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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A HIGHER-ORDER SINGULAR BOUNDARY VALUE PROBLEM WITH A NONLOCAL CONDITION

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Abstract. This study addresses the analysis of a mixed-type singular differential problem exhibiting an n-order singularity at the origin. The problem is governed by Dirichlet boundary conditions and includes an integral constraint with variable limits. We begin by formulating the appropriate functional framework to rigorously investigate the problem. A two-sided a priori estimate is derived based on energy inequality techniques. To establish the existence of a solution, we demonstrate the density of the image of the associated operator, employing regularization operators as a key analytical tool.

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1. Introduction

Nonlocal boundary value problems have attracted significant attention in mathematical analysis due to their extensive applications in physics, engineering, and applied sciences. Unlike classical boundary value problems, which impose conditions at specific points, nonlocal problems involve boundary conditions dependent on integral constraints or solution values over a region. This nonlocality introduces additional mathematical challenges, particularly in establishing existence and uniqueness results.

In recent years, numerous physical phenomena have been modeled using nonlocal mathematical formulations with integral boundary conditions. These conditions arise in scenarios where direct measurements of data at specific points are impractical, but average values over a domain can be obtained. For instance, in some applications, describing the solution $u = \frac{1}{2}$ (e.g., pressure or temperature) at individual points is unfeasible, whereas only its average value along a boundary or a subregion is accessible. Such models appear in various engineering fields, including heat conduction, plasma

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physics, thermoelasticity, electrochemistry, chemical diffusion, and groundwater flow. The significance of nonlocal problems has been highlighted in works such as Samarski [1].

The first study addressing second-order partial differential equations with nonlocal integral conditions was conducted by Cannon [10]. Since then, extensive research has been devoted to boundary value problems involving mixed Dirichlet, Neumann, and integral conditions. Such problems have been studied in the context of parabolic equations Denche and Marhoune, [13], Marhoune [7], Marhoune and Ameur [8], Abdelkader Djerad et al [2,3] hyperbolic equations, and mixed-type equations. Additionally, elliptic equations with nonlocal operator conditions have been explored by Mikhailov and Gushin [5] and Skubachevski and Steblov [6].

In this study, we investigate the existence and uniqueness of solutions to a nonlocal boundary value problem governed by a differential equation with prescribed integral constraints. Analyzing such problems requires specialized techniques, including a priori estimates, functional analysis methods, and regularization techniques, to address potential singularities and ensure well-posedness.

Our approach begins by formulating the problem within an appropriate functional framework. We derive necessary conditions and establish a priori bounds to control solution behavior. Existence and uniqueness are then proved using density arguments and properties of the associated operator. By leveraging energy inequalities and regularization methods, we demonstrate that the problem admits a unique solution under suitable conditions.

The results presented in this work contribute to the theoretical development of nonlocal boundary value problems and highlight their applications across scientific and engineering disciplines. Furthermore, the techniques developed herein may be extended to broader classes of nonlocal problems, enriching the study of differential equations with integral constraints.

2. Position of Problem

In the rectangle $\Omega = (0,T) \times (0,k)$, we consider the equation:

$$Lx = \frac{\partial x}{\partial t} - \frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n \frac{\partial x}{\partial u}\right) = F\left(t,x\right) \tag{2.1}$$

We associate the initial condition with equation (2.1)

$$x(0,u) = \varphi(u), \quad u \in (0,k) \tag{2.2}$$

and the Dirichlet boundary condition:

$$x(0,k) = 0 \tag{2.3}$$

and the integral condition:

$$\int_{k_1}^{k} x(t,\xi)\, d\xi = 0, \quad 0 \le k_1 \le k, \quad t \in (0,T) \tag{2.4}$$

where the function φ satisfies the following compatibility conditions:

$$\varphi(k) = 0, \quad \int_{k_1}^{k} \varphi(u)\, du = 0. \tag{2.5}$$

To study the problem posed, we need the following functional spaces: Let E be the Banach space of functions $x \in L_2(\Omega)$ verifying conditions (2.3) and (2.4) and equipped with the norm:

$$\|x\|_E^2 = \int_\Omega \theta(u) \left|\frac{\partial x}{\partial t}\right|^2 du\, dt + \int_\Omega \frac{\theta(u)}{u^n} \left|\frac{\partial}{\partial u}\left(u^n \frac{\partial x}{\partial u}\right)\right| du + \sup_{0 \le t \le T} \int_0^k \varphi(u) \left|\frac{\partial u}{\partial t}\right|^2 du. \tag{2.6}$$

