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INVERSE RINGS DOMINATION

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Abstract. Let $X = V(G) \setminus S$, where S is a γ_{ri}-set of G. A subset $Y \subseteq X$ is said to be an **inverse rings dominating set** of G with respect to S if Y is a rings dominating set of G. The minimum cardinality of an inverse rings dominating set of G with respect to a γ_{ri}-set S of G is said to be an **inverse rings domination number** of G with respect to a γ_{ri}-set S of G, and we denote such as $\gamma_{ri}^{-1}(G)\big|_S$. An inverse rings dominating set of G with respect to S whose cardinality is equal to $\gamma_{ri}^{-1}(G)\big|_S$ is said to be a $\gamma_{ri}^{-1}\big|_S$-set of G. In this study, we show that the graph may have a rings dominating set and no inverse rings dominating set. Moreover, we study the inverse rings dominating set in the join and corona of graphs.

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1. Introduction

The growing research of domination in graphs is significantly impressive for many researchers have been introducing different varieties every year. From 1958 when Claude Berge the coefficient of external stability which is now known today as domination [3], and from 1962 when Oystein Ore introduced the terms “dominating set" and “domination number", up to when Abed and Al-Harere introduced the rings domination in 2022, the highlights of newfound variants of domination have always been the research interest of many researchers. This is unsurprisingly celebrated due to its impact and applications to many allied areas such as when L. Kelleher and M. Cozzens presents study of social networking through aids of the application of dominating sets [16], Yu, et.al. provides a comprehensive survey review on a type of domination in graphs, called connected domination, to wireless ad hoc and sensor networks [17], and Caay and Maza provides the representation of domination into decomposition of graphs into a unique form [7].

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The rings dominating set, as one of the newest variants introduced by Abed and Al-Harere [1], is one of the interestings areas, and so despite of being new, many researchers have explored on the extensions of this notion. In 2024, Caay introduced the notion of equitable rings domination where a rings dominating set is also equitable dominating set [4]. The same year, Caay and Dondoyano explored the extension of domination into a perfect rings domination in graphs where a rings dominating set if also a perfect dominating set [5], and in the following year, the same set of authors provided results on isolate rings domination in graphs [6]. Moreover, Necesito and Caay studied the rings convex domination in graphs where a rings dominating set is also a convex set [14].

An interesting variant in the timeline of domination is when getting its inverse. Kulli and Sigarkanti introduced the notion of inverse domination in graph in 1991 [18]. The authors defined that a an inverse dominating set with respect to a given dominating set is a subset of the complement of a given dominating that also forms a dominating set. Mathematically speaking, given a minimal dominating set D of G, an inverse dominating set E with respect to D is $E \subseteq V(G) \setminus D$ such that E also forms a dominating set.

There have been many explorations done in the inverse domination. Some of these are in 2020 when Al-Emrany, et.al. studied the inverse domination in Bipolar Fuzzy graphs [21], and in 2024 when Shalini and Rajasingh determined the inverse domination number in X-trees and and Sibling Trees in relation to independent domination numbers and connected domination numbers [20]. Moreover, there are a lot of variants that rose up from this inverse domination study, and one of these is in 2016 when Salve and Enriquez introduced the inverse perfect domination in graphs [19].

One of the nicest applications of the rings domination in graphs is when we consider the installment of surveillance camera. Suppose we want to install camera but due to the restrained budget, we only install cameras to some selected locations and the uncovered locations are adjacent to atleast other uncovered locations. The purpose of studying the inverse rings domination is that with the goal of achieving zero crime rate, some of the uncovered places are installed with police stations such that those places with no stations are adjacent to atleast two other places with no stations. This is to ensure that the unsecured places can be looked out by the station or can be monitored by camera system. The mathematical notion of this goal is to study the rings inverse domination in graphs which is motivated by the study of rings domination and the inverse domination. To give clarity of this study, we show the preliminary notions of the study and the construction of inverse rings domination in Section 2. In Section 3, we present the general structure of what is inverse rings domination, and we extend the results of inverse rings domination of the join and corona of graphs in Section 4.

