

ADVANCEMENTS IN NONLINEAR EQUATION SOLVERS: ZHANG HOMOTOPY CONTINUATION METHOD AND ITS BASINS OF ATTRACTION

MOHD SYAFIQ ABDUL RAHMAN^{1,2,*}, HAFIZUDIN MOHAMAD NOR¹, MUHAMMAD ZAINI AHMAD¹, NOR HANIM ABD RAHMAN²

¹Department of Mathematical Sciences, Faculty of Intelligent Computing, Universiti Malaysia Perlis, Malaysia

²Department of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Pulau Pinang, Malaysia

*Corresponding author: mohdsyafiq5400@uitm.edu.my

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ABSTRACT. Iterative methods play a crucial role in solving nonlinear equations that are too complex for analytical solutions. However, one primary challenge associated with iterative methods is their potential to converge to a local solution or fail to converge at all. This limitation can significantly restrict their applicability and effectiveness in solving complex mathematical problems. To tackle the issue, a global convergence method known as the homotopy continuation method (HCM) was developed. This approach involves gradually deforming the simpler function into the target function. It was demonstrated that using HCM with a classical iterative method for solving roots of nonlinear equations can improve performance. In this study, we developed Zhang-HCM to solve several nonlinear equations. Its performance was demonstrated by presenting numerical examples compared to the original Zhang method. The results showed that the Zhang-HCM algorithm achieved convergence in all numerical situations. Also, the basins of attraction of Zhang's and Zhang-HCM were generated to discuss the stability of the methods.

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Key words and phrases. nonlinear equation; homotopy continuation method; Zhang method; fifth order method; basins of attraction.

1. INTRODUCTION

Nonlinear equations arise in many scientific and engineering problems like computer science, chemistry, engineering, and physics [1,2]. Consider the following nonlinear equation:

$$f(x) = 0. \quad (1)$$

Finding the exact solution of the equation is often challenging or impossible due to the complexity and non-linearity of the functions involved. Therefore, numerical methods are usually employed to obtain approximate solutions with a desired accuracy [3]. However, not all numerical methods are equally

effective or efficient in solving nonlinear equations. Some methods may suffer from slow convergence, divergence, or instability, depending on the initial guess, function properties, and problem size. Hence, developing and analysing new numerical methods that can overcome these difficulties and provide reliable and fast solutions is important.

One of the most popular and powerful numerical methods for solving nonlinear equations is Newton's method, which is based on the linearization of the function at each iteration. Newton's method has a quadratic convergence rate, which means that it can achieve high accuracy with few iterations [4,5]. However, Newton's method has some drawbacks, such as the requirement of the first derivative of the function, the possibility of divergence or oscillation, the sensitivity to the initial guess, and is locally convergent [2,6,7]. It is well known that most local convergence methods will diverge when the denominator is or close to zero. For example, Newton's method diverges if $f'(x_0) = 0$. Multiple enhancements and expansions have been proposed to enhance the effectiveness of Newton's method such as higher-order methods [8–10], secant methods, and homotopy methods [11].

Higher-order methods are numerical methods with a convergence rate higher than quadratic, which means they can achieve higher accuracy with fewer iterations. For example, the fifth-order method is a high-order method that can provide even greater precision than lower-order methods, such as second or third-order methods. Several well-known fifth-order methods are included by [12], [13], [8] and the recent method of [14]. The fifth-order method allows a faster and more accurate solution to complex numerical problems, making it a valuable tool in computational mathematics and scientific computing. However, higher-order methods usually require higher function derivatives, which may be difficult or costly to compute [15]. Moreover, higher-order methods may not guarantee a faster convergence in practice due to the increased computational cost per iteration and the possible loss of precision due to rounding errors [16]. This study utilised the method by [12]. However, even though the method was the fifth-order method, a poorly chosen initial guess could lead to divergence. In such instances, the homotopy continuation method (HCM) provided a solution to address the divergence issues.