Furthermore, let F be the Hilbert space of vector functions $F = (f, \varphi)$, obtained as the completion of the space $L_2(\Omega) \times W_2^2(0,k)$ with respect to the norm:

$$\|F\|_E^2 = \int_\Omega \theta(u) |f(t,u)|^2 du\, dt + \int_0^k \theta(u) \left|\frac{\partial \varphi}{\partial u}\right| du. \tag{2.7}$$

where $\theta(u)$ is defined as:

$$\theta(u) = \begin{cases} k_1 u^n, & \text{if } 0 \le u \le k_1, \\ u^{n+1}, & \text{if } k_1 \le u \le k. \end{cases} \tag{2.8}$$

To the problem (2.1)-(2.4), we associate the operator $\pounds = (L, l)$ defined from E to F.

Definition 2.1. *We call a solution of problem* (2.1)-(2.4) *any solution of the operational equation:*

$$Lx = F. \tag{2.9}$$

Definition 2.2. *The Cauchy Inequality* (ε*-inequality*) *For any* $(a,b) \in \mathbb{R}^2$*, the Cauchy inequality states:*

$$\operatorname{Re}(a,b) \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2. \tag{2.10}$$

3. Priori Estimation

Theorem 3.1. *For each function* $x \in E$*, we have the following a priori estimate:*

$$\|Lx\|_F \le C \|x\|_E. \tag{3.1}$$

where C *is a constant independent of* x.

Proof. From equation (2.1), we have:

$$|Lx|^2 \le 2\left[\left|\frac{\partial x}{\partial t}\right|^2 + \frac{1}{u^{2n}}\left|\frac{\partial}{\partial u}\left(u^n \frac{\partial x}{\partial u}\right)\right|^2\right]. \tag{3.2}$$

Multiplying both sides by $\theta(u)$, we obtain:

$$\theta(u)|Lx|^2 \le 2\theta(u)\left[\left|\frac{\partial x}{\partial t}\right|^2 + \frac{\theta(u)}{u^{2n}}\left|\frac{\partial}{\partial u}\left(u^n \frac{\partial x}{\partial u}\right)\right|^2\right]. \tag{3.3}$$

Integrating over Ω, we get:

$$\int_{\Omega} \theta(u)|Lx|^2 du\, dt \leq \int_{\Omega} 2\theta(u) \left[\left| \frac{\partial x}{\partial t} \right|^2 + \frac{\theta(u)}{u^{2n}} \left| \frac{\partial}{\partial u} \left(u^n \frac{\partial x}{\partial u} \right) \right|^2 \right] du\, dt. \tag{3.4}$$

On the other hand, we have:

$$\int_0^k \theta(u) \left| \frac{\partial lx\,(t,u)}{\partial u} \right|^2 du = \int_0^k \theta(u) \left| \frac{\partial x(0,u)}{\partial u} \right| du \leq \sup_{0\leq t\leq T} \int_0^k \varphi(u) \left| \frac{\partial x}{\partial u} \right|^2 du. \tag{3.5}$$

