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ON SQUARE LINDLEY PARETO DISTRIBUTION: SIMULATIONS AND APPLICATIONS

MOHAMMED EL-ARBI KHALFALLAH[1,*], FAROUK METIRI[2]

[1]LAM2SIN Laboratory, Department of Mathematics, Badji Mokhtar-Annaba University, Box 12, Annaba 23000, Algeria

[2]LaPS Laboratory, Department of Mathematics, Badji Mokhtar-Annaba University, Box 12, Annaba 23000, Algeria

[*]Corresponding author: mohammed-el-arbi.khalfallah@univ-annaba.dz

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Abstract. In this work, we introduce a new lifetime distribution called the Square Lindley Pareto (SLP) distribution and explore its fundamental properties. Particular attention is given to the cumulative distribution function, as well as the monotonic behavior of the probability density and hazard rate functions. We also derive expressions for the moment-generating function, moments (including mean and variance), stress-strength reliability, mean deviations, Rényi and Shannon entropy, the quantile function, and extreme order statistics. Furthermore, a comprehensive simulation study is conducted to evaluate the performance of five parameter estimation methods: maximum likelihood estimation (MLE), Anderson–Darling, Cramér–von Mises, maximum product of spacings, and least squares. The estimators are compared based on average absolute bias, mean squared error, and mean absolute relative error. To illustrate the flexibility and practical applicability of the proposed distribution, we analyze four different real-world data sets. The results confirm the SLP distribution's strong modeling capability and its usefulness in various applied fields such as reliability and survival analysis.

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1. Introduction

The Lindley distribution, originally introduced by Lindley (1958) in the context of Bayesian statistics (see [13]), has garnered significant attention in recent years due to its mathematical tractability and flexibility in modeling lifetime and reliability data. Its ability to model diverse hazard rate shapes has led to a wide array of extensions and generalizations, expanding its applicability in areas such as reliability engineering, survival analysis, and biomedical research.

To enhance the modeling capacity of the Lindley distribution, researchers have proposed various modified and extended versions. Notably, Sankaran [18] combined the Poisson and Lindley distributions to introduce the discrete Poisson–Lindley distribution. A key strategy in the construction of

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new distributions has been the use of mixing processes. For instance, Sharma and Shanker (2013) [23] proposed a two-parameter Lindley distribution by mixing a gamma distribution with parameters $(2, \theta)$ and an exponential distribution with parameter θ. Similarly, Zakerzadeh and Dolati (2010) [26] developed the generalized Lindley distribution using a mixture of gamma distributions with parameters (α, θ) and $(\alpha + 1, \theta)$.

Further generalizations such as the Pareto Poisson–Lindley and zero-truncated Poisson–Lindley distributions have been explored by Ghitany et al. (2008) [8,9] and Asgharzadeh et al. (2013) [4]. Other notable contributions include the weighted Lindley distribution by Mazucheli et al. (2013) [14] and the transmuted Lindley–Geometric distribution introduced by Merovci and Elbatal (2014) [15].

Zeghdoudi and Nedjar (2016) [16,27] proposed the gamma–Lindley distribution, based on a one-parameter Lindley distribution and a gamma $(2, \theta)$ mixture. In a related study, Zeghdoudi and Lazri (2016) [28] introduced the Lindley–Pareto distribution, further enriching the Lindley family of distributions.

More recent developments highlight the growing importance of Lindley-based models in contemporary statistical applications. For example, Gomes-Silva et al. (2017) [10] presented the odd Lindley–G family—a novel class of distributions with enhanced flexibility. Al-Babtain et al. (2022) [1] proposed a modified Lindley distribution tailored for modeling COVID-19 data, while Hussein et al. (2023) [11] introduced a truncated Lindley–G family, reinforcing the relevance and adaptability of Lindley-type distributions in modern applied statistics.

In 2013, Alzaatreh et al. [2] introduced the cumulative distribution function (CDF) for the T-X family of distributions, defined as follows:

$$G(t) = \int_0^{W(F(t))} r(x; \beta)\, dx, \tag{1}$$

where

$$r(x; \beta) = \frac{\beta^2}{1+\beta}(1+x)\exp(-\beta x), \quad x > 0,\ \beta > 0,$$

is the probability density function of the Lindley distribution with parameter β, and $W(F(t))$ is a monotonic, differentiable function that satisfies the following conditions:

- $W(F(t))$ is monotonically non-decreasing and differentiable,
- $W(F(t)) \in [a, b]$,
- $W(F(t)) \to a$ as $t \to -\infty$ and $W(F(t)) \to b$ as $t \to \infty$.

In this paper, we propose a new and more flexible class of continuous distributions, called the **Square Lindley Pareto (SLP)** distribution. It is constructed by choosing

$$W(F(t)) = \frac{F(t)}{1 - F(t)},$$

where $F(t)$ is the CDF of the standard Pareto distribution defined as:

$$F(t) = 1 - \left(\frac{\sigma}{t}\right)^2, \quad t > \sigma,\, \sigma > 0.$$

Substituting into equation (1), the cumulative distribution function (CDF) of the Square Lindley Pareto (SLP) distribution is given by:

$$G_{SLP}(t;\beta,\sigma) = 1 - \frac{\sigma^2 + \beta t^2}{(\beta+1)\sigma^2} \exp\left[-\beta\left(\frac{t^2}{\sigma^2} - 1\right)\right], \quad t > \sigma. \tag{2}$$

The corresponding probability density function (PDF) is:

$$g_{SLP}(t;\beta,\sigma) = \frac{2\beta^2 e^{\beta} t^3}{(\beta+1)\sigma^4} \exp\left[-\beta\left(\frac{t}{\sigma}\right)^2\right], \quad t > \sigma. \tag{3}$$

Henceforth, we denote a random variable T that follows the SLP distribution with parameters β and σ as $T \sim \mathrm{SLP}(\beta,\sigma)$.

2. Main Properties

2.1. **Variation of the density.** The derivative of the PDF is

$$\frac{dg_{SLP}(t;\beta,\sigma)}{dt} = \frac{d\left(\frac{2\beta^2 e^{\beta} t^3}{(\beta+1)\sigma^4} \exp\left(-\beta\left(\frac{t}{\sigma}\right)^2\right)\right)}{dt}$$

$$\frac{dg_{SLP}(t;\beta,\sigma)}{dt} = \frac{2e^{\beta} t^2 \beta^2 e^{-\beta\frac{t^2}{\sigma^2}}\left(3\sigma^2 - 2t^2\beta\right)}{(\beta+1)\,\sigma^6}$$

For $t \geq \sqrt{\frac{3}{2\beta}}\sigma$ the probability density function of Square Lindley Pareto (SLP) is decreasing. For $t < \sqrt{\frac{3}{2\beta}}\sigma$, $t_0 = \sqrt{\frac{3}{2\beta}}\sigma$ is the unique critical point which SLP distribution is maximum.

2.2. **Survival and hazard functions.** The survival functions $\bar{G}_{SLP}(x)$ corresponding to the CDF defined in (1) are given by

$$\bar{G}_{SLP}(t;\beta,\sigma) = 1 - G_{SLP}(t;\beta,\sigma) = \frac{\left(\sigma^2 + t^2\beta\right)}{(\beta+1)\,\sigma^2} \exp\left(-\beta\left(\frac{t^2}{\sigma^2} - 1\right)\right) \tag{4}$$

and the hazard function is given by

$$h(t;\beta,\sigma) = \frac{g_{SLP}(t;\beta,\sigma)}{\bar{G}_{SLP}(t;\beta,\sigma)} = \frac{2\beta^2 t^3}{\sigma^2(\beta t^2 + \sigma^2)}. \tag{5}$$

Proposition 1. The hazard function $h(t;\sigma,\beta)$ is increasing.

Proof. The derivative of hazard function is given by:

$$\frac{dh(t;\beta,\sigma)}{dt} = 2\frac{t^2}{\sigma^2}\beta^2 \frac{\beta t^2 + 3\sigma^2}{\left(\beta t^2 + \sigma^2\right)^2} > 0.$$

then, the $h(t;\beta,\sigma)$ is an increasing function.