HCMs are numerical methods that transform a difficult or complicated problem into a simpler one, that later gradually deforms the simpler problem into the original problem [17] while tracking the solution path [2]. The HCM has the capability to prevent traditional methods from diverging or stagnating, and it can determine all the solutions of the nonlinear equation, even complex ones, with a high level of precision [17,18]. The equation of HCM consists of the homotopy function, the original function (also known as the target function), and the auxiliary function (also known as the start function). According to Wayburn and Seader [19], the most widely used functions are the Newton and fixed-point functions. Additionally, they introduced a new function called the affine function. Furthermore, [20] combined the Newton and fixed-point functions in the study. Inspired by the idea of the De Casteljau algorithm, [21] presented a new homotopy function called the quadratic Bezier

homotopy function. [22] extended the study and proposed a higher degree of homotopy function called the cubic Bezier homotopy function. [23] then simplified the quadratic Bezier homotopy function called the quadratic parameter homotopy function. The rules for choosing the appropriate auxiliary homotopy function or the start system was discussed by Wu [24]. In the study, he modified the classical ancient Chinese algorithm by the HCM. Later, Wu also presented two more HCMs, which were the Adomian-HCM and the secant-HCM [25,26]. Using similar techniques as Wu, [27] presented the Ostrowski-HCM to solve polynomial equations. Recently, [18] proposed a Halley-HCM. Thus, regarding the works of literature discussed, it is perceptible that Newton's second-order, Halley's third-order and Ostrowski's fourth-order methods were all combined with homotopy.

Motivated by this, we provided a novel numerical method for solving nonlinear equations. Our approach combined a fifth-order method, previously introduced by Zhang et al. [12], with an HCM. While the existing approaches, such as the Ostrowski-HCM [27], are powerful, there is still a potential for enhancing their efficiency, robustness, and adaptation to other problem domains. Consequently, we persist in investigating novel variations and improvements to the HCM in the numerical analysis. Zhang's method is a three-step methods that offer a low computing cost and a high convergence rate. The HCM in this study used a fixed-point homotopy approach, utilising a straightforward and controllable auxiliary function to modify the original equation.

This study is structured accordingly, whereby the following section provides a brief explanation of the homotopy continuation approach. The subsequent section demonstrates the transformation of the Zhang homotopy continuation method (Zhang-HCM). In the subsequent section, we discuss and analyse the stability using the basins of attraction, where we compare the proposed method with its original method. In addition, we included numerical examples to demonstrate the efficiency of the proposed method in solving various types of nonlinear equations. Lastly, the conclusion remarks are included in the final section.

2. HOMOTOPY CONTINUATION METHOD

The HCM is a numerical technique for solving nonlinear equations. It starts with transforming the simpler equation to the harder-to-solve equation. A parameter t controls the transformation, and the solution is tracked along the way. We must define a homotopy function that satisfies some conditions to apply the HCM.

For instance, in case we need to find the solution of the nonlinear Equation (1). The homotopy function H can be defined as:

$$H(x, t) = (1 - t)g(x) + tf(x) = 0, \quad (2)$$

where $t \in [0, 1]$, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is an auxiliary function. From the Equation (2), two boundary conditions are obtained;

$$\begin{aligned} H(x, 0) &= g(x) = 0, \\ H(x, 1) &= f(x) = 0. \end{aligned} \quad (3)$$

The boundary conditions (3) show the deformation process from $t = 0$ to $t = 1$. The deformation converts the initial equation, $g(x) = 0$, into $f(x) = 0$. Typically, $g(x)$ is defined similarly to $f(x)$; hence, it is simpler to find the solution x_0 to be $g(x) = 0$.

To solve the original equation, $f(x)$, we begin by choosing an initial guess, represented as x_0 , and then apply an iterative method, such as Newton's method, to solve the homotopy equation at $t = 0$. As we adjust a parameter in the homotopy equation, we trace a path towards the target solution. The homotopy method is versatile and effective, working well with various nonlinear equations. However, its success hinges on choosing the right homotopy function and initial guess. There are several types $g(x)$ even though the choice of the function is arbitrary. We selected the fixed-point function as our auxiliary homotopy function for this study because it was easier to work with than other options. The expression for the fixed-point function is

$$g(x) = x - x_0 \quad (4)$$

where x_0 refers to the initial guess. Most importantly, the function $g(x)$ must be controllable and easily solved.