By adding (3.4) and (3.5), we obtain:

$$\begin{aligned} \int_{\Omega} \theta(u)|f(t,u)|^2 du\, dt + \int_0^k \theta(u) \left| \frac{\partial lx}{\partial t} \right|^2 du \quad &\leq \quad 2\int_{\Omega} \theta(u) \left| \frac{\partial x}{\partial t} \right|^2 du\, dt \\ &+ \quad \int_{\Omega} \frac{\theta(u)}{u^n} \left| \frac{\partial}{\partial u} \left(u^n \frac{\partial x}{\partial u} \right) \right|^2 du\, dt \\ &+ \quad \sup_{0\leq t\leq T} \int_0^k \varphi(u) \left| \frac{\partial x}{\partial u} \right|^2 du. \end{aligned} \tag{3.6}$$

From this, we conclude that:

$$\|Lx\| \leq C\|x\|_E. \tag{3.7}$$

Thus, the theorem is proven. $\square$

Theorem 3.2. *For any function $x \in E$, we have the estimate:*

$$\|x\|_E \leq C\|Lx\|_F. \tag{3.8}$$

Proof. We define:

$$D(L) = \left\{ x \in E : u^n \frac{\partial^2 x\,(t,u)}{\partial u\, \partial t} \in L_2(\Omega) \right\}. \tag{3.9}$$

Let:

$$Ju = \int_u^k x(t,\xi) d\xi. \tag{3.10}$$

And define Mx as:

$$Mx = \begin{cases} k_1 u^n \frac{\partial x(t,u)}{\partial t}, & 0 < u \leq k_1, \\ \\ u^{n+1} \frac{\partial x(t,u)}{\partial t} + u^n J\left(\frac{\partial x(t,u)}{\partial t}\right), & k_1 < u \leq k. \end{cases} \tag{3.11}$$

Multiplying equation (2.1) by $\overline{Mx}$ and integrating over $\Omega^\tau = (0,\tau) \times (0,k)$, where $0 < \tau \leq T$, with respect to u, and then taking the real part, we obtain:

$$\operatorname{Re} \int_0^\tau \int_0^k (Lu)\, \overline{Mx}\, du\, dt = \int_0^\tau \int_0^k \left(\frac{\partial x}{\partial t} \right) \overline{Mx}\, du\, dt - \int_0^\tau \int_0^k \left(\frac{1}{u^n} \frac{\partial}{\partial u} \left(u^n \frac{\partial x}{\partial u} \right) \right) \overline{Mx}\, du\, dt. \tag{3.12}$$

By integrating by parts each term on the right-hand side of (3.12) and using conditions (2.1)-(2.4), we obtain:

$$\operatorname{Re}\int_0^\tau\int_0^k\left(\frac{\partial x(t,u)}{\partial t}\right)\overline{Mx}\,du\,dt = \int_0^\tau\int_0^k k_1u^n\left|\frac{\partial x}{\partial t}\right|^2du\,dt+\int_0^\tau\int_0^k u^{n+1}\left|\frac{\partial x}{\partial t}\right|^2du\,dt + \int_0^\tau\int_{k_1}^k u^n\left(\frac{\partial x}{\partial t}\right)J\left(\frac{\partial x}{\partial t}\right)du\,dt. \tag{3.13}$$

Thus, we can rewrite this as:

$$\operatorname{Re}\int_0^\tau\int_0^k\left(\frac{\partial x}{\partial t}\right)\overline{Mx}\,du\,dt=\operatorname{Re}\int_0^\tau\int_0^k\theta(x)\left|\frac{\partial x}{\partial t}\right|^2du\,dt+\int_0^\tau\int_{k_1}^k u^n\left(\frac{\partial x}{\partial t}\right)J\left(\frac{\partial x}{\partial t}\right)du\,dt. \tag{3.14}$$

By integrating the second term on the right-hand side over Ω^τ, we get:

$$\operatorname{Re}\int_0^\tau\int_{k_1}^k u^n\frac{\partial x}{\partial t}J\frac{\partial x}{\partial t}du\,dt=\frac{n}{2}\int_0^\tau\int_{k_1}^k u^{n-1}\left|J\frac{\partial x}{\partial t}\right|^2du\,dt. \tag{3.15}$$

