization, and knowledge representation. The referenced works provide a robust foundation for further exploration in this interdisciplinary domain, where algebra meets uncertainty in both theory and practice.

This research article develops a unified framework for neutrosophic $\hat{Z}$-ideals within the structure of $\hat{Z}$-algebras by introducing formal definitions, establishing key theorems, and providing illustrative examples. The study further extends these concepts through the analysis of their behavior under $\hat{Z}$-homomorphisms and by formulating the strongest neutrosophic relation, thereby enriching the algebraic modeling of uncertainty with greater structural depth.

2. Preliminaries

This section establishes the foundational definitions and notational framework essential for the development of neutrosophic $\hat{Z}$-ideals. Core concepts such as fuzzy sets, neutrosophic sets, and $\hat{Z}$-algebras are introduced alongside algebraic structures like $\hat{Z}$-ideals, homomorphisms, and level subsets. These definitions serve as the mathematical backbone for the theoretical results that follow, ensuring precision and consistency throughout the exploration of uncertainty within algebraic systems.

Definition 2.1. [15] *Let $\mathfrak{M}$ be a non-empty set. A* fuzzy set *ζ on $\mathfrak{M}$ is defined by a membership function $\mu_\zeta : \mathfrak{M} \to [0,1]$, where for each element $\mathscr{E} \in \mathfrak{M}$, the value $\mu_\zeta(\mathscr{E})$ denotes the degree of membership of $\mathscr{E}$ in ζ. The fuzzy set ζ can thus be represented as:*

$$\zeta = \{\mathscr{E} : \mu_\zeta(\mathscr{E}) \mid \mathscr{E} \in \mathfrak{M}\}.$$

Definition 2.2. [3] *A $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ is defined as a structure, where $\mathfrak{M}$ is a non-empty set with constant 0 and $*$ is a binary operation on $\mathfrak{M}$, satisfying the following conditions:*

(i) $\mathscr{E} * 0 = 0$

(ii) $0 * \mathscr{E} = \mathscr{E}$

(iii) $\mathscr{E} * \mathscr{E} = \mathscr{E}$

(iv) $\mathscr{E} * \delta = \delta * \mathscr{E}$, *when* $\mathscr{E} \neq 0$ *and* $\delta \neq 0$, *for all* $\mathscr{E}, \delta \in \mathfrak{M}$.

Example 2.3. *Consider the Cayley table defined on the set $\mathfrak{M} = \{0, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4\}$, equipped with a binary operation $*$ and a distinguished constant element 0.*

$*$	0	$\mathfrak{N}_1$	$\mathfrak{N}_2$	$\mathfrak{N}_3$	$\mathfrak{N}_4$
0	0	$\mathfrak{N}_1$	$\mathfrak{N}_2$	$\mathfrak{N}_3$	$\mathfrak{N}_4$
$\mathfrak{N}_1$	0	$\mathfrak{N}_1$	$\mathfrak{N}_3$	$\mathfrak{N}_2$	$\mathfrak{N}_4$
$\mathfrak{N}_2$	0	$\mathfrak{N}_3$	$\mathfrak{N}_2$	$\mathfrak{N}_4$	$\mathfrak{N}_3$
$\mathfrak{N}_3$	0	$\mathfrak{N}_2$	$\mathfrak{N}_4$	$\mathfrak{N}_3$	$\mathfrak{N}_2$
$\mathfrak{N}_4$	0	$\mathfrak{N}_4$	$\mathfrak{N}_1$	$\mathfrak{N}_1$	$\mathfrak{N}_4$

Then $(\mathfrak{M}, *, 0)$ is a $\hat{Z}$-algebra.

Definition 2.4. [10] *Let the neutrosophic set $\zeta = \{\mathscr{E} : T_\zeta(\mathscr{E}), I_\zeta(\mathscr{E}), F_\zeta(\mathscr{E}) \mid \mathscr{E} \in \mathfrak{M}\}$. If it holds the following conditions, then it is known to be a neutrosophic $\hat{Z}$-subalgebra in a $\hat{Z}$-algebra $\mathfrak{M}$: for all $\mathscr{E}, \delta \in \mathfrak{M}$,*

(i) $T_\zeta(\mathscr{E} * \delta) \geq \min\{T_\zeta(\mathscr{E}), T_\zeta(\delta)\}$

(ii) $I_\zeta(\mathscr{E} * \delta) \geq \min\{I_\zeta(\mathscr{E}), I_\zeta(\delta)\}$

(iii) $F_\zeta(\mathscr{E} * \delta) \leq \max\{F_\zeta(\mathscr{E}), F_\zeta(\delta)\}$.

Definition 2.5. [3] *Let $\mathfrak{M}$ be a $\hat{Z}$-algebra and $\Im \subseteq \mathfrak{M}$. Then, $\Im$ is called a $\hat{Z}$-ideal of $\mathfrak{M}$ if the following conditions hold: for all $\mathscr{E}, \delta \in \mathfrak{M}$,*

(i) $0 \in \Im$

(ii) $\mathscr{E} * \delta \in \Im$ *and* $\delta \in \Im$ *implies* $\mathscr{E} \in \Im$.

Definition 2.6. [11] *Let $(\mathfrak{M}, *, 0)$ and $(\mathfrak{Y}, *', 0')$ be two $\hat{Z}$-algebras. Then, the mapping $\mathbb{h} : (\mathfrak{M}, *, 0) \to (\mathfrak{Y}, *', 0')$ is known as a $\hat{Z}$-homomorphism of $\hat{Z}$-algebras if*

$$\mathbb{h}(\mathscr{E} * \delta) = \mathbb{h}(\mathscr{E}) *' \mathbb{h}(\delta), \forall \mathscr{E}, \delta \in \mathfrak{M}.$$

Definition 2.7. [11] *Take into consideration a $\hat{Z}$-homomorphism $\mathbb{h}$ that maps the $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ onto another $\hat{Z}$-algebra $(\mathfrak{Y}, *', 0')$. The properties listed below are met by this mapping:*

(i) $\mathbb{h}$ *is called a $\hat{Z}$-monomorphism of $\hat{Z}$-algebras if it is injective (one-to-one)*

(ii) $\mathbb{h}$ *is called a $\hat{Z}$-epimorphism of $\hat{Z}$-algebras if it is surjective (onto).*