3. ZHANG HOMOTOPY CONTINUATION METHOD

Newton's method is well-known for solving nonlinear equations. It is widely acknowledged that Newton's method shows quadratic convergence. The numerical iterative formula for Newton's method is expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (5)$$

From the Equation (5), researchers expanded and modified it to achieve a higher convergence order, and more accurate results. Recently, several researchers presented their iterative methods as order five. One of the methods is by Zhang et al. [12]. The following equation shows the Zhang's method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f'(x_n)} \left[\frac{f(y_n)}{f(x_n) - 2f(y_n)} \right], \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}. \end{aligned} \quad (6)$$

The method in Equation (6) has a three-step function. Besides that, it has four functions to be evaluated. So, according to Kung & Traub [28], the efficiency index for this method is $5^{\frac{1}{4}} \approx 1.495348781$. However, the method (6) will face issues when the initial guess x_0 is selected at $f'(x_0) = 0$ or $f(x_0) - 2f(y_0) = 0$. When the initial guess is poorly chosen, it can lead to divergence. In this study, Zhang's method was transformed into the HCM for this purpose as follows:

$$\begin{aligned} y_n &= x_n - \frac{H(x, t)}{H'(x, t)}, \\ z_n &= y_n - \frac{H(x, t)}{H'(x, t)} \left[\frac{H(y, t)}{H(x, t) - 2H(y, t)} \right], \\ x_{n+1} &= z_n - \frac{H(z, t)}{H'(x, t)} \end{aligned} \quad (7)$$

where H is as in Equation (2). The new proposed HCM is named as Zhang homotopy continuation method (Zhang-HCM). The pseudocode for the algorithm is presented as in Algorithm 1.

Algorithm 1: Algorithm for Zhang-HCM

Input: $f : \mathbb{R} \rightarrow \mathbb{R}$ - function; x_0 - initial guess; $g : \mathbb{R} \rightarrow \mathbb{R}$ - auxiliary function; ε - stopping criterion.

Output: n - number of iterations needed; x - roots of f ; $|f(x)|$ - accuracy.

```

1 Set  $x = x_0; t = 0; n = 1$ 
2  $H(x, t) = (1 - t)g(x) + tf(x)$ 
3  $y_n = x_n - \frac{H(x_n, t)}{H'(x_n, t)}$ 
4  $z_n = y_n \frac{H(x_n, t)}{H'(x_n, t)} \left[ \frac{H(y_n, t)}{H(x_n, t) - 2H(y_n, t)} \right]$ 
5  $x_{n+1} = z_n - \frac{H(z_n, t)}{H'(x_n, t)}$ 
6 while  $|f(x)| \geq \varepsilon$  do
7    $n = n + 1$ 
8   for  $i = 1, 2, \dots, n$  do
9      $t = t + \frac{1}{n}$ 
10     $x = x_{n+1}$ 
11    for  $j=1$  do
12       $x = x_{n+1}$ 
13 return  $x$ 
```

4. BASINS OF ATTRACTION

The basins of attraction of an iterative method refer to the set of all initial points from which the iterative method converges to a solution of an equation [29]. Understanding basins of attraction helps