2.3. **Moment Generating Function and Moments.** For a continuous random variable T, the moment generating function $M_T\left(x\right)$ of the SLP distribution is given by

$$\begin{aligned} M_T\left(x\right) &= E\left(e^{xT}\right) = \int_0^\infty e^{xt} g_{SLP}(t;\beta,\sigma)dt \\ &= \int_0^\infty e^{xt} \frac{2\beta^2 e^\beta t^3}{\left(\beta+1\right)\sigma^4} \exp\left(-\beta\left(\frac{t}{\sigma}\right)^2\right) dt \\ &= \frac{e^\beta}{\left(\beta+1\right)} \sum_{i=0}^\infty \frac{\sigma^i x^i\, \Gamma\left(\frac{i}{2}+2,\beta\right)}{i!\beta^{\frac{i}{2}}} \end{aligned}$$

The kth Moments for the SLP distribution is defined as follows:

$$\begin{aligned} E\left(T^x\right) &= \int_0^\infty t^x g_{SLP}(t;\beta,\sigma)dt \\ &= \frac{\sigma^x e^\beta\, \Gamma\left(\frac{x}{2}+2,\beta\right)}{\left(\beta+1\right)\beta^{\frac{t}{2}}} \end{aligned}$$

The mean

$$E\left(T\right) = \frac{\sigma e^\beta\, \Gamma\left(\frac{5}{2},\beta\right)}{\left(\beta+1\right)\beta^{\frac{t}{2}}}$$

The variance of the Square Lindley Pareto distribution is

$$VaR\left(T\right) = \frac{\sigma^2 e^\beta \left(\left(\beta+1\right)\Gamma\left(3,\beta\right) + e^\beta \Gamma^2\left(\frac{5}{2},\beta\right)\right)}{\left(\beta+1\right)^2 \beta}$$

We obtained now the coefficient of variation λ, skewness and kurtosis of the SLP distribution

$$C.V = \lambda = \frac{\sqrt{VaR\left(T\right)}}{E\left(T\right)} = \frac{e^{-\frac{\beta}{2}\left(\left(\beta+1\right)\Gamma\left(3,\beta\right)+e^\beta\Gamma^2\left(\frac{5}{2},\beta\right)\right)}}{\Gamma\left(\frac{5}{2},\beta\right)}$$

$$skewness = \frac{E\left(T^3\right)}{\left(VaR\left(T\right)\right)^{\frac{3}{2}}} = \frac{\frac{1}{e^{\frac{1}{2}\beta}}\left(\beta+1\right)^{\frac{3}{2}}\Gamma\left(\frac{7}{2},\beta\right)}{\left(\left(\beta+1\right)\Gamma\left(3,\beta\right)+e^\beta\Gamma^2\left(\frac{5}{2},\beta\right)\right)^{\frac{3}{2}}}$$

$$kurtosis = \frac{E\left(T^4\right)}{\left(VaR\left(T\right)\right)^2} = \frac{\frac{1}{e^\beta}\left(\beta+1\right)^3\Gamma\left(4,\beta\right)}{\left(\left(\beta+1\right)\Gamma\left(3,\beta\right)+e^\beta\Gamma^2\left(\frac{5}{2},\beta\right)\right)^2}$$

All these expressions are independent of the parameter σ and depend upon the parameter β.

2.4. **Stress-Strength Reliability.** The measure of stress-strength reliability has many applications, especially in the area of engineering. The lifespan of a component fails at the instant that the random stress T_2 applied to it exceeds the random strength T_1, and the component will function satisfactorily whenever $T_1 > T_2$. Hence, $R = P\left[T_2 < T_1\right]$ is a measure of component stress-strength reliability. We derive the reliability R when T_1 and T_2 have independent $SLP(\beta_1,\sigma)$ and $SLP(\beta_2,\sigma)$ distributions. The reliability is defined by

$$\begin{aligned} R = P\left[T_2 < T_1\right] &= \int_0^\infty g_{SLP1}\left(t;\beta_1,\sigma\right) G_{SLP2}\left(t;\beta_2,\sigma\right) dt \\ &= \int_0^\infty \frac{2\beta_1^2 e^{\beta_1} t^3}{\left(\beta_1+1\right)\sigma^4} \exp\left(-\beta_1\left(\frac{t}{\sigma}\right)^2\right)\left(1-\frac{\left(\sigma^2+t^2\beta_2\right)}{\left(\beta_2+1\right)\sigma^2}\exp\left(-\beta_2\left(\frac{t^2}{\sigma^2}-1\right)\right)\right) dt \\ &= \sum_{i,j,l=0} \frac{p_{i,j}\left(\beta_1\right) q_l\left(\beta_2\right)}{i+j+l+4}, \end{aligned}$$

where

$$p_{i,j}\left(\beta_1\right) = \frac{\left(-1\right)^j \beta_1^{2+j}\Gamma\left(i+j+3\right)}{i!j!\left(\beta_1+1\right)\Gamma\left(j+3\right)}$$

and

$$q_l\left(\beta_2\right) = \frac{\left(-1\right)^l \beta_2^{2+l}\Gamma\left(l+5\right)}{4l!\left(\beta_2+1\right)\left(l+3\right)\Gamma\left(l+3\right)}.$$

2.5. **Mean deviations.** The deviation from the median and the mean are used to measure the dispersion and spread in a population from the centre. On denote the median by M, then the mean deviation from the median, $D(M)$ and the mean deviation from the mean, $D(\mu)$, is as follows

$$D\left(M\right) = \int_\sigma^\infty \left|t-M\right| \, g_{SLP}\left(t;\beta,\sigma\right) dt = \mu - 2\int_\sigma^M t\, g_{SLP}\left(t;\beta,\sigma\right) dt.$$

$$D\left(\mu\right) = \int_\sigma^\infty \left|t-\mu\right| \, g_{SLP}\left(t;\beta,\sigma\right) dt = 2\mu G_{SLP}\left(\mu;\beta,\sigma\right) - 2\int_\sigma^\mu t\, g_{SLP}\left(t;\beta,\sigma\right) dt,$$

where the integral

$$\int_\alpha^b t\, g_{SLP}\left(t;\beta,\sigma\right) dt = \int_\sigma^b \frac{2\beta^2 e^\beta}{\left(\beta+1\right)\sigma^4} t^4 \exp\left(-\beta\left(\frac{t}{\sigma}\right)^2\right) dt \;= \left(-\frac{e^\beta}{\left(\beta+1\right)}\frac{\sigma\Gamma(\frac{5}{2},\beta\left(\frac{t}{\sigma}\right)^2)}{\beta^{\frac{1}{2}}}\right|_\sigma^b$$

we obtain,

$$D\left(M\right) = \mu - 2\int_\sigma^M t\, g_{SLP}\left(t;\beta,\sigma\right) dt. = \mu - 2\frac{\sigma e^\beta}{\left(\beta+1\right)\beta^{\frac{1}{2}}}\left(\Gamma(\frac{5}{2},\beta) - \Gamma(\frac{5}{2},\beta\left(\frac{M}{\sigma}\right)^2)\right).$$

$$\begin{aligned} D\left(\mu\right) &= 2\mu G_{SLP}\left(\mu;\beta,\sigma\right) - \int_\sigma^\mu t\, g_{SLP}\left(t;\beta,\sigma\right) dt, \\ &= 2\mu G_{SLP}\left(\mu;\beta,\sigma\right) - \frac{\sigma e^\beta}{\left(\beta+1\right)\beta^{\frac{1}{2}}}\left(\Gamma(\frac{5}{2},\beta) - \Gamma(\frac{5}{2},\beta\left(\frac{\mu}{\sigma}\right)^2)\right). \end{aligned}$$

2.6. **The Rényi and Shannon Entropy.** The Rényi entropy of a random variable X of the Square Lindley Pareto distribution (SLP) is a measure of the uncertainty's variation (see [17]), is defined as follows

$$I_R\left(s\right) = \frac{1}{1-s}\ln\left\{\int_0^\infty g_{SLP}^s\left(t;\beta,\sigma\right) dt\right\}$$

where s(integer) > 0 ans $s \neq 1$, we have

$$I_R(s) = \frac{1}{1-s}\ln\left(\frac{2^s\beta^{2s}e^{s\beta}}{\beta^{\frac{1-s}{2}}s^{\frac{3s+1}{2}}(\beta+1)^s\sigma^{s-1}}\frac{\Gamma\left(\frac{3s+1}{2},\beta s\right)}{2}\right).$$

For a random variable T with density $g_{SLP}(t;\beta,\sigma)$, the Shannon entropy (see [19]) is defined as follows $E\{-\ln(g_{SLP}(T;\beta,\sigma))\}$, are given by

$$E\{-\ln(g_{SLP}(T;\beta,\sigma))\} = \ln 2 + 2\ln\beta + \beta - \ln(\beta+1) - 4\ln\sigma + 4E(\ln t) - \frac{\beta}{\sigma^2}E\left(t^2\right),$$

$$E\{-\ln(g_{SLP}(T;\beta,\sigma))\} = \beta + \ln 2 + 2\ln\beta - \ln(\beta+1) + \frac{2\left(1+Ei(\beta)e^{-\beta}\right)-\left(\beta^2+2\beta+2\right)}{(\beta+1)}.$$

where, *Ei* is the exponential integral function.

