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ON THE BOUNDEDNESS OF PERIODIC FOURIER INTEGRAL OPERATORS IN LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. The aim of this paper is to investigate the boundedness of periodic Fourier integral operators in Lebesgue spaces with variable exponent $L^{p(\cdot)}$ on the n-dimensional torus. We deal with operators of type (ρ, δ) which symbols belong to the Hörmander class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ for $0 \leq \delta < \rho \leq 1$.

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1. Introduction

A periodic Fourier integral operator (also called Fourier series operator) is defined by providing a symbol and a phase function. Such operator can be expressed as follows

$$A_{\phi,a}f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i\phi(x,\xi)}a(x,\xi)(\mathcal{F}_{\mathbb{T}^n}f)(\xi), \quad \forall f \in C^\infty(\mathbb{T}^n),$$

where $(\mathcal{F}_{\mathbb{T}^n}f)(\xi)$ is the Fourier transform on the torus $\mathbb{T}^n$, $a(x,\xi)$ denotes the symbol and $\phi(x,\xi)$ is the phase function. These operators were first introduced by M. Rushansky and V. Turunen [13]. They naturally emerged in the solutions of hyperbolic Cauchy problems with periodic conditions, as can be seen, for example in [[13], pages 410-411].

The boundedness of Fourier integral operators in a functional space is contingent upon conditions on the symbol $a(x,\xi)$ and the phase $\phi(x,\xi)$. Several authors have established results on the extension of Fourier integral operators in the $L^p(\mathbb{R}^n)$ spaces depending on the values of the real order of the symbol (see [14] and [9]). Furthermore, D. Ferreira and W. Staubach [15] investigated the regularity of Fourier integral operators within weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$, where the weight function belongs to the

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Muckenhoupt space A_p, for $1 < p < \infty$. In [18], K. Alexei Yu and S. Ilya M. studied the boundedness of pseudo differential operators associated to a symbol in certain class of Hörmander.

Also, the study of periodic Fourier integral operators gives rise to a fundamental issue pertaining to a topological property of these operators, namely the question of their boundedness in functional spaces. For example, the study of the L^p-boundedness of periodic Fourier integral operators with a symbol $a(x,\xi) \in S^m_{1,0}(\mathbb{T}^n \times \mathbb{Z}^n)$ i.e.

$$|\partial_x^\beta \triangle_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},$$

with a positively homogeneous phase function of degree 1 (for $\xi \neq 0$), belonging to $C^\infty(\mathbb{T}^n \times \mathbb{R}^n \backslash \{0\})$ whose Hessian matrix is non-degenerate in the spaces $L^p(\mathbb{T}^n)$, was studied by D. Cardona, R. Messiouene and A. Senoussaoui [4]. Moreover, D. Cardona in [3] studied the particular case when the phase function $\phi(x,\xi) = x \cdot \xi$ and established sufficient conditions on the symbol $a(x,\xi)$ to ensure boundedness of periodic pseudo-differential operators in the spaces $L^p(\mathbb{T}^n)$. However, it is obvious that Lebesgue spaces with a constant exponent are not sufficient for modelling complex physical phenomena, in particular those exhibiting spatial variation in properties, such as heterogeneous materials or non-Newtonian fluids. This led to the generalization to the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Some authors such as [2], [1] focused on the extension of differential operators and pseudo-differential operators in $L^{p(\cdot)}(\mathbb{R}^n)$ and $L^{p(\cdot)}_w(\mathbb{R}^n)$.

In this paper, we focus on the study of periodic Fourier integral operators in the Lebesgue spaces with variable exponent on the n-dimensional torus $L^{p(\cdot)}(\mathbb{T}^n)$. We first establish the boundedness of periodic Fourier integral operators in $L^{p_0}_w(\mathbb{T}^n)$, when $1 < p_0 < \infty$ and deduce the boundedness of these operators in $L^{p(\cdot)}(\mathbb{T}^n)$, using the technique developed by V. Rabinovich and S. Samko [11]. We also establish boundedness results for periodic Fourier integral operators in $L^{p(\cdot)}_w(\mathbb{T}^n)$. The rest of the paper is organized as follows. The Section 2 is devoted to preliminaries on variable exponent and weight function in the torus. In section 3, we provide basic tools on periodic Fourier integral operators

2. Preliminaries

The two first sections present the basic definitions and useful results in the sequel. For more informations, see references [7], [2], [6], [13].

2.1. **On the torus.**

(1) The Torus is the quotient space

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n,$$

obtained by the equivalence relation $x \sim y \Longleftrightarrow x - y \in \mathbb{Z}^n$, where $\mathbb{Z}^n$ denotes the additive group of integral coordinate.

(2) We can identify $\mathbb{T}^n$ with the cube $[0,1)^n \subset \mathbb{R}^n$, where the measure on the torus coincides with the restriction of the Euclidean measure on the cube.

(3) A function $f : \mathbb{R}^n \to \mathbb{C}$ is 1-periodic if $f(x+l) = f(x) \quad \forall x \in \mathbb{R}^n$ and $l \in \mathbb{Z}^n$. This definition shows that there is a correspondence between functions defined on $\mathbb{R}^n$ and those defined on $\mathbb{T}^n$.

To define Fourier integral operators, as well as operator series, we need the notion of Fourier transform.