2. Preliminaries and the Construction

Unless stated, the graph we consider is a simple connected graph. For some terminologies and theoretic discussion of graphs, the reader may refer to [12] and [13]

Definition 2.1. [1] A dominating set $S \subset V(G)$ is said to be a **rings dominating set** of G if for every $x \in V(G) \setminus S$, there exist $y, z \in V(G) \setminus S$ with $y \neq z$ such that $xy, xz \in E(G)$. The minimum cardinality of a rings dominating set of G is called the **rings domination number** of G, and is denoted by $\gamma_{ri}(G)$. A rings dominating set S of G such that $|S| = \gamma_{ri}(G)$ is called a γ_{ri}-set of G.

Proposition 2.2. *Let $n \geq 4$. Then $S = \{u\}$ is a γ_{ri}-set of K_n, for some $u \in V(K_n)$. In general, any subset S of $V(K_n)$ such that $|S| \leq n-3$ forms a rings dominating set of K_n.*

Corollary 2.3. *Let G b any graph such that $\delta(G) \geq 2$. Then $\gamma_{ri}(G) = 1$ if and only if there exists a universal vertex of G.*

Theorem 2.4. *Let G_1 and G_2 be any graphs such that $\delta(G_2) \geq 2$. Then $\gamma_{ri}(G_1 \circ G_2) = |V(G_1)|$.*

Theorem 2.5. *Let C and D be some disjoint subgraphs of G with $C \cong C_k \cong D$, for some k, and assume that for every $u \in C$, there exists $v \in D$ such that $uv \in E(G)$ (and vice versa). Then C and D are both rings dominating sets of G if and only if either of the following holds:*

(i) $C \cup D = V(G)$

(ii) $C \cup D \neq V(G)$, *and for every $u \notin C$ and $u \notin D$, u is adjacent to atleast two vertices in C or adjacent to atleast two vertices in D.*

Proof. The forward is obvious. Conversely, if $C \cup D = V(G)$, then the proof is done. Supposed $C \cup D \neq V(G)$. Then $C \cup D \subset V(G)$. Thus, there exist $u \in V(G)$ such that $u \notin C$ and $u \notin D$. Since C and D are rings dominating sets, C and D rings dominate u. If C rings dominates u, then u must be adjacent to at least two vertices in D or must be a vertex of some arbitrary induced subgraph of C_i different from C and D. Similarly, if D rings dominates u, then u must be adjacent to at least two vertices in C or must be a vertex of some arbitrary induced subgraph of C_j different from C and D. This proves the claim. □

Note that in Theorem 2.5, C is a rings dominating set in the complement of D, and D is also a rings dominating set in the complement of C. With this, we now have our formal definition of the study.

Definition 2.6. Let $X = V(G) \setminus S$, where S is a γ_{ri}-set of G. A subset $Y \subseteq X$ is said to be an **inverse rings dominating set** of G with respect to S if Y is a rings dominating set of G. The minimum cardinality of an inverse rings dominating set of G with respect to a γ_{ri}-set S of G is said to be an **inverse rings domination number** of G with respect to a γ_{ri}-set S of G, and we denote such as $\gamma_{ri}^{-1}(G)\big|_S$. An inverse rings dominating set of G with respect to S whose cardinality is equal to $\gamma_{ri}^{-1}(G)\big|_S$ is said to be a $\gamma_{ri}^{-1}\big|_S$-set of G.

Example 2.7. Consider the graph in Figure 1. Note that $S = \{u_3, u_4\}$ forms a rings dominating set and it can be verified that S is a $\gamma_{ri}-$-set of G_1. Thus, we have

$$V(G_1) \setminus S = \{u_1, u_2, u_5, u_6, u_7, u_8\}.$$

Now $T_1 = \{u_1, u_5, u_6\} \subset V(G_1) \setminus S$ also forms a rings dominating set of G_1. Thus, T_1 is an inverse rings dominating set of G_1 with respect to S. Also, $T_2 = \{u_2, u_7\} \subset V(G_1) \setminus S$ forms an inverse rings dominating set of G_1 with respect to S. It can also verified that T_2 is a $\gamma_{ri}^{-1}\big|_S$-set. Hence, $\gamma_{ri}^{-1}(G_1)\big|_S = 2$.