2.7. **Quantile function.** The quantile function T of the SLP distribution is

$$t_\gamma = \sigma\left(-\frac{1}{\beta}-\frac{1}{\beta}\mathrm{LAMBERTW(T)}\left(-1,(\gamma-1)(\beta+1)e^{-\beta-1}\right)\right)^{\frac{1}{2}},\ 0<\gamma<1, \tag{6}$$

where $\beta,\sigma>0$ and $\mathrm{LAMBERTW(T)}$ denotes the negative branch of the $\mathrm{LAMBERTW(T)}$ function ($W(z)\exp(W(z)) = z$, where z is a complex number).

2.8. **Extreme order statistics of SLP.** Let $T_1, T_2, ..., T_n$ a sample of n random variables that follows the SLP distribution, we can derive the asymptotic law of extreme $\min(T_1, T_2, ..., T_n)$, by using the theorem 8.3.6 of Arlond and al (1989) [3]. For the cumulative distribution function (CDF) of SLP distribution, we find that the asymptotic distribution of $T_{\overline{1,n}}$ is Weibull type with shape paramater $\beta > 0$

$$\lim_{x\to 0}\frac{G_{SLP}(xt;\beta,\sigma)}{G_{SLP}(x;\beta,\sigma)} = t^4$$

and

$$\lim_{x\to\infty}\frac{1-G_{SLP}(x+t;\beta,\sigma)}{1-G_{SLP}(x;\beta,\sigma)} = e^{-\beta\left(\frac{t}{\sigma}\right)^2}.$$

3. Estimation of SLPD Parameters

This section presents five methods to estimate the parameters β and σ of the Square Lindley Pareto Distribution (SLPD).

Let $T_1, T_2, \ldots, T_n$ be a random sample from the SLPD.

3.1. **Maximum Likelihood Estimation (MLE).** The likelihood function is:

$$L(\beta,\sigma) = \prod_{i=1}^{n} g_{\mathrm{SLP}}(t_i;\beta,\sigma),$$

and the log-likelihood function becomes:

$$\ln L(\beta,\sigma) = n\ln 2 + 2n\ln\beta + n\beta + 3\sum_{i=1}^{n}\ln t_i \quad - n\ln(\beta+1) - 4n\ln\sigma - \beta\sum_{i=1}^{n}\left(\frac{t_i}{\sigma}\right)^2. \tag{7}$$

Since this expression is nonlinear, numerical methods such as Newton-Raphson or BFGS are used to find the MLEs of β and σ.

3.2. **Anderson-Darling Estimation (AD).** Let $G(t;\beta,\sigma)$ be the CDF of the SLPD. The Anderson-Darling estimator minimizes:

$$A(\beta,\sigma) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log G(t_i) + \log(1 - G(t_{n+1-i}))\right]. \tag{8}$$

3.3. **Cramér–von Mises Estimation (CVM).** The CVM estimator minimizes:

$$C(\beta,\sigma) = -\frac{1}{12n} + \sum_{i=1}^{n}\left[G(t_i) - \frac{2i-1}{2n}\right]^2. \tag{9}$$

3.4. **Maximum Product of Spacings (MPS).** Define spacings as $M_i = G(t_{(i)}) - G(t_{(i-1)})$, where $t_{(0)} = \sigma$. The MPS estimator maximizes:

$$T(\beta,\sigma) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log M_i. \tag{10}$$

3.5. **Ordinary Least Squares Estimation (OLS).** The OLS method minimizes the squared difference between theoretical and empirical CDF values:

$$S(\beta,\sigma) = \sum_{i=1}^{n}\left[G(t_i) - \frac{i}{n+1}\right]^2. \tag{11}$$

All the methods require evaluating the cumulative distribution function $G(t;\beta,\sigma)$, which may not have a closed form. In such cases, numerical integration can be employed to approximate it.

4. Numerical Simulation

This portion of the study undertakes an examination to assess the efficacy of various estimation methods detailed in the previous section. Random datasets were created utilizing our suggested model, after which these estimation techniques were utilized to obtain the model estimators $\hat{\theta} = \left(\hat{\beta},\hat{\sigma}\right)$. The evaluation of estimation method efficacy in this investigation is based on three specific metrics, delineated as follows:

(1) The average of absolute bias (BIAS):

$$\left|\text{bias}\left(\hat{\theta}\right)\right| = \frac{1}{N}\sum_{i=1}^{N}\left|\left(\hat{\theta}_i - \theta_0\right)\right|.$$

(2) The mean squared error (MSE):

$$\text{MSE}\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_0\right)^2.$$

(3) The mean absolute relative error (MRE):

$$\mathrm{MRE}\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|\hat{\theta}_i - \theta_0\right|}{\theta_0}.$$

Another aim of this simulation investigation is to pinpoint the most effective estimation approach for computing our suggested model estimators. The simulation encompasses creating random samples varying in size from our model, establishing the metrics we employ, and iterating this procedure numerous times.

Tables 1 to 4 exhibit our simulation results. The data from these tables is further illustrated in Figures 1 to 4. The significance of each datum is determined by its rank compared to all estimation methods. These tables showcase the outcomes of simulating the proposed model parameters using five distinct estimation techniques. Key observations from these tables encompass the following:

(1) It is notable that all parameter estimation methods for the proposed model exhibit high reliability and precisely approximate their true values.

(2) The computed metrics consistently decrease across all examined scenarios as the sample size (n) increases.

(3) Each estimation technique excels in determining the parameters of the proposed model.

(4) According to our analysis, the MLE estimation method emerges as the most efficient method for evaluating the parameter values.

ID	Sample Size	Method	$\hat{\beta}$	$\hat{\sigma}$	Bias($\hat{\beta}$)	Bias($\hat{\sigma}$)	MSE($\hat{\beta}$)	MSE($\hat{\sigma}$)	MRE($\hat{\beta}$)	MRE($\hat{\sigma}$)
1	25	MLE	1.097588	0.561775	0.116155	0.061904	0.044421	0.006513	0.116155	0.123807
2	50		1.055018	0.533456	0.085970	0.033459	0.037186	0.003251	0.085970	0.066918
3	100		1.001820	0.511687	0.039754	0.011719	0.010858	0.001015	0.039754	0.023438
4	200		0.975457	0.502545	0.027205	0.002548	0.004052	0.000141	0.027205	0.005096
5	300		0.974142	0.501241	0.026527	0.001245	0.003924	0.000041	0.026527	0.002491
6	500		0.975098	0.500479	0.024902	0.000499	0.003791	0.000001	0.024902	0.000997
7	25	AD	1.112852	0.538120	0.256492	0.044357	0.116703	0.003364	0.256492	0.088714
8	50		1.009273	0.523491	0.185575	0.027001	0.058040	0.001971	0.185575	0.054002
9	100		0.927460	0.508970	0.141032	0.010653	0.031406	0.000675	0.141032	0.021306
10	200		0.892308	0.500827	0.109668	0.001483	0.018376	0.000027	0.109668	0.002966
11	300		0.887857	0.500658	0.113439	0.000964	0.018904	0.000015	0.113439	0.001928
12	500		0.896368	0.500328	0.103632	0.000509	0.017283	0.000001	0.103632	0.001019
13	25	CVM	1.176991	0.548667	0.267903	0.052386	0.128729	0.003622	0.267903	0.104773
14	50		1.124405	0.546970	0.185131	0.047988	0.057395	0.002818	0.185131	0.095976
15	100		1.115588	0.547236	0.143018	0.047255	0.032103	0.002500	0.143018	0.094510
16	200		1.113715	0.547112	0.119716	0.047112	0.020436	0.002332	0.119716	0.094224
17	300		1.110993	0.547323	0.116112	0.047323	0.018320	0.002333	0.116112	0.094645
18	500		1.109979	0.547113	0.110750	0.047113	0.015374	0.002270	0.110750	0.094225
19	25	MPS	1.099033	0.536943	0.320630	0.060434	0.150635	0.004782	0.320630	0.120869
20	50		0.994494	0.522012	0.260768	0.037102	0.094423	0.002893	0.260768	0.074204
21	100		0.897365	0.505987	0.189812	0.013784	0.046553	0.000978	0.189812	0.027567
22	200		0.859987	0.498509	0.143862	0.002425	0.023362	0.000043	0.143862	0.004850
23	300		0.869651	0.498812	0.130349	0.001387	0.018895	0.000003	0.130349	0.002774
24	500		0.874030	0.499324	0.125970	0.000804	0.018133	0.000001	0.125970	0.001608
25	25	OLS	1.090656	0.539277	0.225355	0.046697	0.091706	0.003083	0.225355	0.093394
26	50		1.102556	0.542448	0.172729	0.044398	0.050041	0.002483	0.172729	0.088796
27	100		1.104359	0.545958	0.135529	0.045984	0.029573	0.002408	0.135529	0.091968
28	200		1.105048	0.545558	0.117004	0.045558	0.020007	0.002217	0.117004	0.091116
29	300		1.097919	0.545419	0.103104	0.045419	0.015263	0.002149	0.103104	0.090838
30	500		1.107614	0.546670	0.108865	0.046670	0.014708	0.002230	0.108865	0.093339

TABLE 1. Simulation values of Estimates, Bias, MSE, and MRE for ($\beta = 1, \sigma = 0.5$).