Definition 2.1. The Fourier transform is defined by

$$\mathcal{F}_{\mathbb{T}^n} : C^\infty(\mathbb{T}^n) \to \mathcal{S}(\mathbb{Z}^n), \quad f \longmapsto \hat{f},$$

where $(\mathcal{F}_{\mathbb{T}^n} f)(\xi) = \hat{f}(\xi) = \displaystyle\int_{\mathbb{T}^n} e^{-2\pi i x\cdot\xi} f(x)dx.$

Note that $\mathcal{F}_{\mathbb{T}^n}$ is a bijection and its inverse $\mathcal{F}_{\mathbb{T}^n}^{-1} : \mathcal{S}(\mathbb{T}^n) \to C^\infty(\mathbb{Z}^n)$ is defined by:

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi).$$

2.2. **On periodic Fourier integral operators.** Given that the notion of partial derivative is no longer valid when $\xi \in \mathbb{Z}^n$, we use the concept of forward and backward difference operators, also known as discrete derivatives.

Definition 2.2. Let $(\delta_j)_{1\le j\le n}$ be the canonical basis of $\mathbb{R}^n$. For a function $a : \mathbb{Z}^n \to \mathbb{C}$ the forward and backward partial difference operators of a are defined respectively by

$$\triangle_{\xi_j} a(\xi) = a(\xi+\delta_j) - a(\xi), \quad \bar{\triangle}_{\xi_j} a(\xi) = a(\xi) - a(\xi+\delta_j). \tag{1}$$

For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$,

$$\triangle_\xi^\alpha = \triangle_{\xi_1}^{\alpha_1} \cdots \triangle_{\xi_n}^{\alpha_n}; \qquad \bar{\triangle}_\xi^\alpha = \bar{\triangle}_{\xi_1}^{\alpha_1} \cdots \bar{\triangle}_{\xi_n}^{\alpha_n}. \tag{2}$$

Lemma 2.3 ([13] Lemma 3.3.10)**.** Assume that $\varphi, \psi : \mathbb{Z}^n \to \mathbb{C}$. Then for all $\alpha \in \mathbb{N}^n$,

$$\sum_{\xi\in\mathbb{Z}^n} \varphi(\xi) \Delta_\xi^\alpha \psi(\xi) = (-1)^{|\alpha|} \sum_{\xi\in\mathbb{Z}^n} \left(\bar{\Delta}_\xi^\alpha \varphi(\xi)\right) \psi(\xi) \tag{3}$$

provided that both series are absolutely convergent.

Definition 2.4. Let $m \in \mathbb{R}, 0 \le \delta, \rho \le 1$. The Hörmander symbol class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ consists of functions $a(x,\xi) \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ which satisfy the estimation: for $\alpha, \beta \in \mathbb{N}^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \forall x \in \mathbb{T}^n, \quad \forall \xi \in \mathbb{Z}^n. \tag{4}$$

The periodic Fourier integral operator (or Fourier series operator) associated to the symbol $a(x,\xi)$ and phase function $\phi(x,\xi)$ denoted by $A_{\phi,a}$ is defined by

$$A_{\phi,a}f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i\phi(x,\xi)}a(x,\xi)(\mathcal{F}_{\mathbb{T}^n}f)(\xi), \quad \forall f\in C^\infty(\mathbb{T}^n), \tag{5}$$

where $\phi:\mathbb{T}^n\times\mathbb{Z}^n\to\mathbb{R}$ is positively homogeneous of degree 1 in $\xi\neq 0$ and the function $x\mapsto e^{2\pi i\phi(x,\xi)}$ is 1-periodic for all $\xi\in\mathbb{Z}^n$.

In the sequel, the operator $A_{\phi,a}$ will be denoted A.

In [13], the authors mentioned the result below which shows that properties of toroidal symbols automatically imply certain properties for differences. The proof follows their Proposition 3.3.4: let $a(x,\xi)\in C^k(\mathbb{T}^n\times\mathbb{Z}^n)$, $k\in\mathbb{N}$. For every $\alpha\in\mathbb{N}^n$, and $\beta\in\mathbb{N}^n$, $|\beta|\leq k$ we have the identity

$$\Delta_\xi^\alpha\partial_x^\beta a(x,\xi) = \sum_{|\gamma|\leq|\alpha|}(-1)^{|\alpha-\gamma|}\binom{\alpha}{\gamma}\partial_x^\beta a(x,\xi+\gamma), \quad \forall(x,\xi)\in\mathbb{T}^n\times\mathbb{Z}^n. \tag{6}$$

2.3. **Some basic tools on variable exponent and weight functions.**

Definition 2.5. Let $\mathcal{P}(\mathbb{T}^n)$ be the set of all measurable and 1-periodic functions $p(\cdot):\mathbb{T}^n\to(0,\infty]$ and let $p_- = ess\inf_{x\in\mathbb{T}^n}p(x)$ and $p_+ = ess\sup_{x\in\mathbb{T}^n}p(x)$. The function $p(\cdot)\in\mathcal{P}(\mathbb{T}^n)$ is said locally log-Hölder continuous, abbreviated $p\in C^{\log}_{\text{loc}}(\mathbb{T}^n)$, if there exists a constant $c_{\log}(p)>0$ such that

$$|p(x)-p(y)|\leq\frac{c_{\log}(p)}{-\log|x-y|}, \quad x,y\in\mathbb{T}^n, |x-y|\leq\frac{1}{2}.$$

Definition 2.6. Let $p(\cdot)\in\mathcal{P}(\mathbb{T}^n)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T}^n)$ is the set of all measurable, 1-periodic functions f on $\mathbb{T}^n$ such that $\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right)<\infty$ for some $\lambda>0$, equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T}^n)} = \inf\left\{\lambda>0:\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right)\leq 1\right\},$$

where $\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) = \int_{\mathbb{T}^n}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx.$

Lemma 2.7. (Theorem 4.3.12, [6]) If $p(\cdot)\in\mathcal{P}(\mathbb{T}^n)$ with $p_+<\infty$, then $C_0^\infty(\mathbb{T}^n)$ is dense in $L^{p(\cdot)}(\mathbb{T}^n)$.