Note 2.8. *If $\mathbb{h} : (\mathfrak{M}, *, 0) \to (\mathfrak{Y}, *', 0')$ is a $\hat{Z}$-homomorphism, then it must satisfy $\mathbb{h}(0) = 0'$.*

Definition 2.9. [15] *Suppose a non-empty set $\mathfrak{M}$. A fuzzy set ζ on $\mathfrak{M}$ is defined by a membership function $\mu_\zeta : \mathfrak{M} \to [0, 1]$ which assigns to each element $\mathscr{E} \in \mathfrak{M}$ a value in the interval $[0, 1]$. The value $\mu_\zeta(\mathscr{E})$ represents the degree or grade of membership of $\mathscr{E}$ in the fuzzy set ζ.*

Definition 2.10. *Consider a $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$. A fuzzy set ζ on $\mathfrak{M}$, characterized by the membership function μ_ζ, is called a fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ if for all $\mathscr{E}, \delta \in \mathfrak{M}$,*

$$\mu_\zeta(\mathscr{E} * \delta) \geq \min\{\mu_\zeta(\mathscr{E}), \mu_\zeta(\delta)\}.$$

Definition 2.11. *Consider a fuzzy set ζ defined on $\mathfrak{M}$. For a fixed $\mathfrak{t} \in [0, 1]$, the set*

$$U(\zeta, \mathfrak{t}) = \{\mathscr{E} \in \mathfrak{M} \mid \mu_\zeta(\mathscr{E}) \geq \mathfrak{t}\}$$

is called the upper-level subset of ζ, also known as the upper-level cut or upper $\mathfrak{t}$-level subset.

Definition 2.12. [9] *A fuzzy set ζ on $\mathfrak{M}$ with membership function μ_ζ is said to satisfy the supremum property if for every non-empty subset $\tau \subseteq \mathfrak{M}$, there exists an element $\mathscr{E}_0 \in \mathfrak{M}$ such that*

$$\mu_\zeta(\mathscr{E}_0) = \sup_{\mathfrak{t} \in \tau} \mu_\zeta(\mathfrak{t}).$$

Definition 2.13. [1] *A fuzzy relation ζ on a non-empty set $\mathfrak{M}$ refers to a fuzzy set characterized by a membership function $\mu_\zeta : \mathfrak{M} \times \mathfrak{M} \to [0, 1]$.*

Definition 2.14. [1] *If ζ is a fuzzy relation on a non-empty set $\mathfrak{M}$, defined by the membership function μ_ζ, and let ξ be a fuzzy set on $\mathfrak{M}$, with membership function μ_ξ, then ζ is said to be a fuzzy relation on ξ if for all $\mathscr{E}, \delta \in \mathfrak{M}$,*

$$\mu_\zeta(\mathscr{E}, \delta) \leq \min\{\mu_\xi(\mathscr{E}), \mu_\xi(\delta)\}.$$

Definition 2.15. [1] *Let ξ be a fuzzy set on a non-empty set $\mathfrak{M}$ with a membership function μ_ξ. Then the strongest fuzzy relation ζ_ξ on $\mathfrak{M}$, a fuzzy relation on ξ, is a fuzzy relation whose membership function $\mu_{\zeta_\xi} : \mathfrak{M} \times \mathfrak{M} \to [0, 1]$ is expressed as*

$$\mu_{\zeta_\xi}(\mathscr{E}, \delta) = \min\{\mu_\xi(\mathscr{E}), \mu_\xi(\delta)\}.$$

Definition 2.16. [1] *Consider two $\hat{Z}$-algebras $(\mathfrak{M}, *, 0)$ and $(\mathfrak{Y}, *', 0')$. Then, the Cartesian product $(\mathfrak{M} \times \mathfrak{Y}, *'', 0'')$ forms a $\hat{Z}$-algebra, where the binary operation $*''$ is defined by*

$$(\mathscr{E}_1, \delta_1) *'' (\mathscr{E}_2, \delta_2) = (\mathscr{E}_1 * \mathscr{E}_2, \delta_1 *' \delta_2), \forall (\mathscr{E}_1, \delta_1), (\mathscr{E}_2, \delta_2) \in \mathfrak{M} \times \mathfrak{Y},$$

and the constant element is given by $0'' = (0, 0')$.

Definition 2.17. [13] *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. A fuzzy set ζ on $\mathfrak{M}$, defined by the membership function μ_ζ, is said to be a fuzzy $\hat{Z}$-ideal of $\mathfrak{M}$ if it satisfies the following conditions: for all $\mathscr{E}, \delta \in \mathfrak{M}$,*

(*i*) $\mu_\zeta(0) \geq \mu_\zeta(\mathscr{E})$

(*ii*) $\mu_\zeta(\mathscr{E}) \geq \min\{\mu_\zeta(\mathscr{E} * \delta), \mu_\zeta(\delta)\}$.

3. Neutrosophic $\hat{Z}$-Ideals in $\hat{Z}$-Algebras

This section lays the foundational framework for neutrosophic $\hat{Z}$-ideals by formally defining their structure within $\hat{Z}$-algebras. Through carefully constructed examples and a sequence of supporting theorems, we characterize their algebraic behavior under operations such as intersection and level subset formation. The results highlight how truth, indeterminacy, and falsity components interact to preserve ideal properties, establishing a solid basis for the subsequent extension of neutrosophic ideals across more complex algebraic and relational contexts.

Definition 3.1. *A neutrosophic set ζ in a $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ with membership function*

$$\zeta = \{\mathscr{E} : \mu_{\zeta_T}(\mathscr{E}), \mu_{\zeta_I}(\mathscr{E}), \mu_{\zeta_F}(\mathscr{E}) \mid \mathscr{E} \in \mathfrak{M}\}$$

is known as a neutrosophic $\hat{Z}$-ideal in $\mathfrak{M}$ if it satisfies the following conditions: for all $\mathscr{E}, \delta \in \mathfrak{M}$,

(*i*) $\mu_{\zeta_T}(0) \geq \mu_{\zeta_T}(\mathscr{E})$ *and* $\mu_{\zeta_T}(\mathscr{E}) \geq \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\}$

(*ii*) $\mu_{\zeta_I}(0) \geq \mu_{\zeta_I}(\mathscr{E})$ *and* $\mu_{\zeta_I}(\mathscr{E}) \geq \min\{\mu_{\zeta_I}(\mathscr{E} * \delta), \mu_{\zeta_I}(\delta)\}$

(*iii*) $\mu_{\zeta_F}(0) \leq \mu_{\zeta_F}(\mathscr{E})$ *and* $\mu_{\zeta_F}(\mathscr{E}) \leq \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\}$.