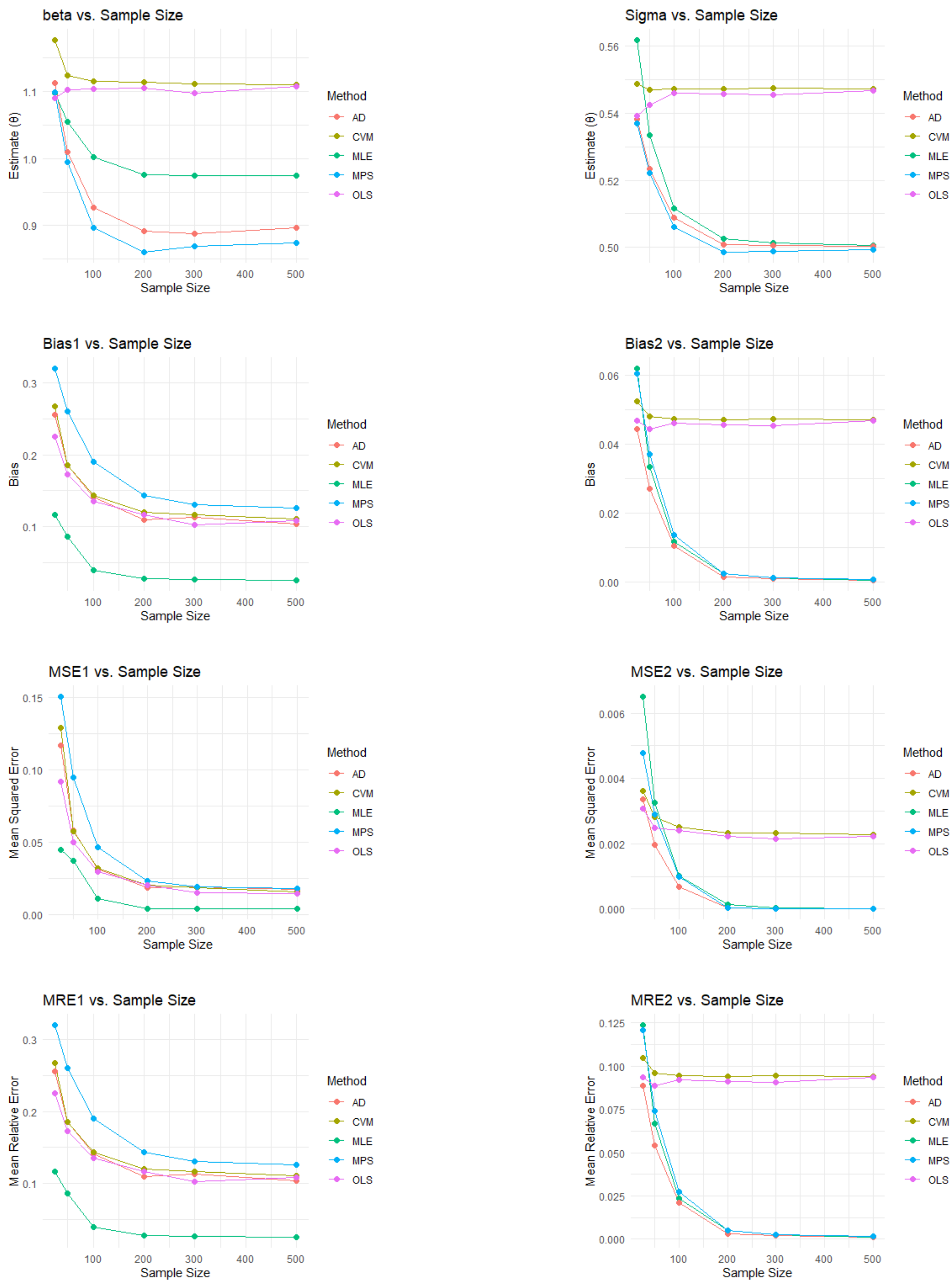

FIGURE 1. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 1.

ID	Sample Size	Method	$\hat{\beta}$	$\hat{\sigma}$	Bias($\hat{\beta}$)	Bias($\hat{\sigma}$)	MSE($\hat{\beta}$)	MSE($\hat{\sigma}$)	MRE($\hat{\beta}$)	MRE($\hat{\sigma}$)
1	25	MLE	1.034335	1.039839	0.050323	0.039839	0.010736	0.002924	0.050323	0.039839
2	50	MLE	1.015740	1.019630	0.029145	0.019630	0.003808	0.000728	0.029145	0.019630
3	100	MLE	1.004983	1.009635	0.013513	0.009635	0.001290	0.000175	0.013513	0.009635
4	200	MLE	1.001977	1.004720	0.007288	0.004720	0.000665	0.000045	0.007288	0.004720
5	300	MLE	1.000418	1.003483	0.005275	0.003483	0.000266	0.000023	0.005275	0.003483
6	500	MLE	0.998692	1.002053	0.003723	0.002053	0.000142	0.000008	0.003723	0.002053
7	25	AD	1.002800	0.990940	0.176663	0.038382	0.053540	0.003176	0.176663	0.038382
8	50	AD	0.997430	0.994431	0.109971	0.022239	0.019498	0.001042	0.109971	0.022239
9	100	AD	0.984974	0.995598	0.073428	0.013415	0.008199	0.000360	0.073428	0.013415
10	200	AD	0.991640	0.997379	0.047437	0.007487	0.003650	0.000113	0.047437	0.007487
11	300	AD	0.996207	0.997648	0.038605	0.005466	0.002358	0.000060	0.038605	0.005466
12	500	AD	0.992648	0.998101	0.030959	0.003680	0.001541	0.000031	0.030959	0.003680
13	25	CVM	1.041175	1.002267	0.202042	0.055059	0.063675	0.005308	0.202042	0.055059
14	50	CVM	1.022654	0.999876	0.142564	0.035026	0.033141	0.002051	0.142564	0.035026
15	100	CVM	1.005616	1.000571	0.091259	0.024765	0.013144	0.000980	0.091259	0.024765
16	200	CVM	1.012246	1.001854	0.069356	0.017494	0.007439	0.000504	0.069356	0.017494
17	300	CVM	1.002199	0.999982	0.054238	0.014364	0.004434	0.000327	0.054238	0.014364
18	500	CVM	1.002803	1.001019	0.040311	0.010898	0.002603	0.000183	0.040311	0.010898
19	25	MPS	0.973629	0.992331	0.161412	0.030693	0.041461	0.001499	0.161412	0.030693
20	50	MPS	0.980091	0.999239	0.094191	0.015358	0.014050	0.000418	0.094191	0.015358
21	100	MPS	0.987207	0.999334	0.064342	0.007372	0.006611	0.000096	0.064342	0.007372
22	200	MPS	0.996103	1.000093	0.045998	0.003923	0.003328	0.000030	0.045998	0.003923
23	300	MPS	0.995387	0.999872	0.036468	0.002495	0.002111	0.000011	0.036468	0.002495
24	500	MPS	0.997366	1.000018	0.027990	0.001485	0.001201	0.000004	0.027990	0.001485
25	25	OLS	0.960056	0.970950	0.221779	0.068442	0.080275	0.009828	0.221779	0.068442
26	50	OLS	0.984724	0.986922	0.146983	0.038944	0.033244	0.002547	0.146983	0.038944
27	100	OLS	0.985285	0.994099	0.100253	0.028900	0.015863	0.001385	0.100253	0.028900
28	200	OLS	0.993286	0.995984	0.068934	0.017674	0.007208	0.000490	0.068934	0.017674
29	300	OLS	1.001008	0.999295	0.054667	0.014644	0.004781	0.000339	0.054667	0.014644
30	500	OLS	1.000008	0.998337	0.041535	0.011310	0.002716	0.000202	0.041535	0.011310

TABLE 2. Simulation values of Estimates, Bias, MSE, and MRE for $(\beta = 1, \sigma = 1)$.