Example 2.8. Consider the graph in Figure 2. The set $S = \{v_4\}$ is a γ_{ri}-set of G_2, and so $V(G_2) \setminus S = \{v_1, v_2, v_3, v_5, v_6\}$. Now $T = \{v_1, v_3\}$ also forms a rings dominating set of G_2. This means that T is an inverse rings dominating set of G_2 with repspect to S. It can also be verified that T is a $\gamma_{ri}^{-1}\big|_S$-set of G_2. Hence, $\gamma_{ri}^{-1}(G_2)\big|_S = 2$.

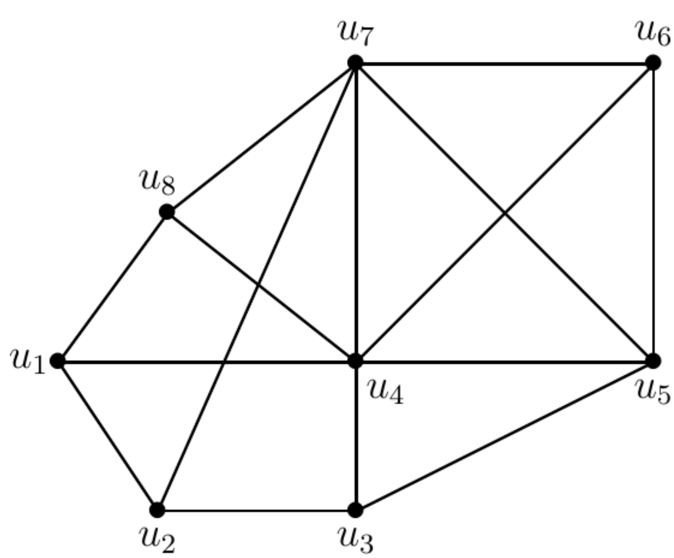

Figure 1. Graph G_1

Remark 2.9. If no confusion arises, we use the notation $\gamma_{ri}^{-1}(G)$ instead of $\gamma_{ri}^{-1}(G)\big|_S$ for the inverse rings domination number of G, and we denote γ_{ri}^{-1}-set instead of $\gamma_{ri}^{-1}\big|_S$-set.

Remark 2.10. It is very obvious to assume that if G does not have a rings dominating set, then G does not have an inverse rings dominating set. In general, the existence of an inverse rings dominating set depends on the presence of rings dominating set. But one has to take note that a graph may have a rings dominating set but no inverse rings dominating set.