Example 3.2. *Consider the Cayley table in Example 2.3. Let $\mu_{\zeta_{(T,I,F)}}$ be the neutrosophic function defined as follows:*

$$\mu_{\zeta_{(T,I)}}(\mathscr{E}) = \begin{cases} 1.0 & \text{if } \mathscr{E} = 0 \\ 0.8 & \text{if } \mathscr{E} = \mathfrak{N}_1, \mathfrak{N}_4 \\ 0.7 & \text{if } \mathscr{E} = \mathfrak{N}_2, \mathfrak{N}_3 \end{cases}$$

$$\mu_{\zeta_F}(\mathscr{E}) = \begin{cases} 0.1 & \text{if } \mathscr{E} = 0 \\ 0.2 & \text{if } \mathscr{E} = \mathfrak{N}_1, \mathfrak{N}_4 \\ 0.3 & \text{if } \mathscr{E} = \mathfrak{N}_2, \mathfrak{N}_3 \end{cases}$$

Hence, ζ is a neutrosophic $\hat{Z}$-ideal in $\mathfrak{M}$.

Theorem 3.3. *The arbitrary intersection of neutrosophic $\hat{Z}$-ideals in a $\hat{Z}$-algebra $\mathfrak{M}$ is itself a neutrosophic $\hat{Z}$-ideal.*

Proof. Let $\{\zeta_i \mid i \in \Omega\}$ be a family of neutrosophic $\hat{Z}$-ideals of a $\hat{Z}$-algebra $\mathfrak{M}$.

Let $\mathscr{E}, \delta \in \mathfrak{M}$. Then

$$\begin{aligned} \mu_{\bigcap_{i\in\Omega}\zeta_{T_i}}(0) &= \inf_{i\in\Omega} \mu_{\zeta_T}(0) \geq \inf_{i\in\Omega} \mu_{\zeta_T}(\mathscr{E}) = \mu_{\bigcap_{i\in\Omega}\zeta_{T_i}}(\mathscr{E}), \\ \mu_{\bigcap_{i\in\Omega}\zeta_{T_i}}(\mathscr{E}) &= \inf_{i\in\Omega} \mu_{\zeta_T}(\mathscr{E}) \\ &\geq \min\{\inf_{i\in\Omega} \mu_{\zeta_{T_i}}(\mathscr{E} * \delta),\ \inf_{i\in\Omega} \mu_{\zeta_{T_i}}(\delta)\} \\ &= \min\{\mu_{\bigcap_{i\in\Omega}\zeta_{T_i}}(\mathscr{E} * \delta),\ \mu_{\bigcap_{i\in\Omega}\zeta_{T_i}}(\delta)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mu_{\bigcap_{i\in\Omega}\zeta_{I_i}}(0) \geq \mu_{\bigcap_{i\in\Omega}\zeta_{I_i}}(\mathscr{E}), \\ &\mu_{\bigcap_{i\in\Omega}\zeta_{I_i}}(\mathscr{E}) \geq \min\{\mu_{\bigcap_{i\in\Omega}\zeta_{I_i}}(\mathscr{E} * \delta),\ \mu_{\bigcap_{i\in\Omega}\zeta_{I_i}}(\delta)\}. \end{aligned}$$

The remaining case,

$$\begin{aligned} \mu_{\bigcap_{i\in\Omega}\zeta_{F_i}}(0) &= \sup_{i\in\Omega} \mu_{\zeta_F}(0) \leq \sup_{i\in\Omega} \mu_{\zeta_F}(\mathscr{E}) = \mu_{\bigcap_{i\in\Omega}\zeta_{F_i}}(\mathscr{E}), \\ \mu_{\bigcap_{i\in\Omega}\zeta_{F_i}}(\mathscr{E}) &= \sup_{i\in\Omega} \mu_{\zeta_F}(\mathscr{E}) \\ &\leq \max\{\sup_{i\in\Omega} \mu_{\zeta_{F_i}}(\mathscr{E} * \delta),\ \sup_{i\in\Omega} \mu_{\zeta_{F_i}}(\delta)\} \\ &= \max\{\mu_{\bigcap_{i\in\Omega}\zeta_{F_i}}(\mathscr{E} * \delta),\ \mu_{\bigcap_{i\in\Omega}\zeta_{F_i}}(\delta)\}. \end{aligned}$$

Hence, $\bigcap_{i\in\Omega}\zeta_i$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. $\square$

Theorem 3.4. *A neutrosophic set ζ of a $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ is a neutrosophic $\hat{Z}$-ideal if and only if for any $\mathfrak{t} \in [0,1]$,*

$$U(\zeta, \mathfrak{t}) = \{\mathscr{E} \in \mathfrak{M} \mid \mu_{\zeta_T}(\mathscr{E}) \geq \mathfrak{t}, \mu_{\zeta_I}(\mathscr{E}) \geq \mathfrak{t}, \mu_{\zeta_F}(\mathscr{E}) \leq \mathfrak{t}\}$$

is a $\hat{Z}$-ideal of $\mathfrak{M}$, provided that $U(\zeta, \mathfrak{t}) \neq \emptyset$.