choose a good initial guess and assess the robustness of an iterative method. A method with well-behaved, large basins is generally considered to be more reliable. Apart from that, with the aid of basins of attraction, the stability of an iterative method can be analysed [30]. Usually, researchers who introduced their new iterative methods will show the basins' visual to show their iterative method's stability. However, in our literature study, we did not find a research study of HCM using basins of attraction. Until recently, [17] exhibited the basins of attraction on HCM of the fourth-order iterative method. From this discovery, we were motivated to illustrate the dynamical behaviour of Zhang's method and Zhang-HCM for solving $p(z) = 0$, where the function $p : \mathbb{C} \rightarrow \mathbb{C}$ is in a complex plane. Three complex polynomials of two to four degrees were used as examples to illustrate the basins of attraction for both methods. The settings to generate the visuals of the geometry in the complex plane with image resolution of 800×800 pixels are as follows: for p_1 the area is set in a square $[-10, 2] \times [-6, 6] \subset \mathbb{C}$, and for p_2 and p_3 the area set to $[-2, 2] \times [-2, 2] \subset \mathbb{C}$, in which the area contains all the polynomial roots of $p(z) = 0$. Different colours of each root were assigned to the basins, which started at z_0 with computations accuracy $\varepsilon = 10^{-7}$, and 30 maximum iterative. For the homotopy version, we used $g(z) = z - z_0$ as the auxiliary function, and three different values for the total number of steps, N as in Algorithm 1. The polynomials, including their roots and the colour of fractals, are shown in Table 1. Mathematica 13 was used to generate the basins of attraction. The programs were run three times for each image, and the average execution time in seconds was recorded as in Table 2. The time taken for each problem is almost similar. When we compare the original method and the number of steps for the homotopy version, the original method is the fastest in producing the image. In contrary, the homotopy version with the highest number of steps, $N = 100$, requires a longer time to generate the basins of attraction.

TABLE 1. Complex polynomials with their roots and colour of fractals generated for basins of attraction

Test problems	Roots	Basin colour
$p_1(z) = z^2 + 8z - 9$	-9	orange
	1	red
$p_2(z) = z^3 - 1$	1	orange
	$-0.5 - 0.866025i$	red
	$-0.5 + 0.866025i$	purple
$p_3(z) = z^4 - 0.75z^2 - 0.25$	-1	orange
	1	red
	$-0.5i$	purple
	$0.5i$	blue

TABLE 2. Execution time in seconds required by Zhang's method and Zhang-HCM for $N = 5, 20$ and 100 for each $p_1(z)$, $p_2(z)$ and $p_3(z)$

	N	$p_1(z)$	$p_2(z)$	$p_3(z)$
Zhang's method		6.782 s	7.701 s	7.770 s
Zhang-HCM	5	7.863 s	8.366 s	9.173 s
	20	13.494 s	11.093 s	12.378 s
	100	25.275 s	19.929 s	25.158 s

Figures 1, 2 and 3 are the graphics generated for $p_1(z)$, $p_2(z)$ and $p_3(z)$ respectively. From these graphics, the behaviour and stability of the iterative methods can be easily seen. Each colour of the basin represents the roots of the polynomials in the complex plane that can be referred to in Table 1. For instance, if the polynomial has two roots, then only two colours will appear in the graphics. The cleaner graphics indicate that the methods are stable.

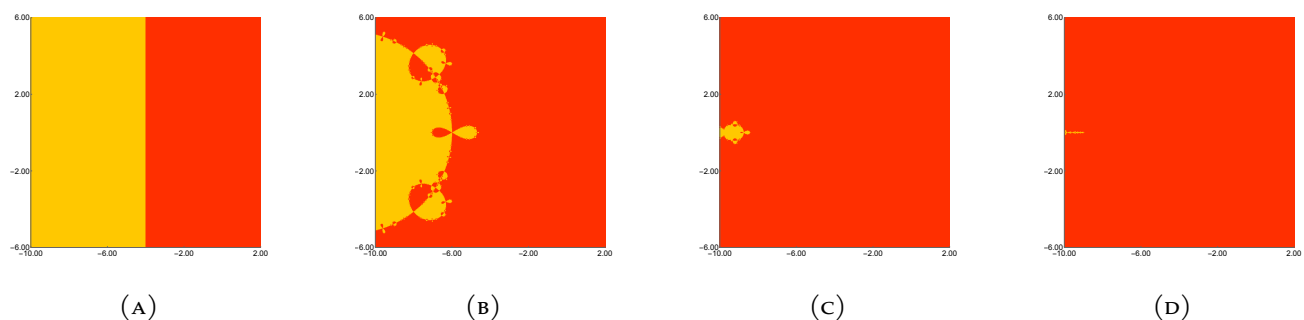


FIGURE 1. Basins of attraction for the complex polynomial $p_1(z)$ generated using (a) Zhang's method and Zhang-HCM version with various numbers of steps: (b) 5, (c) 20, (d) 100.