From this, we obtain:

$$\operatorname{Re}\int_0^\tau\int_0^k\left(\frac{\partial x}{\partial t}\right)\overline{Mx}\,du\,dt=\operatorname{Re}\int_0^\tau\int_0^k\theta(u)\left|\frac{\partial x}{\partial t}\right|^2du\,dt+\frac{n}{2}\int_0^\tau\int_{k_1}^k u^{n-1}\left|J\frac{\partial x}{\partial t}\right|^2du\,dt. \tag{3.16}$$

We have

$$-\operatorname{Re}\int_0^\tau\int_0^k\frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\overline{Mx}\,du\,dt=-\int_0^\tau\int_0^{k_1}k_1\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial t}du\,dt$$

$$-\int_0^\tau\int_{k_1}^k u\left(\frac{\partial}{\partial u}u^n\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial t}du\,dt-\int_0^\tau\int_{k_1}^k\frac{\partial}{\partial u}(u^n\frac{\partial x}{\partial u})J\frac{\partial x}{\partial t}du\,dt. \tag{3.17}$$

Integrating each term on the right-hand side by parts, we obtain:

$$-\int_0^\tau\int_0^{k_1}k_1\frac{\partial}{\partial u}(u^n\frac{\partial x}{\partial u})\frac{\partial x}{\partial t}du\,dt=\int_0^\tau\int_0^{k_1}k_1u^n\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial^2 x}{\partial u\partial t}\right)du\,dt. \tag{3.18}$$

$$-\int_0^\tau\int_{k_1}^k u\left(\frac{\partial}{\partial u}u^n\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial t}du\,dt=\int_0^\tau\int_{k_1}^k u^{n+1}\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\partial t}du\,dt+\int_0^\tau\int_{k_1}^k u^n\frac{\partial x}{\partial u}\frac{\partial x}{\partial t}du\,dt. \tag{3.19}$$

and

$$-\int_0^\tau\int_{k_1}^k\frac{\partial}{\partial u}(u^n\frac{\partial x}{\partial u})J\frac{\partial x}{\partial t}du\,dt=-\int_0^\tau\int_{k_1}^k u^n\frac{\partial x}{\partial u}\frac{\partial x}{\partial t}du\,dt. \tag{3.20}$$

By adding (3.18), (3.19), and (3.20), we obtain:

$$-\operatorname{Re}\int_0^\tau\int_0^k\frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\overline{Mx}\,du\,dt=\operatorname{Re}\int_0^\tau\int_0^k\theta(u)\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\partial t}du\,dt. \tag{3.21}$$

By adding (3.17) and (3.21) again, we obtain:

$$\operatorname{Re}\int_0^\tau\int_0^k Lx\,\overline{Mx}\,du\,dt = \int_0^\tau\int_0^k\theta(u)\left|\frac{\partial x}{\partial t}\right|^2du\,dt + \frac{n}{2}\int_0^\tau\int_{k_1}^k u^{n-1}\left|J\frac{\partial x}{\partial t}\right|^2du\,dt + \int_0^\tau\int_0^k\theta(u)\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\partial t}du\,dt. \tag{3.22}$$

By integrating the last term and replacing its expression in (3.21), we obtain:

$$\begin{aligned}\operatorname{Re}\int_0^\tau\int_0^k Lx\,\overline{Mx}\,du\,dt+\frac{1}{2}\int_0^k\theta(u)\left|\frac{\partial x(0,u)}{\partial u}\right|^2 du &= \int_0^\tau\int_0^k\theta(u)\left|\frac{\partial x}{\partial t}\right|^2 du\,dt\\ &+\frac{n}{2}\int_0^\tau\int_{l_1}^k u^{n-1}\left|J\frac{\partial x}{\partial t}\right|^2 du\,dt\\ &+\frac{1}{2}\int_0^k\theta(u)\left|\frac{\partial x(\tau,u)}{\partial u}\right|^2 du. \qquad (3.23)\end{aligned}$$