Proof. Let ζ be a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ and $U(\zeta, \mathfrak{t}) \neq \emptyset$ for any $\mathfrak{t} \in [0,1]$. If $\mathscr{E} \in U(\zeta, \mathfrak{t})$, then it follows that $\mu_{\zeta_T}(\mathscr{E}) \geq \mathfrak{t}, \mu_{\zeta_I}(\mathscr{E}) \geq \mathfrak{t}$, and $\mu_{\zeta_F}(\mathscr{E}) \leq \mathfrak{t}$. This directly follows from the definition of a neutrosophic $\hat{Z}$-ideal, we get

$$\mu_{\zeta_T}(0) \geq \mu_{\zeta_T}(\mathscr{E}) \geq \mathfrak{t}, \text{ so } 0 \in U(\zeta, \mathfrak{t}).$$

Let $\mathscr{E} * \delta \in U(\zeta, \mathfrak{t})$ and $\delta \in U(\zeta, \mathfrak{t})$. Then $\mu_{\zeta_T}(\mathscr{E} * \delta) \geq \mathfrak{t}$ and $\mu_{\zeta_T}(\delta) \geq \mathfrak{t}$. Thus,

$$\mu_{\zeta_T}(\mathscr{E}) \geq \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\} \geq \min\{\mathfrak{t}, \mathfrak{t}\} = \mathfrak{t}.$$

Similarly,

$$\mu_{\zeta_I}(\mathscr{E}) \geq \min\{\mu_{\zeta_I}(\mathscr{E} * \delta), \mu_{\zeta_I}(\delta)\} \geq \min\{\mathfrak{t}, \mathfrak{t}\} = \mathfrak{t}.$$

If $\mathscr{E} * \delta \in U(\zeta, \mathfrak{t})$ and $\delta \in U(\zeta, \mathfrak{t})$. Then $\mu_{\zeta_F}(\mathscr{E} * \delta) \leq \mathfrak{t}$ and $\mu_{\zeta_F}(\delta) \leq \mathfrak{t}$. Thus,

$$\mu_{\zeta_F}(\mathscr{E}) \leq \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\} \leq \max\{\mathfrak{t}, \mathfrak{t}\} = \mathfrak{t}.$$

Therefore, $\mathscr{E} \in U(\zeta, \mathfrak{t})$. Hence, $U(\zeta, \mathfrak{t})$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.

Conversely, suppose that for every $\mathfrak{t} \in [0,1]$, the set $U(\zeta, \mathfrak{t})$ is either empty or a $\hat{Z}$-ideal of $\mathfrak{M}$. For any $\mathscr{E} \in \mathfrak{M}$, $\mu_{\zeta_T}(\mathscr{E}) = \mathfrak{t}$, then $\mathscr{E} \in U(\zeta, \mathfrak{t})$. Since $U(\zeta, \mathfrak{t}) \neq \emptyset$ is a $\hat{Z}$-ideal of $\mathfrak{M}$, we have $0 \in U(\zeta, \mathfrak{t})$ and $\mu_{\zeta_T}(0) \geq \mathfrak{t} = \mu_{\zeta_T}(\mathscr{E})$. Thus,

$$\mu_{\zeta_T}(0) \geq \mu_{\zeta_T}(\mathscr{E}), \forall \mathscr{E} \in \mathfrak{M}.$$

Assume that $\mu_{\zeta_T}(\mathscr{E}) \geq \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\}$ for all $\mathscr{E}, \delta \in \mathfrak{M}$ is not true. Then, there exist $\mathscr{E}_0, \delta_0 \in \mathfrak{M}$ such that

$$\mu_{\zeta_T}(\mathscr{E}_0) < \min\{\mu_{\zeta_T}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_T}(\delta_0)\}.$$

Let

$$\mathfrak{t}_0 = \frac{1}{2}[\mu_{\zeta_T}(\mathscr{E}_0) + \min\{\mu_{\zeta_T}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_T}(\delta_0)\}].$$

Then

$$\mu_{\zeta_T}(\mathscr{E}_0) < \mathfrak{t}_0 < \min\{\mu_{\zeta_T}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_T}(\delta_0)\}.$$

So, $\mathscr{E}_0 * \delta_0 \in U(\zeta, \mathfrak{t}_0)$ and $\mathscr{E}_0 \notin U(\zeta, \mathfrak{t}_0)$. But $U(\zeta, \mathfrak{t}_0)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$, we have $\mathscr{E}_0 \in U(\zeta, \mathfrak{t}_0)$, so $\mu_{\zeta_T}(\mathscr{E}_0) \geq \mathfrak{t}_0$. This is a contradiction. Hence, $\mu_{\zeta_T}(\mathscr{E}) \geq \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\}$. Similarly,

$$\mu_{\zeta_I}(0) \geq \mu_{\zeta_I}(\mathscr{E}),$$

$$\mu_{\zeta_I}(\mathscr{E}) \geq \min\{\mu_{\zeta_I}(\mathscr{E} * \delta), \mu_{\zeta_I}(\delta)\}.$$

Now, $\mu_{\zeta_F}(0) \leq \mathfrak{t} = \mu_{\zeta_F}(\mathscr{E})$. Thus,

$$\mu_{\zeta_F}(0) \leq \mu_{\zeta_F}(\mathscr{E}), \forall \mathscr{E} \in \mathfrak{M}.$$

Assume that $\mu_{\zeta_F}(\mathscr{E}) \leq \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\}$ for all $\mathscr{E}, \delta \in \mathfrak{M}$ is not true. Then, there exist $\mathscr{E}_0, \delta_0 \in \mathfrak{M}$ such that

$$\mu_{\zeta_F}(\mathscr{E}_0) < \max\{\mu_{\zeta_F}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_F}(\delta_0)\}.$$

Let

$$\mathfrak{t}_0 = \frac{1}{2}[\mu_{\zeta_F}(\mathscr{E}_0) + \max\{\mu_{\zeta_F}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_F}(\delta_0)\}].$$

Then

$$\mu_{\zeta_F}(\mathscr{E}_0) < \mathfrak{t}_0 < \max\{\mu_{\zeta_F}(\mathscr{E}_0 * \delta_0), \mu_{\zeta_F}(\delta_0)\}.$$

So, $\mathscr{E}_0 * \delta_0 \in U(\zeta, \mathfrak{t}_0)$ and $\mathscr{E}_0 \notin U(\zeta, \mathfrak{t}_0)$. But $U(\zeta, \mathfrak{t}_0)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$, we have $\mathscr{E}_0 \in U(\zeta, \mathfrak{t}_0)$, so $\mu_{\zeta_F}(\mathscr{E}_0) \leq \mathfrak{t}_0$. This is a contradiction. Hence, $\mu_{\zeta_F}(\mathscr{E}) \leq \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\}$. Therefore, ζ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. $\square$

Definition 3.5. *Let ζ be a neutrosophic $\hat{Z}$-ideal of a $\hat{Z}$-algebra $\mathfrak{M}$. For each $\mathfrak{t} \in [0, 1]$, a $\hat{Z}$-ideal $U(\zeta, \mathfrak{t})$ is referred to as the upper-level $\hat{Z}$-ideal of ζ.*