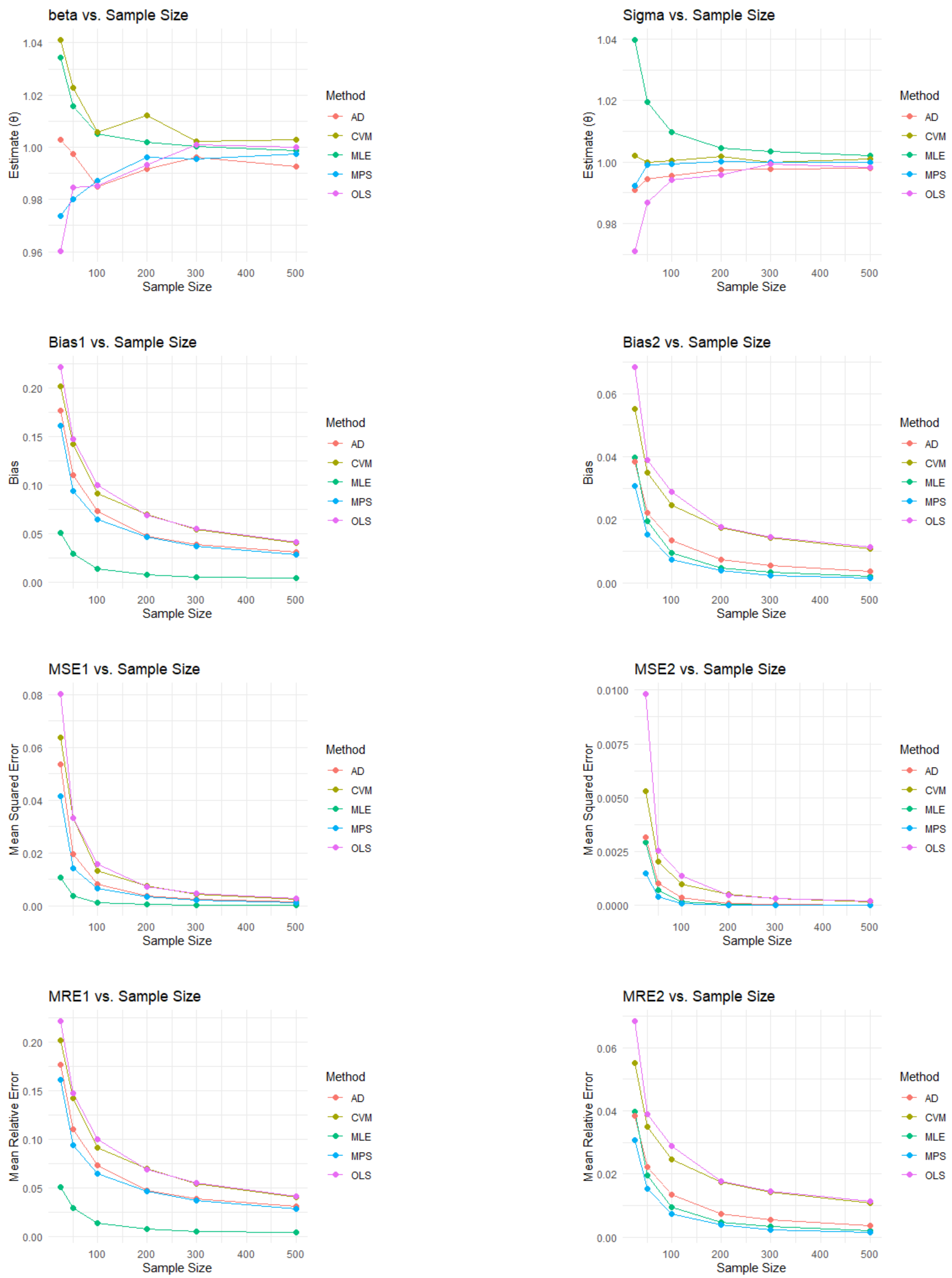

FIGURE 2. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 2.

ID	Sample Size	Method	$\hat{\beta}$	$\hat{\sigma}$	Bias($\hat{\beta}$)	Bias($\hat{\sigma}$)	MSE($\hat{\beta}$)	MSE($\hat{\sigma}$)	MRE($\hat{\beta}$)	MRE($\hat{\sigma}$)
1	25	MLE	0.621555	1.097660	0.129266	0.097792	0.030738	0.016548	0.258532	0.097792
2	50	MLE	0.559822	1.053205	0.066056	0.053205	0.008324	0.004951	0.132113	0.053205
3	100	MLE	0.529772	1.028686	0.035465	0.028686	0.002672	0.001566	0.070931	0.028686
4	200	MLE	0.511720	1.013942	0.017139	0.013942	0.000763	0.000375	0.034277	0.013942
5	300	MLE	0.506698	1.009320	0.010723	0.009320	0.000354	0.000161	0.021445	0.009320
6	500	MLE	0.503687	1.005956	0.006418	0.005956	0.000138	0.000067	0.012837	0.005956
7	25	AD	0.479364	0.942131	0.149854	0.145488	0.040365	0.058396	0.299708	0.145488
8	50	AD	0.483579	0.975047	0.091330	0.076793	0.015441	0.016320	0.182661	0.076793
9	100	AD	0.480303	0.976682	0.057350	0.045021	0.006599	0.006939	0.114700	0.045021
10	200	AD	0.486041	0.985466	0.039371	0.027085	0.002662	0.001579	0.078742	0.027085
11	300	AD	0.489752	0.991088	0.026047	0.017488	0.001178	0.000682	0.052095	0.017488
12	500	AD	0.492341	0.993572	0.020305	0.012086	0.000716	0.000360	0.040610	0.012086
13	25	CVM	0.517579	0.949061	0.205708	0.202654	0.068824	0.090939	0.411417	0.202654
14	50	CVM	0.491399	0.964048	0.133507	0.126993	0.030898	0.038328	0.267014	0.126993
15	100	CVM	0.509684	1.001791	0.090748	0.076484	0.013271	0.009468	0.181496	0.076484
16	200	CVM	0.496481	0.989456	0.066535	0.055831	0.006996	0.005444	0.133070	0.055831
17	300	CVM	0.497430	0.997197	0.053981	0.044813	0.004545	0.003257	0.107963	0.044813
18	500	CVM	0.500502	0.997335	0.041804	0.035550	0.002922	0.002127	0.083608	0.035550
19	25	MPS	0.478059	0.975952	0.120439	0.091072	0.022296	0.013022	0.240878	0.091072
20	50	MPS	0.500098	1.000798	0.071100	0.046770	0.008421	0.003609	0.142200	0.046770
21	100	MPS	0.491645	0.996825	0.040119	0.022621	0.002603	0.000859	0.080238	0.022621
22	200	MPS	0.498224	0.999419	0.023252	0.010663	0.000852	0.000192	0.046503	0.010663
23	300	MPS	0.498212	0.999569	0.019056	0.007443	0.000582	0.000100	0.038113	0.007443
24	500	MPS	0.497872	0.999892	0.014921	0.004551	0.000347	0.000037	0.029842	0.004551
25	25	OLS	0.434538	0.850843	0.221440	0.255426	0.074748	0.136779	0.442881	0.255426
26	50	OLS	0.460852	0.927225	0.148291	0.153857	0.036281	0.055255	0.296583	0.153857
27	100	OLS	0.486808	0.976041	0.097797	0.090415	0.016230	0.017398	0.195594	0.090415
28	200	OLS	0.491066	0.987051	0.063998	0.055909	0.006640	0.005349	0.127996	0.055909
29	300	OLS	0.493502	0.991571	0.055952	0.048374	0.004934	0.003743	0.111903	0.048374
30	500	OLS	0.491665	0.990860	0.043925	0.037635	0.003059	0.002348	0.087851	0.037635

TABLE 3. Simulation values of Estimates, Bias, MSE, and MRE for $(\beta = 0.5, \sigma = 1)$.