The following results are extremely useful. They are known in the literature for the Euclidean space $\mathbb{R}^n$. Let's denote by M the maximal operator and $M^\#$ the sharp operator:

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dx,$$

$$M^\#(f)(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_B(x)|\,dy,$$

where $f_B(x) = \sup_{r>0}\frac{1}{B(x,r)}\int_{B(x_0,r)}|f(y)|dy.$

Theorem 2.8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{T}^n)$. Suppose $1 < p_- \leq p_+ < \infty$. Then the following properties are equivalent:*

(1) *The maximal operator M is bounded in $L^{p(\cdot)}(\mathbb{T}^n)$;*

(2) *The maximal operator M is bounded in $L^{p'(\cdot)}(\mathbb{T}^n)$, with $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$;*

(3) *There exists $p_0 > 1$ such that the maximal operator is bounded in $L^{(p(\cdot)/p_0)}(\mathbb{T}^n)$.*

Since $\mathbb{T}^n$ is identified with the cube $[0;1)^n \subset \mathbb{R}^n$, the above theorem is similar to the Theorem 3.35 of [5] where Ω or $\mathbb{R}^n$ is replaced by $\mathbb{T}^n$.

Definition 2.9. An operator T is of weak type $(1,1)$ if there is a constant $C > 0$ such that for every $\lambda > 0$ we have

$$meas\,\{x \in \mathbb{T}^n : |Tu(x)| > \lambda\} \leq C\frac{\|u\|_{L^1(\mathbb{T}^n)}}{\lambda}.$$

Theorem 2.10 ([1] Theorem 2.1)**.** *Let T be a linear operator associated to a kernel K that satisfies the following conditions*

$$\sup_{|\alpha|=1}\sup_{x,y\in\mathbb{T}^n}\|y\|^{n+1}\left|\partial_x^\alpha K(x,y)\right| < \infty, \tag{7}$$

$$\sup_{|\beta|=1}\sup_{x,y\in\mathbb{T}^n}\|x\|^{n+1}\left|\partial_y^\beta K(x,y)\right| < \infty \tag{8}$$

and T is of weak type $(1,1)$. Then for $0 < s < 1$, there exists a constant $C_s > 0$ such that

$$M_s^{\#}\left(Tf\right)(x) \leq C_s Mf(x), \quad \forall f \in C_0^\infty(\mathbb{T}^n). \tag{9}$$

The weight functions in L^p spaces are useful in the proof of general and precise regularity results.

Definition 2.11. Let $w \in L^1_{loc}(\mathbb{T}^n)$ a non-negative function. Then w belongs to the Muckenhoupt weights space A_{p_0} for $1 < p_0 < \infty$ if

$$[w]_{p_0} := \sup_{\mathcal{Q}}\left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} w(x)dx\right)\left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} w(x)^{-\frac{1}{p_0-1}}dx\right)^{p_0-1} < \infty, \tag{10}$$

where $\mathcal{Q}$ is a cube in $\mathbb{T}^n$.
By definition $w \in A_1$ if there exists a constant $C > 0$ such that $Mw(x) \leq Cw(x)$ for all $x \in \mathbb{T}^n$.

Example 2.1 ([15] Example 1)**.** The function $|x|^\alpha$ is an A_p weighted, for $1 < p < \infty$, if and only if $-n < \alpha < n(p-1)$.

Lemma 2.12 ([8] Property 2)**.** Suppose that w is in A_p for some $p \in [1,\ \infty]$ and $0 < \delta < 1$. Then w belongs to A_q where $q = \delta p + 1 - \delta$. Moreover, $[w^\delta]_p \leq [w]_p^\delta$.

Next, we state the extrapolation theorem of Rubio de Francia [5] applied to the torus.

Theorem 2.13. *Suppose that for $p_0 > 1$ and $\mathcal{F}$ is a family of pairs non-negative measurable functions such that for all $w \in A_1$*

$$\int_{\mathbb{T}^n} F(x)^{p_0} w(x)dx \le c_{p_0} \int_{\mathbb{T}^n} G(x)^{p_0} w(x)dx, \qquad (F,G) \in \mathcal{F}. \tag{11}$$

If $p(\cdot) \in \mathcal{P}(\mathbb{T}^n)$, $p_0 \le p_- \le p_+ < \infty$ and the maximal operator M is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\mathbb{T}^n)$, then there exists a constant $C > 0$ such that

$$\|F\|_{L^{p(\cdot)}(\mathbb{T}^n)} \le C \|G\|_{L^{p(\cdot)}(\mathbb{T}^n)}, \qquad (F,G) \in \mathcal{F}. \tag{12}$$

Proof. Since the torus is identified to the cube $[0;1)^n \subset \mathbb{R}^n$, we replace $\mathbb{R}^n$ by $\mathbb{T}^n$ in the Theorem 4.24 [5]. □

Definition 2.14. Let $w \in L^1_{loc}(\mathbb{T}^n)$ be a weight.