Remark 3.6. *Henceforth, upper-level $\hat{Z}$-ideals will be designated as level $\hat{Z}$-ideals.*

Theorem 3.7. *A neutrosophic set ζ of a $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ if and only if every non-empty level subset $U(\zeta, q)$, where $q \in \mathrm{Im}(\zeta)$ forms a $\hat{Z}$-ideal of $\mathfrak{M}$.*

Proof. Let ζ be a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. Then $\mu_{\zeta_T}(0) \geq \mu_{\zeta_T}(\mathscr{E}), \forall \mathscr{E} \in \mathfrak{M}$. Hence, $\mathscr{E} \in U(\zeta_T, q)$, $\mu_{\zeta_T}(0) \geq q \implies 0 \in U(\zeta_T, q)$. Now, for any $\mathscr{E}, \delta \in \mathfrak{M}$, assume that $\mathscr{E} * \delta \in U(\zeta_T, q)$ and $\delta \in U(\zeta_T, q)$. Then $\mu_{\zeta_T}(\mathscr{E} * \delta) \geq q$ and $\mu_{\zeta_T}(\delta) \geq q \implies \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\} \geq q$. Thus, $\mu_{\zeta_T}(\mathscr{E}) \geq \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\} \geq q$. Thus, $\mathscr{E} \in U(\zeta_T, q)$, so $U(\zeta_T, q)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$. Similarly, $U(\zeta_I, q)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$. Now, $\mu_{\zeta_F}(0) \leq \mu_{\zeta_F}(\mathscr{E})\ \forall \mathscr{E} \in \mathfrak{M}$. Hence, $\mathscr{E} \in U(\zeta_F, q)$, $\mu_{\zeta_F}(0) \leq q \implies 0 \in U(\zeta_F, q)$. Now, for any $\mathscr{E}, \delta \in \mathfrak{M}$, assume that $\mathscr{E} * \delta \in U(\zeta_F, q)$ and $\delta \in U(\zeta_F, q)$. Then $\mu_{\zeta_F}(\mathscr{E} * \delta) \leq q$ and $\mu_{\zeta_F}(\delta) \leq q \implies \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\} \leq q$. Thus, $\mu_{\zeta_F}(\mathscr{E}) \leq \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\} \leq q$. Thus, $\mathscr{E} \in U(\zeta_F, q)$. Hence, $U(\zeta_F, q)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$.

Conversely, let $U(\zeta_T, q)$ be a $\hat{Z}$-ideal of $\mathfrak{M}$ for $q \in \mathrm{Im}(\zeta_T)$. Let $\mathscr{E}, \delta \in \mathfrak{M}$. For any $q \in \mathrm{Im}(\zeta_T)$ and $q = \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\}$. This indicates $\mathscr{E} * \delta, \delta \in U(\zeta_T, q)$. Since $U(\zeta_T, q)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$, we have $\mathscr{E} \in U(\zeta_T, q)$. This confirms that $\mu_{\zeta_T}(\mathscr{E}) \geq q = \min\{\mu_{\zeta_T}(\mathscr{E} * \delta), \mu_{\zeta_T}(\delta)\}$. Similarly, $\mu_{\zeta_I}(\mathscr{E}) \geq \min\{\mu_{\zeta_I}(\mathscr{E} * \delta), \mu_{\zeta_I}(\delta)\}$. Let $\mathscr{E}, \delta \in \mathfrak{M}$. For any $q \in \mathrm{Im}(\zeta_F)$ and $q = \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\}$. This indicates $\mathscr{E} * \delta, \delta \in U(\zeta_F, q)$. Since $U(\zeta_F, q)$ is a $\hat{Z}$-ideal of $\mathfrak{M}$, we have $\mathscr{E} \in U(\zeta_F, q)$. This confirms that $\mu_{\zeta_F}(\mathscr{E}) \leq q = \max\{\mu_{\zeta_F}(\mathscr{E} * \delta), \mu_{\zeta_F}(\delta)\}$. Hence, ζ_F is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. $\square$

Theorem 3.8. *Let ζ be a neutrosophic $\hat{Z}$-ideal of a $\hat{Z}$-algebra $\mathfrak{M}$ and let $\mathscr{E} \in \mathfrak{M}$. Then $\mu_{\zeta_{(T,I,F)}}(\mathscr{E}) = t$ if and only if $\mathscr{E} \in U(\zeta_{(T,I,F)}, t)$ but $\mathscr{E} \notin U(\zeta_{(T,I,F)}, q)\ \forall q > t$.*

Proof. Assume $\mu_{\zeta_T}(\mathscr{E}) = t$, so $\mathscr{E} \in U(\zeta_T, t)$. If $\mathscr{E} \in U(\zeta_T, q)$ for $q > t$, then $\mu_{\zeta_T}(\mathscr{E}) \geq q > t$. This leads to contradiction, $\mu_{\zeta_T}(\mathscr{E}) = t$. Hence, $\mathscr{E} \notin U(\zeta_T, q), \forall q > t$. Similarly, $\mu_{\zeta_I}(\mathscr{E}) = t$. Hence, $\mathscr{E} \notin U(\zeta_I, q)$ $\forall q > t$. Assume $\mu_{\zeta_F}(\mathscr{E}) = t$, so $\mathscr{E} \in U(\zeta_F, t)$. If $\mathscr{E} \in U(\zeta_F, q)$ for $q < t$, then $\mu_{\zeta_F}(\mathscr{E}) \leq q < t$. This opposes the fact that $\mu_{\zeta_F}(\mathscr{E}) = t$. Hence, $\mathscr{E} \notin U(\zeta_F, q), \forall q < t$.