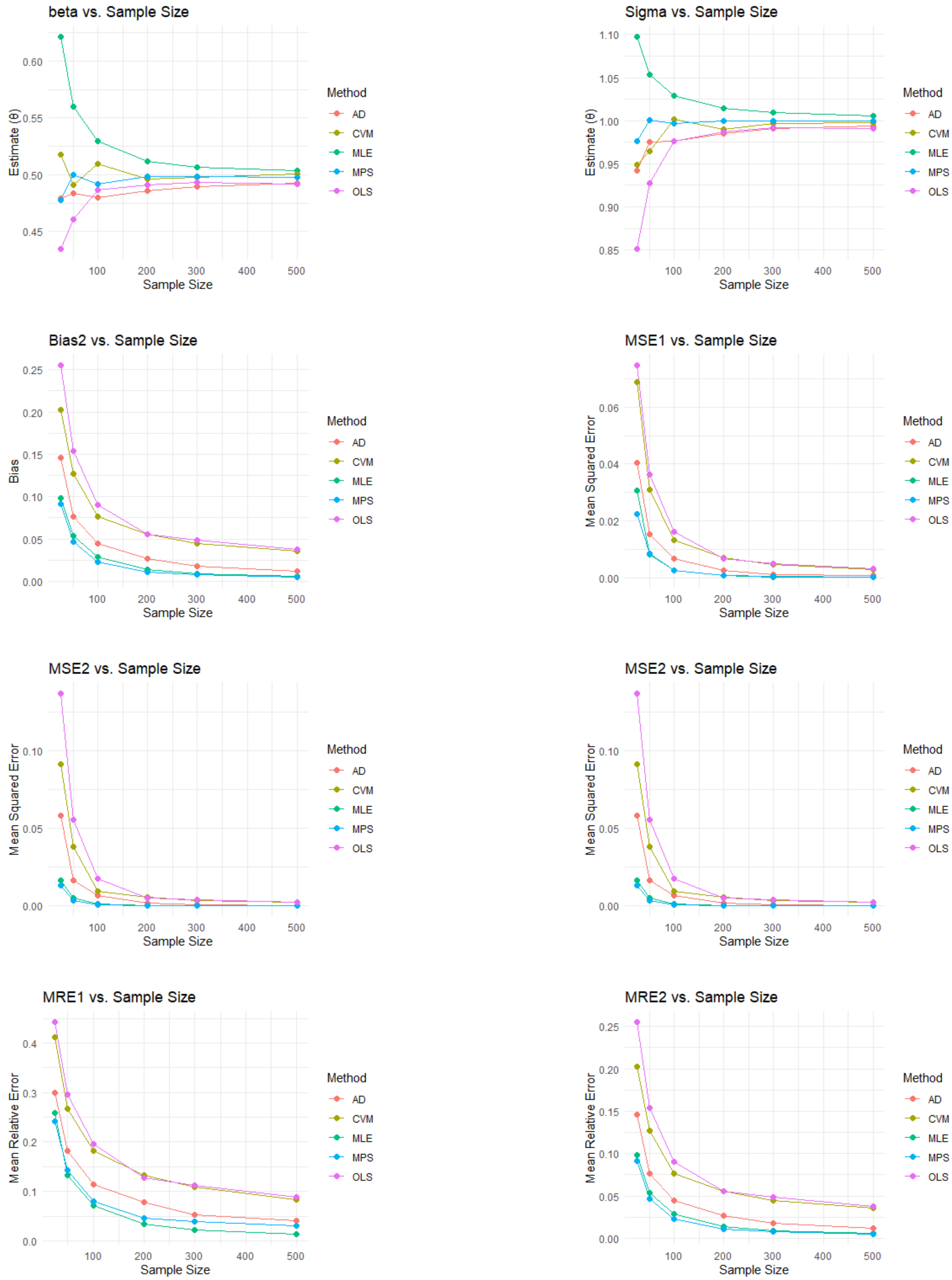

FIGURE 3. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 3.

ID	Sample Size	Method	$\hat{\beta}$	$\hat{\sigma}$	Bias($\hat{\beta}$)	Bias($\hat{\sigma}$)	MSE($\hat{\beta}$)	MSE($\hat{\sigma}$)	MRE($\hat{\beta}$)	MRE($\hat{\sigma}$)
1	25	MLE	2.035448	3.043246	0.061169	0.043246	0.033675	0.003769	0.030585	0.014415
2	50	MLE	2.013317	3.021934	0.028012	0.021934	0.008871	0.001000	0.014006	0.007311
3	100	MLE	2.004523	3.011444	0.023387	0.011444	0.004330	0.000267	0.011694	0.003815
4	200	MLE	2.003375	3.005331	0.015242	0.005335	0.002005	0.000058	0.007621	0.001778
5	300	MLE	1.998975	3.003764	0.009626	0.003764	0.001055	0.000030	0.004813	0.001255
6	500	MLE	2.001430	3.002330	0.008036	0.002330	0.000564	0.000010	0.004018	0.000777
7	25	AD	2.035141	2.994035	0.303887	0.041565	0.148781	0.003247	0.151943	0.013855
8	50	AD	1.992560	2.993065	0.196201	0.025008	0.061209	0.001129	0.098100	0.008336
9	100	AD	2.004769	2.996415	0.133863	0.013763	0.028562	0.000381	0.066931	0.004588
10	200	AD	1.993854	2.996057	0.093479	0.008157	0.013924	0.000153	0.046740	0.002719
11	300	AD	1.989677	2.996541	0.077633	0.006489	0.009954	0.000090	0.038817	0.002163
12	500	AD	1.997432	2.997241	0.054945	0.004378	0.005034	0.000046	0.027473	0.001459
13	25	CVM	2.068099	3.007935	0.343654	0.063163	0.205670	0.006551	0.171827	0.021054
14	50	CVM	2.049305	3.007114	0.239706	0.043983	0.095696	0.003128	0.119853	0.014661
15	100	CVM	2.011693	2.999175	0.159131	0.029581	0.040411	0.001386	0.079565	0.009860
16	200	CVM	2.011266	3.001588	0.114016	0.021417	0.021043	0.000727	0.057008	0.007139
17	300	CVM	2.010742	2.999875	0.090640	0.016535	0.013602	0.000435	0.045320	0.005512
18	500	CVM	2.006246	3.000559	0.071517	0.012712	0.008247	0.000249	0.035758	0.004237
19	25	MPS	1.939203	2.991624	0.282350	0.034681	0.124329	0.002049	0.141175	0.011560
20	50	MPS	1.961935	2.997947	0.191737	0.016719	0.058957	0.000467	0.095869	0.005573
21	100	MPS	1.988185	3.000096	0.128631	0.008649	0.028098	0.000133	0.064315	0.002883
22	200	MPS	1.993837	3.000052	0.090475	0.004404	0.012818	0.000034	0.045238	0.001468
23	300	MPS	1.984277	2.999917	0.071088	0.003028	0.007961	0.000018	0.035544	0.001009
24	500	MPS	1.995524	3.000001	0.055894	0.001674	0.005010	0.000005	0.027947	0.000558
25	25	OLS	1.946748	2.974978	0.331790	0.067163	0.166924	0.007422	0.165895	0.022388
26	50	OLS	1.968614	2.984940	0.238444	0.045947	0.088106	0.003407	0.119222	0.015316
27	100	OLS	1.991005	2.994482	0.164958	0.030967	0.041323	0.001516	0.082479	0.010322
28	200	OLS	1.991396	2.996579	0.106842	0.019926	0.018554	0.000624	0.053421	0.006642
29	300	OLS	2.000653	2.998053	0.087635	0.016981	0.011906	0.000454	0.043818	0.005660
30	500	OLS	1.996876	2.998794	0.070669	0.013373	0.008120	0.000273	0.035334	0.004458

TABLE 4. Simulation values of Estimates, Bias, MSE, and MRE for $(\beta = 2, \sigma = 3)$.