By replacing the expression of $\overline{Mx}$ in the term $\operatorname{Re}\int_0^\tau\int_0^k Lx\,\overline{Mx}\,du\,dt$, we get:

$$\begin{aligned}\operatorname{Re}\int_0^\tau\int_0^k Lx\,\overline{Mx}\,du\,dt &= \int_0^\tau\int_0^{k_1} k_1u^n\frac{\partial x}{\partial t}\,Lx(t,u)\,du\,dt\\ &+\int_0^\tau\int_{k_1}^k u^{n+1}\frac{\partial x}{\partial t}\,Lx(t,u)\,du\,dt\\ &+\int_0^\tau\int_{k_1}^k u^n Lx\,J\frac{\partial x}{\partial t}\,du\,dt. \qquad (3.24)\end{aligned}$$

Using the ε inequality in each term, taking the values of ε respectively ($\varepsilon=2,2,$ and $\varepsilon=1/n$), and adding the three inequalities, we find:

$$\begin{aligned}\operatorname{Re}\int_0^\tau\int_0^k Lx\,\overline{Mx}\,du\,dt &\le \frac{2n+1}{2n}\int_0^\tau\int_{k_1}^k\theta(u)|Lx|^2\,du\,dt\\ &+\frac{1}{4}\int_0^\tau\int_{k_1}^k\theta(u)\left|\frac{\partial x}{\partial t}\right|^2 du\,dt\\ &+\frac{n}{2}\int_0^\tau\int_{k_1}^k u^{n-1}\left|\frac{\partial x}{\partial t}\right|^2 du\,dt. \qquad (3.25)\end{aligned}$$

Using the relation:

$$\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)=u^n\frac{\partial x}{\partial t}-u^nLx\,(t,u)\,,$$

and applying the ε-inequality, we obtain:

$$\left|\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\right|^2\le 2\left(u^{2n}|Lx|^2+u^{2n}\left|\frac{\partial x}{\partial t}\right|^2\right).$$

Thus, we have:

$$\frac{\theta(u)}{4u^{2n}}\left|\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\right|^2\le\frac{\theta(u)}{2}|Lx|^2+\frac{\theta(u)}{2}\left|\frac{\partial x}{\partial t}\right|^2.$$

By integrating over the domain, it follows that:

$$\begin{aligned}\frac{1}{4}\int_0^\tau\int_0^k\theta(u)u^n\left|\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\right|^2 dudt &\le \frac{1}{2}\int_0^\tau\int_0^k\theta(u)|Lx|^2dudt\\ &+\frac{1}{2}\int_0^\tau\int_0^k\theta(u)\left|\frac{\partial x}{\partial t}\right|^2 dudt. \qquad (3.26)\end{aligned}$$

By combining equations (3.24) and (3.25), we obtain:

$$\frac{3}{4}\int_0^\tau\int_0^k \theta(u)\left|\frac{\partial x}{\partial t}\right|^2 dudt+\frac{1}{2}\int_0^k\theta(u)\left|\frac{\partial x(\tau,u)}{\partial u}\right|^2 du \leq \frac{(2n+1)}{2n}\int_0^\tau\int_{k_1}^k \theta(u)|Lx|^2 dudt + \frac{1}{2}\int_0^k \theta(u)\left|\frac{\partial lx}{\partial u}\right|^2 du. \quad (3.27)$$

Adding equation (3.26) to the above inequality, we get:

$$\begin{aligned}&\frac{3}{4}\int_0^\tau\int_0^k \theta(u)\left|\frac{\partial x}{\partial t}\right|^2 dudt+\frac{1}{2}\int_0^k\theta(u)\left|\frac{\partial x(\tau,u)}{\partial u}\right|^2 du\\ &+\frac{1}{4}\int_0^\tau\int_0^k \theta(u)u^n\left|\frac{\partial}{\partial u}\left(u^n\frac{\partial x}{\partial u}\right)\right|^2 dudt\\ \leq\ &\frac{(3n+1)}{2n}\int_0^\tau\int_{k_1}^k\theta(u)|Lx|^2dudt+\frac{1}{2}\int_0^k\theta(u)\left|\frac{\partial lx}{\partial u}\right|^2 du. \quad (3.28)\end{aligned}$$