Conversely, let $\mathscr{E} \in U(\zeta_T, t)$, but $\mathscr{E} \notin U(\zeta_T, q)\ \forall q > t$. Thus, $\mathscr{E} \in U(\zeta_T, t) \implies \mu_{\zeta_T}(\mathscr{E}) \geq t$. Since $\mathscr{E} \notin U(\zeta_T, q), \forall q > t$, we have $\mu_{\zeta_T}(\mathscr{E}) = t$. Similarly, $\mu_{\zeta_I}(\mathscr{E}) = t$. Let $\mathscr{E} \in U(\zeta_F, t)$, but $\mathscr{E} \notin U(\zeta_F, q), \forall q < t$. Thus, $\mathscr{E} \in U(\zeta_F, t) \implies \mu_{\zeta_F}(\mathscr{E}) \leq t$. Since $\mathscr{E} \notin U(\zeta_F, q), \forall q < t$, we have $\mu_{\zeta_F}(\mathscr{E}) = t$. □

4. $\hat{Z}$-Homomorphism of Neutrosophic $\hat{Z}$-Ideal in $\hat{Z}$-Algebra

In this section, we investigate how neutrosophic $\hat{Z}$-ideals behave under structural transformations between $\hat{Z}$-algebras through the lens of $\hat{Z}$-homomorphisms. By examining the pre-image and endomorphic images of these ideals, we demonstrate that key neutrosophic properties are preserved under both surjective and internal mappings. Theoretical results are supported with precise formulations, confirming that the algebraic integrity of neutrosophic $\hat{Z}$-ideals remains intact under homomorphic operations—thereby reinforcing their robustness in abstract algebraic frameworks and potential for structural generalization.

Definition 4.1. *Let $\mathfrak{M}$ and $\mathfrak{Y}$ be two $\hat{Z}$-algebras and $\mathbb{h} : \mathfrak{M} \to \mathfrak{Y}$ be a function. If ζ is a neutrosophic set in $\mathfrak{Y}$, then the pre-image of ζ under $\mathbb{h}$ denoted by $\mathbb{h}^{-1}(\zeta)$ is the neutrosophic set in $\mathfrak{M}$, is defined by*

$$\mathbb{h}^{-1}(\zeta) = \{\mathscr{E} : \mathbb{h}^{-1}(\zeta_T(\mathscr{E})), \mathbb{h}^{-1}(\zeta_I(\mathscr{E})), \mathbb{h}^{-1}(\zeta_F(\mathscr{E})) \mid \mathscr{E} \in \mathfrak{M}\},$$

where

$$\mathbb{h}^{-1}(\zeta_T(\mathscr{E})) = \zeta_T(\mathbb{h}(\mathscr{E})), \mathbb{h}^{-1}(\zeta_I(\mathscr{E})) = \zeta_I(\mathbb{h}(\mathscr{E})), \text{ and } \mathbb{h}^{-1}(\zeta_F(\mathscr{E})) = \zeta_F(\mathbb{h}(\mathscr{E})).$$

Theorem 4.2. *Let $\mathbb{h} : \mathfrak{M} \to \mathfrak{Y}$ be a $\hat{Z}$-epimorphism of $\hat{Z}$-algebras. If ζ is a neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{Y}$, then the pre-image of ζ under $\mathbb{h}$ is a neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{M}$.*

Proof. Let ζ be a neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{Y}$, and let $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 \in \mathfrak{M}$. Then

$$\begin{aligned}\mathbb{h}^{-1}(\zeta_T(0)) &= \zeta_T(\mathbb{h}(0)) \\ &\geq \zeta_T(\mathbb{h}(\mathscr{E}_1)) \\ &= \mathbb{h}^{-1}(\zeta_T(\mathscr{E}_1)).\end{aligned}$$

Similarly, $\mathbb{h}^{-1}(\zeta_I(0)) \geq \mathbb{h}^{-1}(\zeta_I(\mathscr{E}_1))$. The remaining case,

$$\mathbb{h}^{-1}(\zeta_F(0)) = \zeta_F(\mathbb{h}(0))$$

$$\geq \zeta_F(\mathbb{h}(\mathscr{E}_1))$$
$$= \mathbb{h}^{-1}(\zeta_F(\mathscr{E}_1)).$$

Now,

$$\begin{aligned}\mathbb{h}^{-1}(\zeta_T(\mathscr{E}_2 * \mathscr{E}_3)) &= \zeta_T(\mathbb{h}(\mathscr{E}_2 * \mathscr{E}_3))\\ &= \zeta_T(\mathbb{h}(\mathscr{E}_2) *' \mathbb{h}(\mathscr{E}_3))\\ &\geq \min\{\zeta_T(\mathbb{h}(\mathscr{E}_2)), \zeta_T(\mathbb{h}(\mathscr{E}_3))\}\\ &= \min\{\mathbb{h}^{-1}(\zeta_T(\mathscr{E}_2)), \mathbb{h}^{-1}(\zeta_T(\mathscr{E}_3))\}.\end{aligned}$$

Similarly, $\mathbb{h}^{-1}(\zeta_I(\mathscr{E}_2 * \mathscr{E}_3)) \geq \min\{\mathbb{h}^{-1}(\zeta_I(\mathscr{E}_2)), \mathbb{h}^{-1}(\zeta_I(\mathscr{E}_3))\}$. The remaining case,

$$\begin{aligned}\mathbb{h}^{-1}(\zeta_F(\mathscr{E}_2 * \mathscr{E}_3)) &= \zeta_F(\mathbb{h}(\mathscr{E}_2 * \mathscr{E}_3))\\ &= \zeta_F(\mathbb{h}(\mathscr{E}_2) *' \mathbb{h}(\mathscr{E}_3))\\ &\leq \max\{\zeta_F(\mathbb{h}(\mathscr{E}_2)), \zeta_F(\mathbb{h}(\mathscr{E}_3))\}\\ &= \max\{\mathbb{h}^{-1}(\zeta_F(\mathscr{E}_2)), \mathbb{h}^{-1}(\zeta_F(\mathscr{E}_3))\}.\end{aligned}$$

Hence, $\mathbb{h}^{-1}(\zeta)$ is a neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

Theorem 4.3. *Let $\mathbb{h} : \mathfrak{M} \to \mathfrak{Y}$ be a $\hat{Z}$-epimorphism of $\hat{Z}$-algebras. If ζ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$, then the pre-image of ζ under $\mathbb{h}$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.*

Proof. The proof is similar to Theorem 4.2. □

Definition 4.4. *Let $\mathbb{h}$ be a $\hat{Z}$-endomorphism of a $\hat{Z}$-algebra $\mathfrak{M}$ and let ζ be a neutrosophic set on $\mathfrak{M}$. A new neutrosophic set $\zeta^{\mathbb{h}}$ on $\mathfrak{M}$ is defined by the membership function:*

$$\mu_{\zeta^{\mathbb{h}}_{(T,I,F)}}(\mathscr{E}) = \mu_{\zeta_{(T,I,F)}}(\mathbb{h}(\mathscr{E})), \forall \mathscr{E} \in \mathfrak{M}.$$