(1) If $1 < p < \infty$, $L^p_w(\mathbb{T}^n)$ is the space of all functions $f : \mathbb{T}^n \longrightarrow \mathbb{C}$ with finite quasi-norm

$$\|f\|_{L^p_w(\mathbb{T}^n)} = \int_{\mathbb{T}^n} |f(x)|^p w(x)dx.$$

(2) If $p(\cdot) \in \mathcal{P}(\mathbb{T}^n)$ such that $1 < p_- \le p(x) \le p_+ < \infty$, $L^{p(\cdot)}_w(\mathbb{T}^n)$ is the space of all functions $f : \mathbb{T}^n \longrightarrow \mathbb{C}$ with finite quasi-norm

$$\|f\|_{L^{p(\cdot)}_w(\mathbb{T}^n)} = \|wf\|_{L^{p(\cdot)}(\mathbb{T}^n)}.$$

Proposition 2.15. *Let $p(\cdot) \in \mathcal{P}(\mathbb{T}^n)$ such that $1 < p_- \le p(x) \le p_+ < \infty$ and $0 < s < p_-$. If $w \in L^1_{loc}(\mathbb{T}^n)$ is a weight, then*

$$\|f\|_{L^{p(\cdot)}_w(\mathbb{T}^n)} = \|f^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}_{w^s}(\mathbb{T}^n)}. \tag{13}$$

The following theorem is proved for constant p in the non-weighted case in [[17], p. 148] and for variable $p(\cdot)$ in the weighted case in [10], Lemma 4.1.

Theorem 2.16. *Let T be an operator with kernel K such that*

$$Tf(x) = \int_{\mathbb{T}^n} K(x, x-y) f(y)dy.$$

Let $p(\cdot) \in C^{\log}_{loc}(\mathbb{T}^n)$ such that $1 < p_- < p_+ < \infty$ and $p(x) = p_\infty$ for $|x| \ge R$ where $R > 0$. Suppose also a weight function $w \in A_{p(\cdot)}$ of the form

$$w(x) = (1+|x|)^\beta \prod_{k=1}^{n} |x - x_k|^{\beta_k}, \quad x_k \in \mathbb{T}^n.$$

Then if

$-\dfrac{n}{p(x_k)} < \beta_k < \dfrac{n}{p'(x_k)}$ *and* $-\dfrac{n}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \dfrac{n}{p'_\infty}$, $k = 1, \cdots, n$, *there exists a constant* $C > 0$ *such that*

$$\|Tf\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} \leq C \left\| M^{\#}(|Tf|) \right\|_{L_w^{p(\cdot)}(\mathbb{T}^n)}, \quad \forall f \in C_0^\infty(\mathbb{T}^n). \tag{14}$$

3. Mains Results

We begin by proving the regularity of periodic Fourier integral operator in the weighted Lebesgue space $L_w^{p_0}(\mathbb{T}^n)$, where the weight function is locally integrable and positive. To this purpose let's set up the following lemma.

Lemma 3.1. For all multi-indices $\alpha \in \mathbb{N}^n$ and $a(x, \xi) \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ we have

$$\sum_{\xi \in \mathbb{Z}^n} e^{2\pi i(x-y)\cdot\xi} a(x, \xi) = (-1)^{|\alpha|} (e^{2\pi i(x-y)} - 1)^{-\alpha} \sum_{\xi \in \mathbb{Z}^n} \left(\bar{\Delta}_\xi^\alpha e^{2\pi i(x-y)\cdot\xi} \right) a(x, \xi).$$

Proof. By using the identity (1) we obtain:

$$\begin{aligned} \bar{\Delta}_{\xi_1}^{\alpha_1} e^{2\pi i(x-y)\cdot\xi} &= e^{2\pi i(x-y)\cdot\xi} - e^{2\pi i(x-y)\cdot(\xi+\delta_1)} \\ &= e^{2\pi i(x-y)\cdot\xi} - e^{2\pi i(x-y)\cdot\xi} \cdot e^{2\pi i(x-y)\cdot\delta_1} \\ &= -e^{2\pi i(x-y)\cdot\xi} \left(e^{2\pi i(x-y)\cdot\delta_1} - 1 \right). \end{aligned}$$

$$\begin{aligned} -\bar{\Delta}_{\xi_1}^{\alpha_1} e^{2\pi i(x-y)\cdot\xi} &= e^{2\pi i(x-y)\cdot\xi} \left(e^{2\pi i(x-y)\cdot\delta_1} - 1 \right) \\ &= e^{2\pi i(x-y)\cdot\xi} \left(e^{2\pi i(x-y)} - 1 \right). \end{aligned}$$

Now the identity (2) with $\alpha = (\alpha_1 \cdots \alpha_n)$ and $\xi = (\xi_1 \cdots \xi_n)$ gives

$$(-1)^{|\alpha|} \bar{\Delta}_\xi^\alpha e^{2\pi i(x-y)\cdot\xi} = \left(e^{2\pi i(x-y)} - 1 \right)^\alpha \cdot e^{2\pi i(x-y)\cdot\xi}.$$

Thus

$$e^{2\pi i(x-y)\cdot\xi} = (-1)^{|\alpha|} \left(e^{2\pi i(x-y)} - 1 \right)^{-\alpha} \bar{\Delta}_\xi^\alpha e^{2\pi i(x-y)\cdot\xi}.$$

This yields the result:

$$\sum_{\xi \in \mathbb{Z}^n} e^{2\pi i(x-y)\cdot\xi} a(x, \xi) = (-1)^{|\alpha|} \left(e^{2\pi i(x-y)} - 1 \right)^{-\alpha} \sum_{\xi \in \mathbb{Z}^n} \left(\bar{\Delta}_\xi^\alpha e^{2\pi i(x-y)\cdot\xi} \right) a(x, \xi).$$