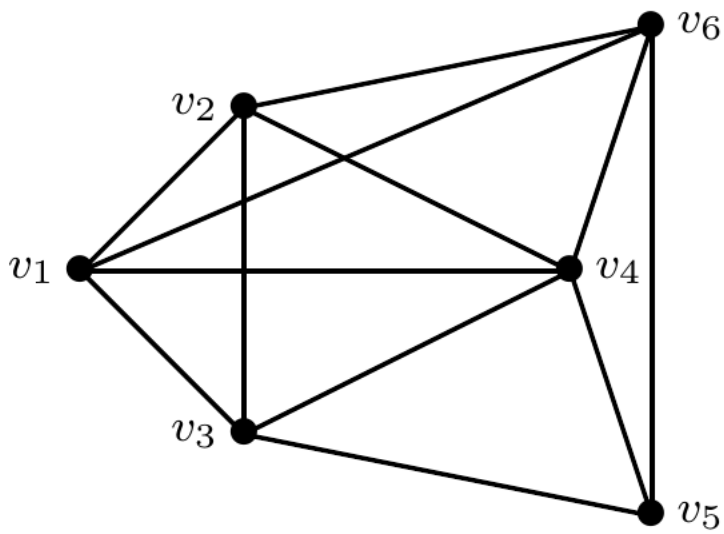

FIGURE 2. Graph G_2

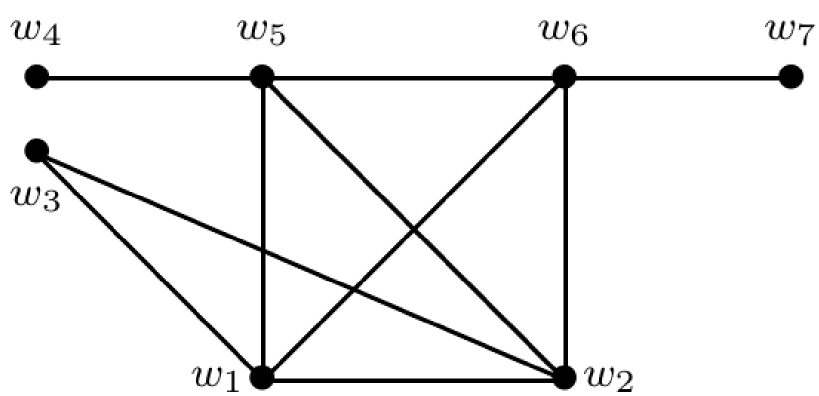

FIGURE 3. Graph G_3

Example 2.11. Consider the graph in Figure 3. Then we can verify that $S = \{w_3, w_4, w_7\}$ is a γ_{ri}-set of G_3. Thus, $V(G_3) \setminus S = \{w_1, w_2, w_5, w_6\}$ and we cannot create a rings dominating set of G_3 from this subset $V(G_3) \setminus S$.

3. The Inverse Rings Domination

Theorem 3.1. *An inverse rings dominating set T of G must not contain a pendant.*

Proof. Let T be an inverse rings dominating set of G such that $T \subseteq V(G) \setminus S$ for some γ_{ri}-set S of G. Suppose T contains a pendant. Let $x \in T$ such that $\deg_T(x) = 1$. This means that $x \in V(G) \setminus S$. This is a contradiction to the definition of a rings dominating set S of G that every element $x \in V(G) \setminus S$ must be adjacent to at least two vertices in $x \in V(G) \setminus S$. Therefore, T must not contain a pendant. $\square$

Corollary 3.2. *Let S be any γ_{ri}-set of G. Then there does not exist an inverse rings dominating set of G if and only if S contains u such that $\deg(u) = 1$.*

Theorem 3.3. *Let G be any graph such that there exists an induced subgraph $H \equiv K_p$ and $H \subseteq V(G) \setminus S$ for some γ_{ri}-set of G, with $|S| \geq 2$. Then $T \subset V(H)$ is a $\gamma_{ri}^{-1}|_S$-set of G with $|T| \leq |V(H)| - 2$.*

Proof. Let S be a γ_{ri}-set of G. Then $|S| = |V(G)| - 3$. Thus, there are at least 3 vertices in $V(G) \setminus S$. Since there exists an induced subgraph $H \equiv K_p$ in $V(G) \setminus S$, it follows that $V(G) \setminus S$ must contain a subset T such that $\delta(T) \geq p - 1$, for some positive integer p. Thus, there exists $w \in H \setminus T$. Note that $u \in T$ can dominate w. Since w is adjacent to at least one element is S so that T can be γ_{ri}-set of G, there exists another $\overline{w} \in H \setminus T$ such that $\overline{w}w \in E(G)$. This means that at least $|H| - 2$ vertices can form a rings dominating set implying that $|T| \leq |H| - 2$. This proves the claim. □

Proposition 3.4. *Let $n \geq 4$. If S is a γ_{ri}-set of K_n. Then there are $n - 1$ γ_{ri}^{-1}-sets S_i of K_n with respect to S. In general, if S is a γ_{ri}-set of K_n , there are $\sum_{i=1}^{n-4} \binom{n-1}{i}$ number of inverse rings dominating sets of K_n with respect to S.*