As the right-hand side is independent of τ, taking the supremum with respect to τ yields:

$$\|x\|_E \leq C\|Lx\|_F.$$

Thus, the theorem is proven. □

4. Solvability of Problem

From estimates (3.1) and (3.8), we see that the operator

$$L : E \to F$$

is continuous and that its image is closed in F. Since the inverse operator L^{-1} exists and is continuous from $R(L)$ to E, it follows that L is a homeomorphism from E to $R(L)$. To establish the existence of the solution, it therefore suffices to show that:

$$R(L) = F.$$

The proof relies on the following lemma:

Lemma 4.1. *Let*

$$D_0(L) = \{x \in D(L) \mid Lx = 0\}.$$

If

$$\int_\Omega \theta(u)\, Lx\, \overline{w}\, du\, dt = 0 \quad (4.1)$$

for all $x \in D_0(L)$ and for some w such that $u^n w \in L_2(\Omega)$, then we must have $w = 0$.

Proof. Posing in (4.1), let $h = \frac{\partial x(t,u)}{\partial t}$, where $h, u^n\frac{\partial x(t,u)}{\partial t}, u^n\frac{\partial x(t,u)}{\partial u} \in L_2(\Omega)$, and h satisfies the boundary conditions (2.3) and (2.4). Then,

$$\int_\Omega \theta(u)\frac{\partial x}{\partial t}\bar{w}\, du\, dt = \int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n\frac{\partial x\,(t,u)}{\partial u}\right)\bar{w}\, du\, dt. \quad (4.2)$$

Again, posing in (4.2) with $x = J_t^\star h = \int_0^t h(\tau, u)\,d\tau$, we have:

$$\int_\Omega \theta(u) h \bar{w}\,du\,dt = \int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u} J_t^\star h\right)\bar{w}\,du\,dt. \tag{4.3}$$

Integrating by parts the right-hand side and setting $J_t w = \int_0^T h(\tau, u)\,d\tau$, we obtain $\frac{\partial}{\partial t} J_t w = -w(t, u)$. Thus,

$$\int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u} J_t^\star h\right)\bar{w}\,du\,dt = \int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial h}{\partial u}\right)\bar{J_t} w\,du\,dt. \tag{4.4}$$

Subsequently, by writing

$$\frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial x}{\partial u}\right) = \frac{1}{u^{2n}}\theta(u)\left[\frac{\partial}{\partial u}u^n \frac{\partial}{\partial u}(u^n h) - \frac{2n}{u}\frac{\partial}{\partial u}(u^n h) + \frac{n}{u^2}(u^n h)\right]. \tag{4.5}$$

Then equation (4.4) becomes, we have the equation:

$$\begin{aligned}\int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u}(J_t^\star h)\right)\bar{w}\,du\,dt &= \int_\Omega \frac{1}{u^{2n}}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u}(u^n h)\right)dudt \\ &- \int_\Omega \frac{2n}{u^{2n+1}}\theta(u)\frac{\partial}{\partial u}(u^n h)dudt \\ &+ \int_\Omega \frac{n}{u^{2n+2}}\theta(u)(u^n h)dudt.\end{aligned} \tag{4.6}$$

The left side of equation (4.4) shows that the application

$$u^n h \to \int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial h}{\partial u}\right) J_t w\,du\,dt$$

is a continuous linear function. From the right-hand side of equation (4.6), we conclude that the function w satisfies the following properties:

$$\frac{\theta(u)}{u^{2n+2}} J_t w, \quad \frac{\theta(u)}{u^{2n+1}}\frac{\partial}{\partial u}(J_t w), \quad J_t w, \quad \frac{\theta(u)}{u^{2n}}\frac{\partial}{\partial u}\left[u^n\left(\frac{\partial}{\partial u} J_t w\right)\right] \in L_2(\Omega).$$