□

Theorem 3.2. *Let* $1 < p_0 < \infty$ *and* $w \in A_{p_0}$. *Let* $A : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ *be the periodic Fourier integral operator defined by*

$$Af(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i\phi(x,\xi)} a(x, \xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi), \quad \forall f \in C^\infty(\mathbb{T}^n),$$

where $\phi(x,\xi) : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{R}$ *is a phase function such that* $x \mapsto e^{2\pi i\phi(x,\xi)}$ *is 1-perodic for all* $\xi \in \mathbb{Z}^n$ *and* $a(x,\xi) : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *is a symbol satisfying the Hörmander condition*

$$|\partial_x^\beta \triangle_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

with $m \leq (\rho-1)|\frac{1}{p_0}-\frac{1}{2}|-\epsilon$ *and* $\alpha,\beta \in \mathbb{N}^n$*;* $\epsilon > \delta$*. Then the periodic Fourier integral operator A is bounded on* $L^{p_0}_w(\mathbb{T}^n)$*.*

Proof. The symbol $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ is continuous and thus admits a Fourier series expansion given by

$$a(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e^{2\pi i x\cdot\eta}\hat{a}(\eta,\xi) \text{ for all } \eta \in \mathbb{Z}^n.$$

Let us assume that $f \in C_0^\infty(\mathbb{T}^n)$. The decomposition of the phase function $\phi(x,\xi) = x\cdot\xi + \psi(\xi)$, where $\psi(\xi)$ is a real values function belonging to $C^\infty(\mathbb{R}^n \setminus \{0\})$ and is positively homogeneous of degree 1 in $\xi \neq 0$ gives

$$\begin{aligned} Af(x) &= \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i\phi(x,\xi)} a(x,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi) \\ &= \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i(x\cdot\xi+\psi(\xi))} a(x,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi) \\ &= \sum_{\xi\in\mathbb{Z}^n}\sum_{\eta\in\mathbb{Z}^n} e^{2\pi i(x\cdot\xi+\psi(\xi))} e^{2\pi i x\cdot\eta}\hat{a}(\eta,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi) \\ &= \sum_{\eta\in\mathbb{Z}^n} e^{2\pi i x\cdot\eta}\left(\sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{a}(\eta,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi) e^{2\pi i\psi(\xi)}\right). \end{aligned}$$

One can see the expression

$$\sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{a}(\eta,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi) e^{2\pi i\psi(\xi)}$$

as the symbol of the product of the two operators $\hat{a}(\eta, D_x)$ and $e^{2\pi i\psi(D_x)}$. Namely

$$\left(\hat{a}(\eta,D_x)e^{2\pi i\psi(D_x)}\right) f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{a}(\eta,\xi)e^{2\pi i\psi(\xi)}(\mathcal{F}_{\mathbb{T}^n} f)(\xi).$$

It follows that

$$Af(x) = \sum_{\eta\in\mathbb{Z}^n} e^{2\pi i x\cdot\eta}\hat{a}(\eta,D_x)f(x)e^{2\pi i\psi(D_x)},$$

where

$$\hat{a}(\eta,D_x)f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{a}(\eta,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi)$$

is the Fourier multiplier.

We now estimate Af with respect to $\hat{a}(\eta, D_x)f$.

$$\begin{aligned}\|Af\|_{L^{p_0}_w(\mathbb{T}^n)} &\leq \left\{\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n}\left|e^{2\pi i x\cdot\xi}\hat{a}(\eta,D_x)f(x)e^{2\pi i\psi(D_x)}\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}\\ &\leq \sum_{\eta\in\mathbb{Z}^n}\left\{\int_{\mathbb{T}^n}|\hat{a}(\eta,D_x)f(x)|^{p_0}\,w(x)dx\right\}^{\frac{1}{p_0}}\\ &\leq \sum_{\eta\in\mathbb{Z}^n}\|\hat{a}(\eta,D_x)f(x)\|_{L^{p_0}_w(\mathbb{T}^n)}\,. \end{aligned}\tag{15}$$

The next step is to estimate the norm $\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)}$.

$$\begin{aligned}\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)} &= \left\{\int_{\mathbb{T}^n}|\hat{a}(\eta,D_x)f(x)|^{p_0}\,w(x)dx\right\}^{\frac{1}{p_0}}\\ &\leq \left\{\int_{\mathbb{T}^n}\left|\sum_{\xi\in\mathbb{Z}^n}e^{2\pi i x\cdot\xi}\hat{a}(\eta,\xi)\hat{f}(\xi)\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}\\ &\leq \left\{\int_{\mathbb{T}^n}\left|\sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{T}^n}e^{2\pi i(x-y)\cdot\xi}\hat{a}(\eta,\xi)f(y)dy\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}\\ &= \left\{\int_{\mathbb{T}^n}\left|\int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n}e^{2\pi i(x-y)\cdot\xi}\hat{a}(\eta,\xi)f(y)dy\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}. \end{aligned}$$

We then apply Lemma 3.1 to deduce:

$$\begin{aligned}&\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq \left\{\int_{\mathbb{T}^n}\left|\int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n}\left[(-1)^{|\alpha|}(e^{2\pi i(y-x)}-1)^{-\alpha}\bar{\triangle}_\xi^\alpha e^{2\pi i(x-y)\cdot\xi}a(\eta,\xi)\right]f(y)dy\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}.\end{aligned}$$