Proof. By Proposition 2.2, if S is a γ_{ri}-set of K_n, $n \geq 4$, then $S = \{u\}$, for some $u \in V(K_n)$. Thus, every singleton set $\{v\}$ of subsets of $V(K_n)$, $v \in V(K_n)$ different from u, forms a rings dominating set for K_n. Thus, for $S = \{u\}$ a γ_{ri}-set of K_n, $u \in V(K_n)$, there are $|V(G) \setminus S| = n - 1$ choices of forming singleton subsets of $V(K_n)$ to form an inverse rings dominating set of K_n with respect to S. Furthermore, by Proposition 2.2, we can shoose any subset T of $V(G) \setminus S$ such that $|T| \leq |V(G) \setminus S| - 3 = n - 4$. Hence, by combinatorial counting, there must be $\sum_{i=1}^{n-4} \binom{n-1}{i}$ number of inverse rings dominating sets of K_n with respect to S. □

4. On Some Binary Operations

4.1. **Join of Graphs.**

Theorem 4.1. *Let G and H be any simple connected graphs of order n and m, respectively, $n \geq 3$, $m \geq 3$. Then $\gamma_{ri}(G + H) = \gamma_{ri}^{-1}(G + H) \leq 2$.*

Proof. Since $|V(G)| = n \geq 3$ and $|V(H)| = m \geq 3$, and G and H are connected graphs, for every $u \in V(G)$, $\deg_{G+H}(u) \geq 4$ and $\deg_{G+H}(v) \geq 4$. Since $uv_i \in V(G + H)$, for all $u \in V(G)$, and $v_i \in V(H)$, $i = 1, \cdots, m$, u dominates all v_i. Since $\deg_{G+H}(v_i) \geq 4$, u rings dominates all $v_i \in V(H)$, $i = 1, \cdots, m$. Similarly, since $u_jv \in V(G + H)$, for all $u_j \in V(G)$, $j = 1, \cdots, n$, and $v \in V(H)$, v dominates all u_j. Since $\deg_{G+H}(u_j) \geq 4$, v rings dominates all $u_j \in V(H)$, $j = 1, \cdots, n$. Thus, $S = \{u, v\}$ forms a rings dominating set of $G + H$. In this case, S is a γ_{ri}-set.

Now let $a \in V(G)$ and $b \in V(H)$ such that $a \neq u$ and $b \neq v$. Using the same argument, it follows that $T = \{a, b\}$ can form a rings dominating set. Hence, T is an inverse rings dominating set with respect to S. Therefore, $\gamma_{ri}^{-1}(G + H) = 2$. □

Corollary 4.2. *If S is a γ_{ri}^{-1}-set of $G + H$, then $S := \{u, v\}$, for some $u \in V(G)$ and $v \in V(H)$.*

4.2. **Corona of Graphs.** In this paper, we mean $V(G_2)^{(i)}$ to be the ith copy of $V(G_2)$ in the coronal product $G_1 \circ G_2$.

Theorem 4.3. *Let G_1 and G_2 be any simple connected graphs such that $\delta(G_2) \geq 2$. Then*

(1) $\gamma_{ri}(G \circ H) \leq \gamma_{ri}^{-1}(G \circ H)$.

(2) $S = \bigcup_{i=1}^{|V(G_1)|} S^{(i)}$, *where $S \subseteq V(G_2)$, and the superscript (i) indicates the subset S in the ith copy of $V(G_2)$, is an inverse rings dominating set of $G \circ H$.*

Proof. (1) Observe that $V(G_1)$ dominates $V(G_1 \circ G_2) \setminus V(G_1)$. Let T be another dominating set of $G_1 \circ G_2$ such that $|T| \leq |V(G_1)|$. Then we have the following cases:

Case 1: Suppose $T \subset V(G_1)$. Then there exists $u_i \in V(G_1)$ such that $u_i \notin T$. Thus, there does not exist a vertex in T that dominates $V(G_2)^{(i)}$. This is a contradiction to the definition of the domination in graphs.