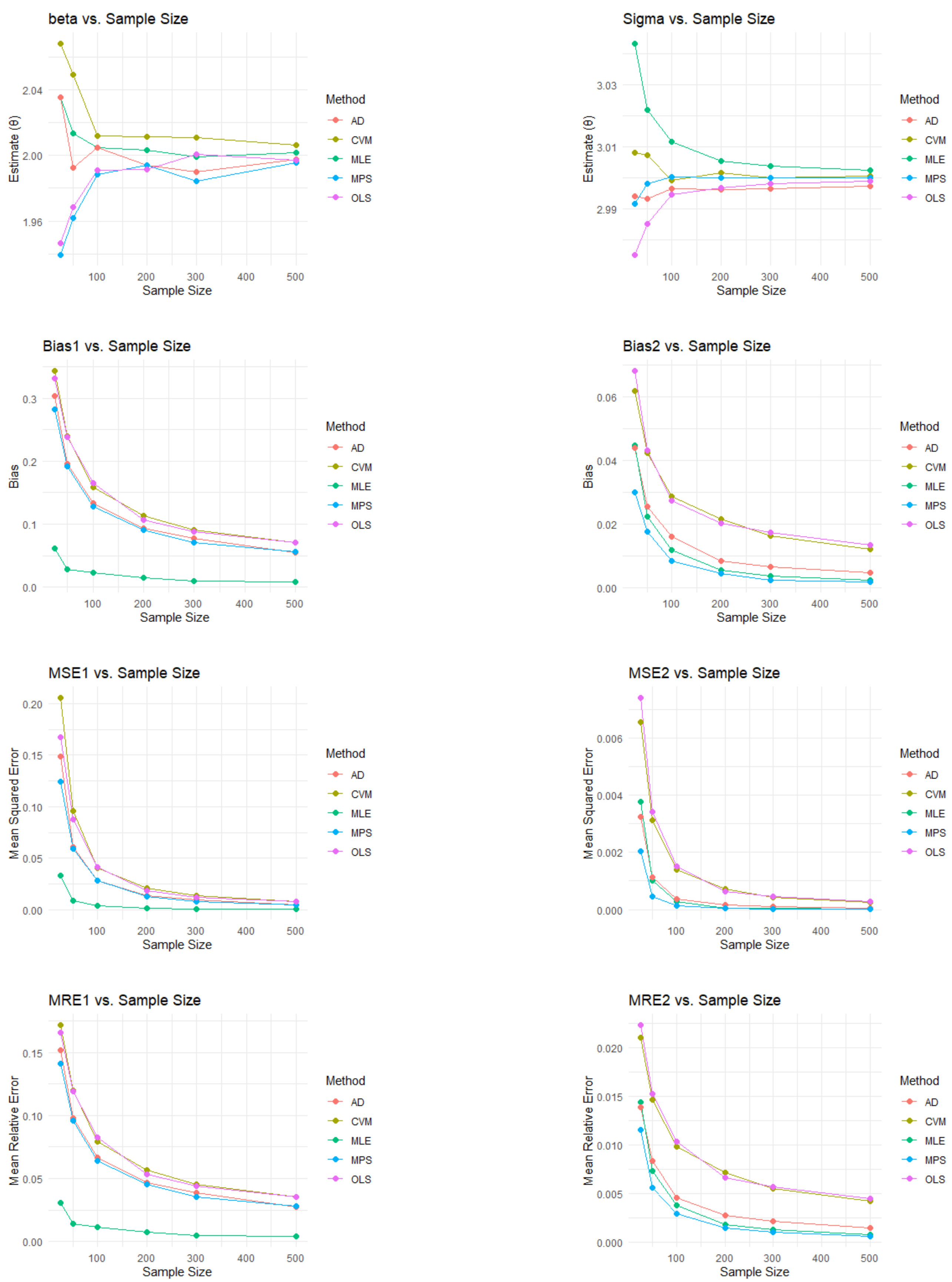

FIGURE 4. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 4.

5. Applications with Real Data

In this section, we have considered four real datasets to assess the proposed model's capability of goodness-of-fit and adequacy of the model. We illustrate that the (SLP) distribution can be a better distribution than the two-parameter L1 [20], gamma Lindley [16,27], quasi Lindley [21], new quasi Lindley [22], two parameter L2 [23], Xlindley [5], and Power XLindley distribution [29], by using these four real datasets. And to compare several distributions, we consider criteria like AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2L (-2Log-Likelihood), and AICC (Consistent Akaike Information Criterion) for the data set. The better distribution corresponds to smaller AIC, BIC, -2L, and AICC values.

Dataset 1: Luteinizing Hormone in Blood Samples. A regular time series giving the luteinizing hormone in blood samples at 10-minute intervals from a human female (48 samples) used by Diggle (1990) [6].

Data:

2.4, 2.4, 2.4, 2.2, 2.1, 1.5, 2.3, 2.3, 2.5, 2.0, 1.9, 1.7, 2.2, 1.8, 3.2, 3.2, 2.7, 2.2, 2.2, 1.9, 1.9, 1.8, 2.7, 3.0, 2.3, 2.0, 2.0, 2.9, 2.9, 2.7, 2.7, 2.3, 2.6, 2.4, 1.8, 1.7, 1.5, 1.4, 2.1, 3.3, 3.5, 3.5, 3.1, 2.6, 2.1, 3.4, 3.0, 2.9

Model	θ	γ	AIC	BIC	-2L	AICC
Two-parameter L1	0.8546036	8.39×10^{-5}	149.545	153.2874	145.545	149.8117
Gamma Lindley	0.8314911	42.62868	149.9641	153.7065	145.9641	150.2308
Quasi Lindley	1.377216	7.19×10^{-4}	147.6536	151.396	143.6536	147.9202
New Quasi Lindley	0.8248596	34.76987	150.3554	154.0978	146.3554	150.6221
Two-parameter L2	0.8281973	42.72697	150.345	154.0874	146.345	150.6116
X-Lindley	0.5772491	/	174.4943	176.3655	172.4943	174.5812
Power X-Lindley	1.599186	0.4308425	291.5391	295.2815	287.5391	291.8058
SLP	β: 0.7498583	σ: 1.399183	81.61614	85.35854	77.61614	81.88281

Table 5. Parameter estimates and the statistics AIC, BIC, -2L, AICC for luteinizing hormone in blood samples data

Dataset 2: US State Facts and Figures. Data related to the 50 states of the United States of America, used by Bureau of the Census (1977) [25].

Data:

32.5901, 49.2500, 34.2192, 34.7336, 36.5341, 38.6777, 41.5928, 38.6777, 27.8744, 32.3329, 31.7500, 43.5648, 40.0495, 40.0495, 41.9358, 38.4204, 37.3915, 30.6181, 45.6226, 39.2778, 42.3645, 43.1361, 46.3943, 32.6758, 38.3347, 46.8230, 41.3356, 39.1063, 43.3934, 39.9637, 34.4764, 43.1361, 35.4195, 47.2517, 40.2210, 35.5053, 43.9078, 40.9069, 41.5928, 33.6190, 44.3365, 35.6767, 31.3897, 39.1063, 44.2508, 37.5630, 47.4231, 38.4204, 44.5937, 43.0504

Model	θ	γ	AIC	BIC	-2L	AICC
Two-parameter L1	0.05075296	8.42×10^{-4}	433.5861	437.4102	429.5861	433.8415
Gamma Lindley	0.05077601	23.78275	433.6841	437.5081	429.6841	433.9394
Quasi Lindley	0.06982123	7.99×10^{-4}	711.172	714.996	707.172	711.4273
New Quasi Lindley	0.05073988	6.256798	433.6043	437.4283	429.6043	433.8596
Two-parameter L2	0.05060517	23.06109	433.6922	437.5163	429.6922	433.9476
X-Lindley	0.04843964	/	436.1108	438.0229	434.1108	436.1942
Power X-Lindley	0.9310586	0.2315416	629.4108	633.2349	625.4108	629.6661
SLP	β: 1.36138	σ: 27.87356	330.4731	334.2971	326.4731	330.7284

Table 6. Parameter estimates and the statistics AIC, BIC, -2L, AICC for US State Facts and Figures data

Dataset 3: DDT in Kale. We consider a numeric vector of 15 measurements by different laboratories of the pesticide DDT in kale, in ppm, using the multiple pesticide residue measurement, these data were analyzed by many authors (see Finsterwalder (1976) [7], Staudte and Sheather (1990) [24]).

Data:

2.79, 2.93, 3.22, 3.78, 3.22, 3.38, 3.18, 3.33, 3.34, 3.06, 3.07, 3.56, 3.08, 4.64, 3.34

Dataset 4: Percentage of Shrimp in Shrimp Cocktail. We consider a numeric vector with 18 determinations by different laboratories of the amount (percentage of declared total weight) of shrimp in shrimp cocktail, used by King and Ryan (1976) [12], Staudte and Sheather (1990) [24] .

Data:

32.2, 33.0, 30.8, 33.8, 32.2, 33.3, 31.7, 35.7, 32.4, 31.2, 26.6, 30.7, 32.5, 30.7, 31.2, 30.3, 32.3, 31.7

Model	θ	γ	AIC	BIC	-2L	AICC
Two-parameter L1	0.6223911	8.55×10^{-4}	58.74148	60.15758	54.74148	59.74148
Gamma Lindley	0.5985468	20.45021	58.96757	60.38367	54.96757	59.96757
Quasi Lindley	0.8977403	6.94×10^{-4}	67.49544	68.91155	63.49544	68.49544
New Quasi Lindley	0.5933078	24.60633	58.90761	60.32371	54.90761	59.90761
Two-parameter L2	0.5959023	23.3598	59.06532	60.48142	55.06532	60.06532
X-Lindley	0.4402955	/	64.57162	65.27967	62.57162	64.87932
Power X-Lindley	1.449654	0.3811038	106.8118	108.2279	102.8118	107.8118
SLP	β: 2.812591	σ: 2.789801	14.48242	15.89852	10.48242	15.48242

Table 7. Parameter estimates and the statistics AIC, BIC, -2L, AICC for DDT in Kale data

Model	θ	γ	AIC	BIC	-2L	AICC
Two-parameter L1	0.0630501	6.03×10^{-4}	150.688	152.4687	146.688	151.488
Gamma Lindley	0.06262739	12.50669	150.7721	152.5528	146.7721	151.5721
Quasi Lindley	0.09715586	5.32×10^{-4}	242.5376	244.3184	238.5376	243.3376
New Quasi Lindley	0.06283553	5.838002	150.6993	152.48	146.6993	151.4993
Two-parameter L2	0.06287901	13.97741	150.7675	152.5483	146.7675	151.5675
X-Lindley	0.05943647	/	150.6912	151.5816	148.6912	150.9412
Power X-Lindley	0.9594836	0.2377622	221.0102	222.7909	217.0102	221.8102
SLP	β: 2.886044	σ: 26.59915	95.64276	97.42351	91.64276	96.44276

Table 8. Parameter estimates and the statistics AIC, BIC, -2L, AICC for percentage of Shrimp in Shrimp Cocktail data

According to Tables 5,6,7,8, we can observe that SLP distribution provide smallest AIC, BIC, -2L, AICC values as compared to two-parameter L1, gamma Lindley, quasi Lindley, new quasi Lindley, two parameter L2, Xlindley, and Power XLindley distributions, and hence best fits the data among all the models considered.