By Lemma 2.3, and the second-order multidimensional Taylor expansion around $(0,0)$ to the function $e^{2\pi i(x-y)}$:

$$\begin{aligned}e^{2\pi i(x-y)} &= 1+\nabla e^0\cdot 2\pi i(x-y)+o(\|x-y\|^2)\\ &= 1+2\pi i(x-y)+o(\|x-y\|^2),\end{aligned}$$

where $\nabla e^0=(1,-1)$ is the gradient of the function $e^{2\pi i(x-y)}$ at the point $(0,0)$, we obtain

$$\begin{aligned}&\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq \left\{\int_{\mathbb{T}^n}\left|\int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n}\left[(2\pi i)^{-|\alpha|}\,(x-y)^{-\alpha}\,e^{2\pi i(x-y)\cdot\xi}\triangle_\xi^\alpha\hat{a}(\eta,\xi)\right]f(y)dy\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}\\ &\leq \left\{\int_{\mathbb{T}^n}\left|\sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\left|(2\pi i)^{-|\alpha|}\,(x-y)^{-\alpha}\,f(y)\right|dy|\triangle_\xi^\alpha\hat{a}(\eta,\xi)|\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}\end{aligned}$$

$$\leq \quad (2\pi)^{-|\alpha|}\left\{\int_{\mathbb{T}^n}\left|\sum_{\xi\in\mathbb{Z}^n}\left|\triangle_\xi^\alpha \hat{a}(\eta,\xi)\right|\int_{\mathbb{T}^n}|(x-y)^{-\alpha}f(y)|dy\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}$$

$$\leq \quad (2\pi)^{-|\alpha|}\left\{\int_{\mathbb{T}^n}\left|\sum_{\xi\in\mathbb{Z}^n}\left|\triangle_\xi^\alpha \hat{a}(\eta,\xi)\right|\left(|(x-\cdot)^{-\alpha}|\star|f(\cdot)|\right)(x)\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}$$

$$\leq \quad (2\pi)^{-|\alpha|}\left(\sum_{\xi\in\mathbb{Z}^n}\left|\triangle_\xi^\alpha \hat{a}(\eta,\xi)\right|^{p_0}\right)^{\frac{1}{p_0}}\left\{\int_{\mathbb{T}^n}\left||(x-\cdot)^{-\alpha}|\star|f(\cdot)|(x)\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}}.$$

Note that the convolution norm on the weighted spaces gives

$$\begin{aligned}\left\{\int_{\mathbb{T}^n}\left||(x-\cdot)^{-\alpha}|\star|f(\cdot)|(x)\right|^{p_0}w(x)dx\right\}^{\frac{1}{p_0}} &\leq \left\||\cdot|^{-|\alpha|}\right\|_{L^1_w(\mathbb{T}^n)}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq C_\alpha\|f\|_{L^{p_0}_w(\mathbb{T}^n)}.\end{aligned}$$

Let's use the estimate

$$\left|\triangle_\xi^\alpha \hat{a}(\eta,\xi)\right|\leq C_{r,\alpha}\langle\eta\rangle^{-r}\langle\xi\rangle^{m-\rho|\alpha|+r\delta},\forall r\in\mathbb{N}_0 \tag{16}$$

established in [13], Lemma 4.2.1. We obtain

$$\begin{aligned}\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)} &\leq (2\pi)^{-|\alpha|}C_\alpha\left(\sum_{\xi\in\mathbb{Z}^n}C_{r,\alpha}\langle\eta\rangle^{-rp_0}\langle\xi\rangle^{(m-\rho|\alpha|+r\delta)p_0}\right)^{\frac{1}{p_0}}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq (2\pi)^{-|\alpha|}C_{r,\alpha}\langle\eta\rangle^{-r}\left(\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{(m-\rho|\alpha|+r\delta)p_0}\right)^{\frac{1}{p_0}}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}.\end{aligned}$$

The series converge for $1<r\leq[\frac{\epsilon}{\delta}]+1$. So, there exists a constant $C_{p_0,r,\alpha}>0$ such that

$$\begin{aligned}\|\hat{a}(\eta,D_x)f\|_{L^{p_0}_w(\mathbb{T}^n)} &\leq (2\pi)^{-|\alpha|}C_{r,\alpha}\langle\eta\rangle^{-r}C_{p_0,r,\alpha}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq C'_{p_0,r,\alpha}\langle\eta\rangle^{-r}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}.\end{aligned}$$

We are now ready to formulate the boundedness of the operator A. If we go back to the estimate (15) we can write

$$\begin{aligned}\|Af\|_{L^{p_0}_w(\mathbb{T}^n)} &\leq \sum_{\eta\in\mathbb{Z}^n}\|\hat{a}(\eta,D_x)f(x)\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq \sum_{\eta\in\mathbb{Z}^n}C'_{p_0,r,\alpha}\langle\eta\rangle^{-r}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}\\ &\leq \left(C'_{p_0,r,\alpha}\sum_{\eta\in\mathbb{Z}^n}\langle\eta\rangle^{-r}\right)\|f\|_{L^{p_0}_w(\mathbb{T}^n)}.\end{aligned}$$

Since $r>1$, the sum $\sum_{\eta\in\mathbb{Z}^n}\langle\eta\rangle^{-r}$ is finite and there exists a constant $C''_{p_0,r,\alpha}>0$ such that

$$\|Af\|_{L^{p_0}_w(\mathbb{T}^n)}\leq C''_{p_0,r,\alpha}\|f\|_{L^{p_0}_w(\mathbb{T}^n)}.$$

□

Now, we present a sufficient condition for the boundedness of periodic Fourier integral operators in generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{T}^n)$, using Rubio de Francia extrapolation theorem on the torus.