Case 2: Suppose $V(G_1) \cap T = \varnothing$. This means that every vertex $u \in T$ belongs to $V(G_2)$. Thus, there should be atleast $|V(G_1)|$ copies of u from T to dominate $V(G_1)$. This implies that $|V(G_1)| \leq |T|$ which is a contradiction.

Case 3: Suppose $T \not\subset V(G_1)$ and $V(G_1) \cap T \neq \varnothing$. Then there exists $u \in V(G_1) \setminus T$. Since $|T| \leq |V(G_1)|$, there exists $v \in V(G_1) \setminus T$. This means that either u dominates v or u does not dominate v. If u dominates v, then all vertices in the jth copy of $V(G_2)$ that are adjacent to v are not dominated by u. Similarly, if u does not dominate v, then also all vertices in the jth copy of $V(G_2)$ that are adjacent to v are not dominated by u. Thus, the assumption shows a contradiction.

Hence, either of those case, we get the contradiction implying that there does not exists T such that $|T| \leq |V(G_1)|$. Hence $V(G_1)$ is a γ-set of $G_1 \circ G_2$. Since $\delta(G_2) \geq 2$, it follows that for all $w \in V(G_1 \circ G_2) \setminus V(G_1)$ has degree atleast 2 in $V(G_1 \circ G_2) \setminus V(G_1)$. Hence, $V(G_1)$ is a γ_{ri}-set of $G_1 \circ G_2$.

Now we have $V(G_1 \circ G_2) \setminus V(G_1) = \bigcup_{i=1}^{|V(G_1)|} V(G_2)^{(i)}$. Thus, there must be at least one dominating vertex in every $|V(G_1)|$ copies of G_2. By Theorem 2.4, we have

$$\gamma_{ri}^{-1}(G_1 \circ G_2) \geq |V(G_1)| = \gamma_{ri}(G_1 \circ G_2).$$

This proves (1).

(2) Following the argument of (1), it follows that every copy of G_2 contains a subset Q of a γ_{ri}^{-1}-set of $G_1 \circ G_2$. Since every Q are not incident, it follows that the γ_{ri}^{-1}-set of $G_1 \circ G_2$ is a union of disjoint subsets of $V(G_2)$. This proves the claim.

□

Corollary 4.4. *Let G_1 and G_2 be any simple connected graphs such that $\delta(G_2) \geq 2$. Then $\gamma_{ri}(G \circ H) = \gamma_{ri}^{-1}(G \circ H)$ if and only if there exists a universal vertex $u \in V(G_2)$.*

Proof. Suppose $\gamma_{ri}(G \circ H) = \gamma_{ri}^{-1}(G \circ H)$. By Theorem 4.3, every copy of G_2 must have a dominating vertex. Since $\gamma_{ri}(G \circ H) = \gamma_{ri}^{-1}(G \circ H)$, it follows that every copy of G_2 must only have one dominating vertex. By Corollary 2.3, there exists a universal vertex in G_2. The converse is obvious and follows directly from Corollary 2.3. □

Corollary 4.5. *Let G_1 and G_2 be any simple connected graphs such that $\delta(G_2) \geq 2$. Then $S = \bigcup_{i=1}^{|V(G_1)|} \{u\}^{(i)}$, where $u \in V(G_2)$, and the superscript (i) indicates the element u in the ith copy of $V(G_2)$, if and only if u is a universal vertex in $V(G_2)$.*

Proof. The proof is very obvious and follows directly from Corllary 4.5. □

5. Conclusion

This study arises from questioning the possibility of the existence of a rings dominating set that is distinct from another rings dominating set. We conclude that the inverse rings dominating set depends on the existence of a γ_{ri}-set; however, there exist graphs that do not have an inverse rings dominating set. Furthermore, we investigate the presence of inverse rings dominating sets in the join and corona of graphs in general. To further extend this study, we recommend exploring the existence of inverse rings dominating sets in certain binary operations not discussed here. However, we anticipate that the exploration may be somewhat similar to the discussion in Section 3, as that section addresses general types of graphs.

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