Theorem 4.5. *Let $\mathbb{h}$ be a $\hat{Z}$-endomorphism of a $\hat{Z}$-algebra $\mathfrak{M}$ and ζ be a neutrosophic set on $\mathfrak{M}$. Then $\zeta^{\mathbb{h}}$ will also be a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$, provided that ζ itself is a neutrosophic $\hat{Z}$-ideal.*

Proof. The proof is straightforward. □

5. Strongest Neutrosophic Relation of Neutrosophic $\hat{Z}$-Ideals in $\hat{Z}$-Algebras

This section introduces a pivotal extension of neutrosophic $\hat{Z}$-ideals through the formulation of their strongest neutrosophic relation. By constructing a binary relation over the Cartesian product $\mathfrak{M} \times \mathfrak{M}$, we explore how the interaction between two distinct neutrosophic $\hat{Z}$-ideals can be captured in a structured and algebraically meaningful way. Specifically, the relation is defined using the infimum (min) for truth and indeterminacy, and the supremum (max) for falsity—reflecting the strongest logical overlap between ideals. A key theorem is established, demonstrating that this combined relation retains the defining properties of a neutrosophic $\hat{Z}$-ideal, thereby extending the ideal structure to product algebras without loss of integrity.

Definition 5.1. *Let ζ_1 and ζ_2 be two neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$. Then the strongest neutrosophic relation $\zeta_1 \times \zeta_2$ is defined by:*

$$\zeta_1 \times \zeta_2 : \mathfrak{M} \times \mathfrak{M} \to [0,1]$$

such that

$$\zeta_1 \times \zeta_2 = \{(\mathscr{E}, \delta) : \mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}, \delta), \mu_{\zeta_{I_1} \times \zeta_{I_2}}(\mathscr{E}, \delta), \mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}, \delta) \mid \mathscr{E}, \delta \in \mathfrak{M}\},$$

where

$$\begin{aligned}
\mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}, \delta) &= \min\{\mu_{\zeta_{T_1}}(\mathscr{E}), \mu_{\zeta_{T_2}}(\delta)\} \\
\mu_{\zeta_{I_1} \times \zeta_{I_2}}(\mathscr{E}, \delta) &= \min\{\mu_{\zeta_{I_1}}(\mathscr{E}), \mu_{\zeta_{I_2}}(\delta)\} \\
\mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}, \delta) &= \max\{\mu_{\zeta_{F_1}}(\mathscr{E}), \mu_{\zeta_{F_2}}(\delta)\}.
\end{aligned}$$

Theorem 5.2. *Let ζ_1 and ζ_2 be neutrosophic $\hat{Z}$-ideals in a $\hat{Z}$-algebra $\mathfrak{M}$. Then $\zeta_1 \times \zeta_2$ is a neutrosophic $\hat{Z}$-ideal in $\mathfrak{M} \times \mathfrak{M}$.*

Proof. Let ζ_1 and ζ_2 be two neutrosophic $\hat{Z}$-ideals in $\mathfrak{M}$. Let $(\mathscr{E}_1, \mathscr{E}_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\begin{aligned}
\mu_{\zeta_{T_1} \times \zeta_{T_2}}(0,0) &= \min\{\mu_{\zeta_{T_1}}(0), \mu_{\zeta_{T_2}}(0)\} \\
&\geq \min\{\mu_{\zeta_{T_1}}(\mathscr{E}_1), \mu_{\zeta_{T_2}}(\mathscr{E}_2)\} \\
&= \mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}_1, \mathscr{E}_2).
\end{aligned}$$

Let $(\mathscr{E}_1, \mathscr{E}_2), (\delta_1, \delta_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\begin{aligned}
\mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}_1, \mathscr{E}_2) &= \min\{\mu_{\zeta_{T_1}}(\mathscr{E}_1), \mu_{\zeta_{T_2}}(\mathscr{E}_2)\} \\
&\geq \min\{\min\{\mu_{\zeta_{T_1}}(\mathscr{E}_1 * \delta_1), \mu_{\zeta_{T_1}}(\delta_1)\}, \min\{\mu_{\zeta_{T_2}}(\mathscr{E}_2 * \delta_2), \mu_{\zeta_{T_2}}(\delta_2)\}\} \\
&= \min\{\min\{\mu_{\zeta_{T_1}}(\mathscr{E}_1 * \delta_1), \mu_{\zeta_{T_2}}(\mathscr{E}_2 * \delta_2)\}, \min\{\mu_{\zeta_{T_1}}(\delta_1), \mu_{\zeta_{T_2}}(\delta_2)\}\} \\
&= \min\{\mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}_1 * \delta_1), (\mathscr{E}_2 * \delta_2), \mu_{\zeta_{T_1} \times \zeta_{T_2}}(\delta_1, \delta_2)\} \\
&= \min\{\mu_{\zeta_{T_1} \times \zeta_{T_2}}(\mathscr{E}_1, \mathscr{E}_2) * (\delta_1, \delta_2), \mu_{\zeta_{T_1} \times \zeta_{T_2}}(\delta_1, \delta_2)\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu_{\zeta_{I_1} \times \zeta_{I_2}}(0,0) &\geq \mu_{\zeta_{I_1} \times \zeta_{I_2}}(\mathscr{E}_1, \mathscr{E}_2), \\
\mu_{\zeta_{I_1} \times \zeta_{I_2}}(\mathscr{E}_1, \mathscr{E}_2) &\geq \min\left\{\mu_{\zeta_{I_1} \times \zeta_{I_2}}((\mathscr{E}_1, \mathscr{E}_2) * (\delta_1, \delta_2)), \mu_{\zeta_{I_1} \times \zeta_{I_2}}(\delta_1, \delta_2)\right\}.
\end{aligned}$$

Let $(\mathscr{E}_1, \mathscr{E}_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\begin{aligned}\mu_{\zeta_{F_1} \times \zeta_{F_2}}(0,0) &= \max\{\mu_{\zeta_{F_1}}(0), \mu_{\zeta_{F_2}}(0)\} \\ &\leq \max\{\mu_{\zeta_{F_1}}(\mathscr{E}_1), \mu_{\zeta_{F_2}}(\mathscr{E}_2)\} \\ &= \mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}_1, \mathscr{E}_2).\end{aligned}$$