6. Conclusion

This article introduces and investigates a new lifetime distribution, the Square Lindley Pareto (SLP) distribution. We explored its key statistical properties, including the probability density function, cumulative distribution function, moment-generating function, quantile function, moments, mean deviation, Rényi and Shannon entropy, stress-strength reliability, and order statistics.

A Monte Carlo simulation study was performed to assess the estimation accuracy of five different methods for estimating the unknown parameters. The simulation results, based on metrics such as average absolute bias, mean squared error, and mean absolute relative error, indicate that the maximum likelihood estimation (MLE) method is the most efficient and reliable approach.

Furthermore, a comparative goodness-of-fit analysis was conducted using real data (window strength data), where the SLP distribution was evaluated against several competing two-parameter models, including the L1, gamma Lindley, quasi Lindley, new quasi Lindley, L2, XLindley, and Power-XLindley distributions. Based on model selection criteria such as the $-2\log$-likelihood, AIC, AICC, and BIC, the SLP distribution consistently provided the best fit to the data.

In summary, the Square Lindley Pareto distribution offers significant flexibility and modeling power, making it a valuable addition to the family of lifetime distributions. We recommend its application in various domains such as medical statistics, reliability engineering, and actuarial science, where modeling skewed or heavy-tailed data is essential.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.A. Al-Babtain, I. Elbatal, E.M. Almetwally, Bayesian and Non-Bayesian Reliability Estimation of Stress-Strength Model for Power-Modified Lindley Distribution, Comput. Intell. Neurosci. 2022 (2022), 1154705. https://doi.org/10.1155/2022/1154705.

[2] A. Alzaatreh, C. Lee, F. Famoye, A New Method for Generating Families of Continuous Distributions, Metron 71 (2013), 63–79. https://doi.org/10.1007/s40300-013-0007-y.

[3] B.C. Arnold, Relations, Bounds and Approximations for Order Statistics, Springer New York, 1989.

[4] A. Asgharzadeh, H.S. Bakouch, L. Esmaeili, Pareto Poisson–lindley Distribution with Applications, J. Appl. Stat. 40 (2013), 1717–1734. https://doi.org/10.1080/02664763.2013.793886.

[5] S. Chouia, H. Zeghdoudi, The Xlindley Distribution: Properties and Application, J. Stat. Theory Appl. 20 (2021), 318–327. https://doi.org/10.2991/jsta.d.210607.001.

[6] P.J. Diggle, Time Series, Oxford University Press, Oxford, 1990. https://doi.org/https://doi.org/10.1093/oso/9780198522065.001.0001.

[7] C.E. Finsterwalder, Collaborative Study of an Extension of the Mills et Al. Method for the Determination of Pesticide Residues in Foods, J. AOAC Int. 59 (1976), 169–171. https://doi.org/10.1093/jaoac/59.1.169.

[8] M. Ghitany, D. Al-Mutairi, S. Nadarajah, Zero-truncated Poisson–lindley Distribution and Its Application, Math. Comput. Simul. 79 (2008), 279–287. https://doi.org/10.1016/j.matcom.2007.11.021.

[9] M. Ghitany, B. Atieh, S. Nadarajah, Lindley Distribution and Its Application, Math. Comput. Simul. 78 (2008), 493–506. https://doi.org/10.1016/j.matcom.2007.06.007.

[10] F.S. Gomes-Silva, A. Percontini, E.D. Brito, M.W. Ramos, R. Venâncio, G.M. Cordeiro, The Odd Lindley-G Family of Distributions, Austrian J. Stat. 46 (2017), 65–87. https://doi.org/10.17713/ajs.v46i1.222.

[11] M. Hussein, G.M. Rodrigues, E.M.M. Ortega, R. Vila, H. Elsayed, A New Truncated Lindley-Generated Family of Distributions: Properties, Regression Analysis, and Applications, Entropy 25 (2023), 1359. https://doi.org/10.3390/e25091359.

[12] F.J. King, J.J. Ryan, Collaborative Study of the Determination of the Amount of Shrimp in Shrimp Cocktail, J. AOAC Int. 59 (1976), 644–649. https://doi.org/10.1093/jaoac/59.3.644.

[13] D.V. Lindley, Fiducial Distributions and Bayes' Theorem, J. R. Stat. Soc. Ser. B: Stat. Methodol. 20 (1958), 102–107. https://doi.org/10.1111/j.2517-6161.1958.tb00278.x.

[14] J. Mazucheli, F. Louzada, M. Ghitany, Comparison of Estimation Methods for the Parameters of the Weighted Lindley Distribution, Appl. Math. Comput. 220 (2013), 463–471. https://doi.org/10.1016/j.amc.2013.05.082.

[15] F. Merovci, I. Elbatal, Transmuted Lindley-Geometric Distribution and Its Applications, J. Stat. Appl. Probab. 3 (2014), 77–91. https://doi.org/10.12785/jsap/030107.

[16] S. Nedjar, H. Zeghdoudi, On Gamma Lindley Distribution: Properties and Simulations, J. Comput. Appl. Math. 298 (2016), 167–174. https://doi.org/10.1016/j.cam.2015.11.047.

[17] A. Rényi, On Measures of Entropy and Information, in: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, pp. 547–561, 1961.

[18] M. Sankaran, 275. Note: the Discrete Poisson-Lindley Distribution, Biometrics 26 (1970), 145–149. https://doi.org/10.2307/2529053.

[19] C.E. Shannon, A Mathematical Theory of Communication, Bell Syst. Tech. J. 27 (1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x.

[20] R. Shanker, A. Mishra, A Two-Parameter Lindley Distribution, Stat. Transit. New Ser. 14 (2013), 45–56. https://doi.org/10.59170/stattrans-2013-003.

[21] R. Shanker, A. Mishra, A Quasi Lindley Distribution, Afr. J. Math. Comput. Sci. Res. 6 (2013), 64–71. https://doi.10.5897/AJMCSR12.067.

[22] R. Shanker, A.H. Ghebretsadik, A New Quasi Lindley Distribution, Int. J. Stat. Syst. 8 (2013), 143–156.

[23] R. Shanker, S. Sharma, R. Shanker, A Two-Parameter Lindley Distribution for Modeling Waiting and Survival Times Data, Appl. Math. 04 (2013), 363–368. https://doi.org/10.4236/am.2013.42056.

[24] R.G. Staudte, S.J. Sheather, Robust Estimation & Testing, John Wiley & Sons, Inc., Hoboken, 1990. https://doi.org/10.1002/9781118165485.

[25] U.S. Bureau of the Census, Statistical Abstract of the United States: 1977. (98th edition.) Washington, D.C., 1977.

[26] H. Zakerzadah, A. Dolati, Generalized Lindley Distribution, J. Math. Ext. 3 (2010), 13–25.

[27] S. Nedjar, H. Zeghdoudi, Gamma Lindley Distribution and Its Application, J. Appl. Probab. Stat. 11 (2016), 129–138.

[28] N. Lazri, H. Zeghdoudi, On Lindley-Pareto Distribution: Properties and Application, GSTF J. Math. Stat. Oper. Res. 3 (2016), 1–7.

[29] B. Meriem, A.M. Gemeay, E.M. Almetwally, Z. Halim, E. Alshawarbeh, A.T. Abdulrahman, M.M.A. El-Raouf, E. Hussam, The Power Xlindley Distribution: Statistical Inference, Fuzzy Reliability, and COVID-19 Application, J. Funct. Spaces 2022 (2022), 9094078. https://doi.org/10.1155/2022/9094078.