Theorem 3.3. *Let $\phi(x,\xi) \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ be a phase function such that $x \mapsto e^{2\pi i \phi(x,\xi)}$ is 1-periodic for all $\xi \in \mathbb{Z}^n$ and let $a(x,\xi) \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ be a symbol which satisfies*

$$|\triangle_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|} \quad \forall \alpha, \beta \in \mathbb{N}^n.$$

For all $p(\cdot) \in \mathcal{P}(\mathbb{T}^n)$ such that $1 < p_- \leq p(\cdot) \leq p_+ < \infty$, there exists a constant $C > 0$ such that the periodic Fourier integral operator A associated to the symbol $a(x,\xi)$ satisfies

$$\|Af\|_{L^{p(\cdot)}(\mathbb{T}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{T}^n)}.$$

Proof. Let $f \in C_0^\infty(\mathbb{T}^n)$. Then $f \in L^{p(\cdot)}(\mathbb{T}^n)$ since $C_0^\infty(\mathbb{T}^n)$ is dense in $L^{p(\cdot)}(\mathbb{T}^n)$ (Lemma 2.7) . Moreover, if $1 \leq p < q < \infty$, there is a continuous embedding of Muckenhoupt classes $A_p \hookrightarrow A_q$ (see Lemma 2.12). Let w be a weight function in A_1. Since $p_0 > 1$, then $w \in A_{p_0}$. By Theorem 3.2, the operator A is bounded in $L_w^{p_0}(\mathbb{T}^n)$. Moreover, the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{T}^n)$ and $(|Af|, |f|)$ is a pair of positive functions. By Theorem 2.13, the periodic Fourier integral operator A is bounded in $L^{p(\cdot)}(\mathbb{T}^n)$, and there exists a constant $C > 0$ such that

$$\|Af\|_{L^{p(\cdot)}(\mathbb{T}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{T}^n)}.$$

□

Hereafter is the result of boundedness for periodic Fourier integral operators in weighted Lebesgue spaces with variable exponent.

Theorem 3.4. *Let $a(x,\xi) : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be a symbol which satisfies the condition*

$$|\partial_x^\beta \Delta_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $0 \leq \delta < \rho \leq 1$, the parameter $m < -(n+1)$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Let $\phi : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{R}$ be a phase function such that the function $x \mapsto e^{2\pi i \phi(x,\xi)}$ is 1-periodic and satisfies the condition: there exist a constant $C > 0$ such that

$$|\partial_x^\alpha \phi(x,\xi)| \leq C.$$

Further, suppose $p(\cdot) \in C_{loc}^{\log}(\mathbb{T}^n)$ such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) = p_\infty$ for $|x| \geq R$ where $R > 0$. Let $w \in A_{p(\cdot)}$ be a Muckenhoupt weight function of the form $w(x) = (1+|x|)^\beta \prod_{k=1}^{n} |x - x_k|^{\beta_k}$ such that for all $x_k \in \mathbb{T}^n$,

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \text{and} \quad -\frac{n}{p_\infty} < \beta + \sum_{k=1}^{m} \beta_k < \frac{n}{p'_\infty}, \quad k = 1, \ldots, n. \tag{17}$$

Then the periodic Fourier integral operator A associated to the symbol a is bounded in $L_w^{p(\cdot)}(\mathbb{T}^n)$, and there exists a constant $C' > 0$ such that

$$\|Af\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} \leq C' \|f\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} .$$

Proof. To simplify, we use the expression $f \lesssim g$ which means that $f \leq cg$ for some independent constant $c > 0$.

Let $f \in C_0^\infty(\mathbb{T}^n)$ and $0 < t < 1$. We have

$$\|Af\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} = \left\||Af|^t\right\|^{\frac{1}{t}}_{L_{w^t}^{\frac{p(\cdot)}{t}}(\mathbb{T}^n)} .$$

In what follows, we will decompose the proof in two steps.

First step: In this step, we establish the estimate

$$\|Af\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} \lesssim \left\|M^{\#}|Af|^t\right\|^{\frac{1}{t}}_{L_{w^t}^{\frac{p(\cdot)}{t}}(\mathbb{T}^n)} .$$

Let $p(\cdot) \in C_{loc}^{\log}(\mathbb{T}^n)$. For all $0 < t < 1$, we have $\frac{p(\cdot)}{t} \in C_{loc}^{\log}(\mathbb{T}^n)$. Multiplying inequality (17) by t gives

$$-\frac{n}{\frac{p(x_k)}{t}} < t\beta_k < \frac{n}{\frac{p'(x_k)}{t}} \quad \textit{and} \quad -\frac{n}{\frac{p_\infty}{t}} < t\beta + \sum_{k=1}^{n} t\beta_k < \frac{n}{\frac{p'_\infty}{t}}, \quad k = 1, ..., n.$$

Furthermore, for all $0 < t < 1$, we have $tp_- + 1 - t < p_-$. Thus by Lemma 2.12, the weight function w^t belongs to A_{p_-}. Since the space $A_{p_-} \subset A_{p(\cdot)}$, then $w^t \in A_{p(\cdot)}$. Now, using Theorem 2.16 and the density of $C_0^\infty(\mathbb{T}^n)$ in $L_w^{p(\cdot)}(\mathbb{T}^n)$ (see Lemma 2.7), for all $f \in C_0^\infty(\mathbb{T}^n)$,

$$\|Af\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} \lesssim \left\|M^{\#}\left(|Af|^t\right)\right\|^{\frac{1}{t}}_{L_{w^t}^{\frac{p(\cdot)}{t}}(\mathbb{T}^n)} .$$

Second step: For the second step we show that

$$\left\|M^{\#}\left(|Af|^t\right)\right\|^{\frac{1}{t}}_{L_{w^t}^{\frac{p(\cdot)}{t}}(\mathbb{T}^n)} \lesssim \|f\|_{L_w^{p(\cdot)}(\mathbb{T}^n)} .$$