From equation (4.4), we obtain:

$$\begin{aligned}\int_\Omega \theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial h}{\partial u}\right) J_t w\,du\,dt &= \int_0^\tau \int_{k_1}^k \frac{\partial}{\partial u}\left(k_1 u^n \frac{\partial h}{\partial u}\right) J_t w\,du\,dt \\ &+ \int_0^\tau \int_{k_1}^k \frac{\partial}{\partial u}\left(u^{n+1}\frac{\partial h}{\partial u}\right) J_t w\,du\,dt.\end{aligned} \tag{4.7}$$

We have:

$$\begin{aligned}\int_\Omega \theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial h}{\partial u}\right) J_t w\,du\,dt &= \int_0^\tau \int_{k_1}^k \frac{\partial}{\partial u}\left(k_1 u^n \frac{\partial h}{\partial u}\right) J_t w\,du\,dt \\ &+ \int_0^\tau \int_{k_1}^k \frac{\partial}{\partial u}\left(u^{n+1}\frac{\partial h}{\partial u}\right) J_t w\left[J_t w - \int_u^k \frac{J_t w}{\zeta}d\zeta\right]dudt.\end{aligned} \tag{4.8}$$

We define:

$$y(t, u) = \begin{cases} J_t w, & 0 < u \le k_1, \\ J_t w - \int_u^k \frac{J_t w}{\zeta}d\zeta, & k_1 \le u < k. \end{cases}$$

Then, we have:

$$\int_u^k y(t,\zeta)d\zeta = \int_u^k J_t w(t,\zeta)d\zeta - \int_u^k d\zeta \left[\int_\zeta^k \frac{J_t w(t,\mu)}{\mu} d\mu\right] = u\int_u^k \frac{J_t w(t,\zeta)}{\zeta} d\zeta.$$

We can write $\theta(u)y(t,u)$ in the following form:

$$\theta(u)y(t,u) = \begin{cases} k_1 u^n J_t w, & 0 < u \leq k_1, \\ \\ u^{n+1}\left(J_t w - u^{n+1}\int_u^k \frac{J_t w(t,\zeta)}{\zeta} d\zeta\right), & k_1 \leq u < k. \end{cases}$$

So,

$$My = \theta(u)y(t,u) = \begin{cases} k_1 u^n y(t,u), & 0 < u \leq k_1, \\ \\ u^{n+1}\left(y(t,u)J_t w - u^n \int_u^k v(t,\zeta)d\zeta\right), & k_1 \leq u < k. \end{cases}$$

And,

$$\int_{k_1}^k y(t,u)du = 0,$$

since

$$u(y(t,u) - J_t w) = u\int_u^l \frac{J_t w(t,\zeta)}{\zeta} d\zeta, \quad \text{for } k_1 \leq u < k.$$

For $u = k_1$, we have $J_t w(t,k_1) = y(t,k_1)$. We define $h = J_t^\star y = \int_0^t y(\tau,u)d\tau$, and using (4.8), we obtain:

$$\int_\Omega \theta(u)u^n J_t^\star yw \, du \, dt = \int_\Omega \frac{1}{u^n}\theta(u)\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u}(J_t^\star y)\right) J_t w \, du \, dt \tag{4.9}$$

But since $\theta(u)w = -\frac{\partial}{\partial t}(My)$, then (4.9) becomes:

$$\int_\Omega J_t^\star y \frac{\partial}{\partial t}(My) \, du \, dt = \int_\Omega \frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n \frac{\partial}{\partial u}(J_t^\star y)\right) My \, du \, dt \tag{4.10}$$

By doing calculations similar to those in 3.2 , we obtain:

$$-\int_\Omega \frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n \frac{d}{du} J_t^\star y\right) My \, du \, dt = \int_\Omega \theta(u)\frac{\partial}{\partial u}(J_t^\star y)\frac{\partial^2}{\partial u \partial t}(J_t^\star y) \, dy \, dt.$$