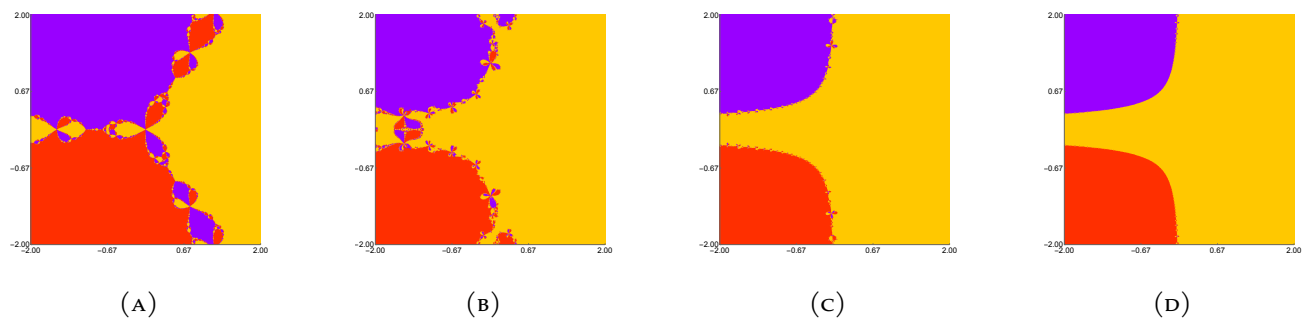


FIGURE 2. Basins of attraction for the complex polynomial $p_2(z)$ generated using (a) Zhang's method and Zhang-HCM version with various numbers of steps: (b) 5, (c) 20, (d) 100.

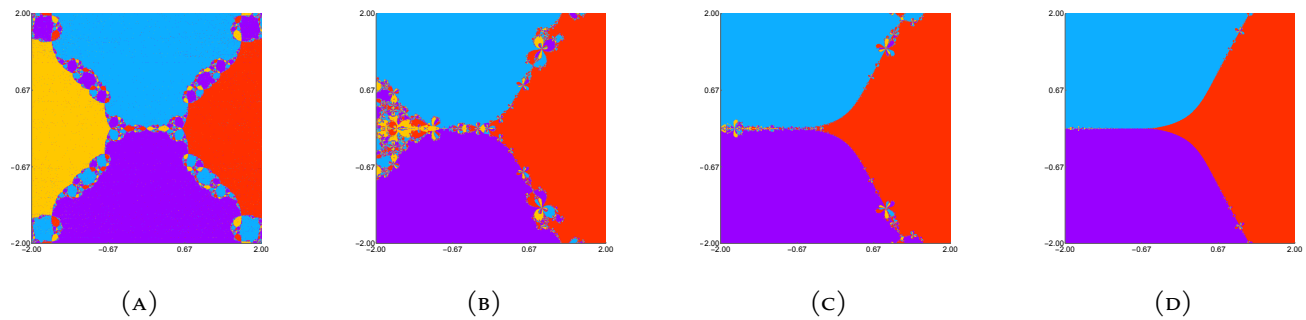


FIGURE 3. Basins of attraction for the complex polynomial $p_3(z)$ generated using (a) Zhang's method and Zhang-HCM version with various numbers of steps: (b) 5, (c) 20, (d) 100.

In Figure 1, only two basin colours appeared, where orange represents root -9 , and red represents root 1 . As seen, the basins of Zhang's method in Figure 1(a) are divided into two colours by a vertical line $z = -4$. This means that if we chose $z_0 < -4$, Zhang's method will converge to the orange root, which is root -9 . The opposite will happen for $z_0 > -4$. However, when the homotopy version is applied, the basin of root -9 (orange) starts shrinking, while the basin of root 1 (red) grows larger, as seen in Figure 1 (b), (c), and (d). The larger the number of steps, the method tends to converge to root 1 only while the root -9 almost disappears in the graphics.