Let's establish the conditions of the Theorem 2.10, where $K(x, y)$ is the kernel associated to the periodic Fourier integral operator A :

$$\sup_{|\alpha'|=1} \sup_{x,y\in\mathbb{T}^n} \|y\|^{n+1} \left|\partial_x^{\alpha'} K(x, y)\right| < \infty,$$

$$\sup_{|\beta'|=1} \sup_{x,y\in\mathbb{T}^n} \|x\|^{n+1} \left|\partial_y^{\beta'} K(x, y)\right| < \infty.$$

By using the Leibniz formula as well as the estimates on the symbol a and the phase ϕ, we obtain

$$\begin{aligned} \partial_x^{\alpha'} K(x, y) &= \partial_x^{\alpha'} \left(\sum_{\xi\in\mathbb{Z}^n} e^{2\pi i(\phi(x,\xi) - y\cdot\xi)} a(x, \xi) \right) \\ &= \sum_{\xi\in\mathbb{Z}^n} \sum_{|\gamma|\leq|\alpha'|} C_{\alpha,\gamma} (2\pi i)^{|\gamma|} \partial_x^\gamma \phi(x, \xi) \partial_x^{\alpha'-\gamma} a(x, \xi) e^{2\pi i(\phi(x,\xi) - y\cdot\xi)}. \end{aligned}$$

Consequently

$$\begin{aligned}\left|\partial_x^{\alpha'} K(x,y)\right| &\leq \sum_{\xi\in\mathbb{Z}^n}\sum_{|\gamma|\leq|\alpha'|}\left|(2\pi i)^{|\gamma|}\partial_x^{\gamma}\phi(x,\xi)\partial_x^{\alpha'-\gamma}a(x,\xi)e^{2\pi i(\phi(x,\xi)-y\cdot\xi)}\right| \\ &\leq \sum_{\xi\in\mathbb{Z}^n}\sum_{|\gamma|\leq|\alpha'|} C_{\alpha',\gamma}\left|\partial_x^{\gamma}\phi(x,\xi)\right|\left|\partial_x^{\alpha'-\gamma}a(x,\xi)\right| \\ &\leq \sum_{\xi\in\mathbb{Z}^n}\sum_{|\gamma|\leq|\alpha'|} C'_{\alpha',\gamma}C\langle\xi\rangle^{m+\delta|\alpha'-\gamma|} < \infty.\end{aligned}$$

Since $m+\delta|\alpha'-\gamma| < -n$, this ensures convergence with respect to ξ. Moreover, by multiplying $\left|\partial_x^{\alpha'} K(x,y)\right|$ by $|y|^{n+1}$ and considering the identification of $\mathbb{T}^n$ with the cube $[0,1)^n$, such that for all $y\in\mathbb{T}^n$ we have $|y|\leq 1$, we can conclude that $\sup_{|\alpha'|=1}\sup_{x,y\in\mathbb{T}^n}\|x\|^{n+1}\left|\partial_y^{\alpha'}K(x,y)\right| < \infty$.

We now give an estimate of $|\partial_y^{\beta'}K(x,y)|$.

$$\begin{aligned}\left|\partial_y^{\beta'} K(x,y)\right| &\leq \sum_{\xi\in\mathbb{Z}^n}|(2\pi i)^{|\beta|}||\xi|^{|\beta|}|a(x,\xi)| \\ &\leq \sum_{\xi\in\mathbb{Z}^n} C'\langle\xi\rangle^{m+|\beta'|}.\end{aligned}$$

Using the precedent idea for $x\in\mathbb{T}^n$ we obtain $\sup_{|\beta'|=1}\sup_{x,y\in\mathbb{T}^n}\|x\|^{n+1}\left|\partial_x^{\beta'}K(x,y)\right| < \infty$.
Note also that if the symbol of integral operators is order $m < -(n+1)$ then this operator is a locally weak $(1,1)$ [Seeger [16]]. Since the kernel satisfy (7) and A is a locally weak $(1,1)$, the Theorem 2.10 yields

$$\left\|M^{\#}\left(|Af|^t\right)\right\|^{\frac{1}{t}}_{L^{\frac{p(\cdot)}{t}}_{w^t}(\mathbb{T}^n)} \lesssim \left\|M\left(|f|^t\right)\right\|^{\frac{1}{t}}_{L^{\frac{p(\cdot)}{t}}_{w^t}(\mathbb{T}^n)} = \|M(f)\|_{L^{p(\cdot)}_w(\mathbb{T}^n)}.$$

Moreover, the maximal operator M is bounded in $L^{p(\cdot)}_w(\mathbb{T}^n)$ (Theorem 2.8). Thus, by the density of $C_0^\infty(\mathbb{T}^n)$ in $L^{p(\cdot)}(\mathbb{T}^n)$ (Lemma 2.7), we have

$$\begin{aligned}\left\|M^{\#}\left(|Af|^t\right)\right\|^{\frac{1}{t}}_{L^{\frac{p(\cdot)}{t}}_{w^t}(\mathbb{T}^n)} &\lesssim \|M(f)\|_{L^{p(x)}_w(\mathbb{T}^n)} \\ &\lesssim \|f\|_{L^{p(\cdot)}_w(\mathbb{T}^n)}.\end{aligned}$$

It follows that

$$\|Af\|_{L^{p(\cdot)}_w(\mathbb{T}^n)} \lesssim \|f\|_{L^{p(\cdot)}_w(\mathbb{T}^n)}.$$

□

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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