Let $(\mathscr{E}_1, \mathscr{E}_2), (\delta_1, \delta_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\begin{aligned}\mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}_1, \mathscr{E}_2) &= \max\{\mu_{\zeta_{F_1}}(\mathscr{E}_1), \mu_{\zeta_{F_2}}(\mathscr{E}_2)\} \\ &\leq \max\{\max\{\mu_{\zeta_{F_1}}(\mathscr{E}_1 * \delta_1), \mu_{\zeta_{F_1}}(\delta_1)\}, \max\{\mu_{\zeta_{F_2}}(\mathscr{E}_2 * \delta_2), \mu_{\zeta_{F_2}}(\delta_2)\}\} \\ &= \max\{\max\{\mu_{\zeta_{F_1}}(\mathscr{E}_1 * \delta_1), \mu_{\zeta_{F_2}}(\mathscr{E}_2 * \delta_2)\}, \max\{\mu_{\zeta_{F_1}}(\delta_1), \mu_{\zeta_{F_2}}(\delta_2)\}\} \\ &= \max\{\mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}_1 * \delta_1, \mathscr{E}_2 * \delta_2), \mu_{\zeta_{F_1} \times \zeta_{F_2}}(\delta_1, \delta_2)\} \\ &= \max\{\mu_{\zeta_{F_1} \times \zeta_{F_2}}(\mathscr{E}_1, \mathscr{E}_2) * (\delta_1, \delta_2), \mu_{\zeta_{F_1} \times \zeta_{F_2}}(\delta_1, \delta_2)\}.\end{aligned}$$

Hence, $\zeta_1 \times \zeta_2$ is a neutrosophic $\hat{Z}$-ideal in $\mathfrak{M} \times \mathfrak{M}$. □

6. Conclusion and Future Directions

This study presents a rigorous generalization of classical $\hat{Z}$-ideals by introducing the concept of neutrosophic $\hat{Z}$-ideals within the algebraic framework of $\hat{Z}$-algebras. By incorporating the independent degrees of truth, indeterminacy, and falsity, the proposed model offers a more expressive and adaptable approach to uncertainty than traditional fuzzy or intuitionistic systems. Key properties such as closure under intersection and preservation under homomorphisms were established, demonstrating both theoretical soundness and operational versatility.

Building upon these foundations, several promising directions for future research emerge. A natural progression involves the classification of neutrosophic $\hat{Z}$-ideals into prime, maximal, and semiprime classes, enhancing the structural granularity of the theory. Extending this framework to broader algebraic systems—such as semigroups, modules, or near-rings—may reveal new algebraic behaviors under neutrosophic uncertainty. Furthermore, the integration of topological or lattice-theoretic perspectives could deepen our understanding of continuity, convergence, and ideal hierarchies in this context.

From an applied standpoint, developing computational methods to detect and manipulate neutrosophic $\hat{Z}$-ideals could accelerate their use in decision-making, data science, and AI systems where ambiguity is inherent. Comparative studies with fuzzy and intuitionistic frameworks may further highlight the strengths and limitations of the neutrosophic approach. Ultimately, this work lays a strong foundation for both theoretical advancement and real-world impact in the algebraic modeling of uncertainty.

Acknowledgements. This work was supported by the revenue budget in 2025, School of Science, University of Phayao (Grant No. PBTSC68018).

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Bhattacharya, N. Mukherjee, Fuzzy Relations and Fuzzy Groups, Inf. Sci. 36 (1985), 267–282. https://doi.org/10.1016/0020-0255(85)90057-x.

[2] A.B. Saeid, M. Prakasam, P. Hemavathi, Bi-Normed Intuitionistic Fuzzy β-Ideals of β-Algebras, J. Uncertain Syst. 13 (2019), 42–55.

[3] M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha, S. Sabarinathan, A Note on Z-Algebra, Ital. J. Pure Appl. Math. 38 (2017), 707–714.

[4] P. Das, Fuzzy Groups and Level Subgroups, J. Math. Anal. Appl. 84 (1981), 264–269. https://doi.org/10.1016/0022-247x(81)90164-5.

[5] Y. Imai, K. Iséki, On Axiom Systems of Propositional Calculi, Xiv, Proc. Jpn. Acad. Ser. Math. Sci. 42 (1966), 19–22. https://doi.org/10.3792/pja/1195522169.

[6] K. Iséki, On BCI-Algebras, Mathematics Seminar Notes, Kobe University, vol. 8, pp. 125–130, 1980.

[7] K. Iséki, S. Tanaka, An Introduction to the Theory of BCK-Algebras, Math. Jpn. 23 (1978), 1–26.

[8] Y.B. Jun, S.M. Hong, S.J. Kim, S.Z. Song, Fuzzy Ideals and Fuzzy Subalgebras of BCK-Algebras, J. Fuzzy Math. 7 (1999), 411–418. https://cir.nii.ac.jp/crid/1573387450300872448.

[9] A. Rosenfeld, Fuzzy Groups, J. Math. Anal. Appl. 35 (1971), 512–517. https://doi.org/10.1016/0022-247x(71)90199-5.

[10] K.P. Shanmugapriya, P. Hemavathi, A Novel Concept of Neutrosophic Fuzzy Sets in Z-Algebra, in: International Conference on Recent Developments in Mathematics, Dubai, 2022. https://doi.org/10.1007/978-3-031-37538-5_5.

[11] S. Sowmiya, P. Jeyalakshmi, Fuzzy Algebraic Structure in Z-Algebras, World J. Eng. Res. Technol. 5 (2019), 74–88.

[12] S. Sowmiya, P. Jeyalakshmi, On Fuzzy H-Ideals in Z-Algebras, Adv. Fuzzy Sets Syst. 27 (2022), 243–251. https://doi.org/10.17654/0973421x22011.

[13] S. Sowmiya, P. Jeyalakshmi, On Fuzzy Z-Ideals in Z-Algebras, Glob. J. Pure Appl. Math. 15 (2019), 505–516. https://doi.org/10.37622/gjpam/15.4.2019.505-516.

[14] O.G. Xi, Fuzzy BCK-Algebras, Math. Jpn. 36 (1991), 935–942. https://cir.nii.ac.jp/crid/1571980074821181440.

[15] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.