In Figure 2, basins of attraction for polynomial $p_2(z)$ are shown. Figure 2(a) is the graphic generated by Zhang's method showing a complex pattern with intricate boundaries between different regions. Three colours of basins represent the real and complex roots. We can visually analyse the presence of several 'islands' within the predominant area, which suggests that the original method is highly sensitive when choosing the initial guess. Choosing z_0 in a region where different basins meet each other is not preferable because it is difficult to guess which root the method will converge [31]. In contrast, in the homotopy version, the spiral-like pattern diminishes with increase in step number. When $N = 100$, clearly, three large blocks of colours can be seen, thus representing the method's stability.

Lastly, in Figure 3, four dominant colours were generated by the original method with small, braided islands on the boundaries of the four large basins. This indicates that the method will struggle to converge to a root when the initial guess is chosen along the borders. On the contrary, the homotopy version shows that the small islands tend to decrease when the number of steps is increased from 5 to 100. When $N = 5$, the complexity in the central region reduces slightly, but a fractal-like boundary between different basins is still maintained. The number of basins seems to have decreased as larger regions of single colours become more prominent. When $N = 20$ (Figure 3(c)), fewer basins are presented. The fractal nature is less evident, and large basins are more noticeable. Only three large

basins are presented when N is increased to 100 as in Figure 3(d), while the orange basin is only located on the boundaries as small regions.

These basins of attraction provide valuable insights into the dynamic behaviour of the methods' convergence patterns. The visuals of the basins of attraction in the three figures indicate that the Zhang-HCM has more stability and less sensitivity than Zhang's method.

5. NUMERICAL EXPERIMENTS

This study seeks to address the issue of solving $H(x, t) = 0$ instead of $f(x) = 0$ by changing the parameter t within the range of 0 to 1 to prevent divergence. To showcase the efficiency of our suggested approach, we solved four examples of nonlinear equations using the original Zhang method versus the new method, Zhang-HCM.

The computations were performed using Mathematica 13 software. The stopping criterion ε was set for both methods. We chose $\varepsilon = 10^{-10}$ as a stopping criterion in this study. When $|f(x)| < \varepsilon$, the iteration stopped, and the number of iterations, n , and the solution x^* were to be determined. These examples were tested for comparison of numerical results on accuracy and the number of iterations n needed to reach the stopping criterion.

Example 1. Consider the quadratic equation [17]

$$f_1(x) = x^2 + 8x - 9. \quad (8)$$

The quadratic equation f_1 has two real distinct roots which are -9 and 1 . Even though f_1 can be solved easily by factorisation, in this study, we wanted to solve the equation using iteration methods. A few initial guesses x_0 were provided to test the convergence of Zhang's method (6) and Zhang-HCM (7). The comparison results are shown in Table 3.

TABLE 3. Comparison of Zhang's method and Zhang-HCM for the problem in Example

1

x_0	Zhang's method			Zhang-HCM		
	n	x_n	$ f(x) $	n	x_n	$ f(x) $
-4		Diverge		3	-9.000000000000000	$4.950\text{E-}37$
4	2	1.000000000000027	$2.68\text{E-}13$	2	1.000000000000005	$1.26\text{E-}36$
-3	3	-9.000000000000001	$1.22\text{E-}13$	2	1.000000000000000	$2.76\text{E-}26$

Zhang's method is expected to have a problem when $f'(x) = 0$. In Equation 8, when $x_0 = -4$ is chosen, Zhang's method is diverged while Zhang-HCM converges to a root. Furthermore, when the other initial guess was chosen, it showed that Zhang-HCM was still superior to Zhang's method. Even

though the number of iterations is the same, Zhang-HCM is still superior as $|f(x)|$ is smaller than Zhang's method.

Example 2. Consider the cubic equation [14]

$$f_2(x) = x^3 + 4x^2 - 10. \quad (9)$$

The cubic equation f_2 has three roots which consist of one real root, and two complex roots. However, the initial guess will only use real numbers in this example. So, the solution is expected to be found in real numbers too. The results of the comparison are shown in Table 4.

TABLE 4. Comparison of Zhang's method and Zhang-HCM for problem in Example 2

x_0	Zhang's method			Zhang-HCM		
	n	x_n	$ f(x) $	n	x_n	$ f(x) $
0		Diverge		4	1.36523001341409	7.95E-45
$-\frac{8}{3}$		Diverge		5	1.3652300134141	8.70E-36
6	3	1.3652300134145	6.73E-12	2	1.36523001341409	1.66E-22

In Table 4, three initial guesses were tested. When $x_0 = 0$ and $x_0 = -\frac{8}{3}$, Zhang's method diverged, while Zhang-HCM managed to converge to the real root of the equation. Zhang's method diverged because 0 and $-\frac{8}{3}$ are the roots of $f'_2(x)$. The initial guesses chosen in this example were numbers that were easy to guess. Therefore, only whole numbers and simple fractions were considered in this example. Since roots for $f(x) - 2f(y) = 0$ in Zhang's method for this example are irrational roots, which are four distinct real roots and four distinct complex roots; hence, the roots are not considered in this example.

Example 3. Consider the transcendental equation [12]

$$f_3(x) = \sin^2 x - x^2 + 1. \quad (10)$$

The transcendental equation f_3 consists of trigonometric and polynomial functions. The equation has two roots, which are $x \approx -1.40449$ and $x \approx 1.40449$. The results of the comparison are shown in Table 5.

In Example 3, the solution for Zhang's method is diverged when $x_0 = 0$ because of the roots of $f'(x) = 0$ while Zhang-HCM can reach the root with three iterations. Note that when $x_0 = 0.1$, although $|f(x)|$ for Zhang's method is less than Zhang-HCM, Zhang's method still needs 15 iterations to satisfy the stopping criterion. While Zhang-HCM is superior, only three iterations are needed. Same with when $x_0 = -0.5$, Zhang-HCM has fewer iterations than Zhang's method.

TABLE 5. Comparison of Zhang's method and Zhang-HCM for problem in Example 3

x_0	Zhang's method			Zhang-HCM		
	n	x_n	$ f(x) $	n	x_n	$ f(x) $
0		Diverge		3	-1.40449164821486	1.81E-12
0.1	15	-1.40449164821534	3.33E-16	3	-1.40449164821534	1.86E-11
-0.5	6	1.40449164821534	3.33E-16	2	-1.40449164820913	1.54E-11

Example 4. Consider the transcendental equation [32]

$$f_4(x) = e^{-x^2} + \cos x - x^2. \quad (11)$$

The transcendental equation f_4 consists of trigonometric, exponential, and polynomial functions. The equation has two roots, which are $x \approx -0.97416$ and $x \approx 0.97416$. The comparison results are shown in Table 6.

TABLE 6. Comparison of Zhang's method and Zhang-HCM for problem in Example 4

x_0	Zhang's method			Zhang-HCM		
	n	x_n	$ f(x) $	n	x_n	$ f(x) $
0		Diverge		2	0.974162305200541	1.41E-25
2	2	0.9741623052	7.99E-16	4	0.974162305200541	1.79E-32
-3	2	-0.974162305223	8.00E-11	2	-0.974162305200541	1.60E-22

In Table 6, Zhang-HCM is superior to Zhang's method. We noted that Zhang's method diverged when $x_0 = 0$ while Zhang-HCM successfully reached the root with two iterations. With other initial guesses, both methods managed to converge to a root. However, Zhang-HCM is still superior to Zhang's method as Zhang-HCM has a smaller value of absolute error $|f(x)|$.

6. CONCLUSION

In this study, we modified a fifth-order iterative method by converting the method to a HCM for solving nonlinear equations. The homotopy used a fixed-point function as the auxiliary function. Apart from observing three examples from basins of attraction, numerical experiments were also conducted by comparing both methods. Four examples with different values of initial guesses were illustrated in this study by considering the initial guesses as rational numbers. According to the findings, the proposed method demonstrated superior stability and efficiency over the original method, particularly in avoiding divergence issues associated with poor initial guesses. The parts where the divergence happened were also discussed in this paper. This work is important in giving us an idea to advance the field of numerical methods for solving nonlinear equations.

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