Then,

$$-\int_\Omega \frac{1}{u^n}\frac{\partial}{\partial u}\left(u^n \frac{d}{du} J_t^\star y\right) My \, du \, dt = \int_\Omega \theta(u)\frac{\partial}{\partial u}(J_t^\star y)\frac{\partial y}{\partial u} \, du \, dt. \tag{4.11}$$

By integrating the first term of (4.10), we obtain:

$$\int_\Omega J_t^\star y \, \frac{\partial}{\partial t}(My) \, du \, dt = -\operatorname{Re}\int_\Omega y \, \overline{My} \, du \, dt.$$

But

$$\int_\Omega y(My) \, du \, dt = \int_\Omega \theta(u)|y|^2 \, du \, dt + \int_0^\tau \int_{k_1}^k yu^n J_t y \, du \, dt.$$

Furthermore,

$$\int_0^\tau \int_{k_1}^k yu^n J_t y \, du \, dt = \frac{n}{2}\int_0^T \int_{k_1}^k u^{n-1}|J_t y|^2 \, du \, dt.$$

Then,

$$\int_{\Omega} J_t^{\star} y \frac{\partial}{\partial t}(My)\, du\, dt = -\left[\int_{\Omega} \theta(u)|y|^2\, du\, dt + \frac{n}{2}\int_0^T \int_{k_1}^{k} u^{n-1}|J_t y|^2\, du\, dt\right]. \tag{4.12}$$

By integrating the last term of (4.10), we obtain:

$$\int_{\Omega} \theta(u) \frac{\partial}{\partial u}(J_t^{\star} y)\frac{\partial y}{\partial u}\, du\, dt = \frac{1}{2}\int_0^T \int_{k_1}^{k} \theta(u) \left| J_t^{\star} y \frac{\partial y}{\partial u}(T,u)\right|^2 dudt.$$

So (4.10) becomes:

$$\int_{\Omega} \theta(u)|y|^2\, du\, dt + \frac{n}{2}\int_0^T \int_{k_1}^{k} u^{n-1}|J_t y|^2\, du\, dt = \frac{1}{2}\int_0^T \int_{k_1}^{k} \theta(u) \left| J_t^{\star} y \frac{\partial y}{\partial x}(T,u)\right|^2 dudt.$$

That is to say,

$$\int_{\Omega} \theta(u)|y|^2\, du\, dt + \frac{n}{2}\int_0^T \int_{k_1}^{k} u^{n-1}|J_t y|^2\, du\, dt \leq 0.$$

Then we obtain $y = 0$, from which $w = 0$. This completes the proof. □

Theorem 4.2. *The range $\overline{R(L)}$ of $\overline{L}$ coincides with F.*

Proof. Since F is a Hilbert space, we have $R(L) = F$ if and only if

$$\int_{\Omega} \theta(u) Lx\overline{f}\, du\, dt + \int_0^k \theta(u)\left(\frac{\partial lx}{\partial u}\right)\left(\frac{\partial \overline{\phi}}{\partial u}\right) du = 0, \tag{4.13}$$

which implies that $F = (f, \phi) = 0$. By taking $x \in D_0(L)$ and $lx = 0$, we obtain

$$\int_{\Omega} \theta(u) Lx\overline{f}\, du\, dt = 0.$$

From Lemma 4.1, we have $f = 0$, from which equation (4.13), becomes:

$$\int_0^k \theta(u)\left(\frac{\partial lx}{\partial u}\right)\left(\frac{\partial \overline{\phi}}{\partial u}\right) du = 0.$$

Since the image of the trace operator l is dense in the Hilbert space equipped with the norm

$$\left(\int_0^k \theta(u)\left|\frac{\partial \phi}{\partial u}\right|^2 du\right)^{1/2},$$

it follows that $\phi = 0$. Thus, we conclude that $F = 0$. □

5. Conclusion

In this work, we analyze the existence and uniqueness of solutions to the proposed problem by deriving two a priori estimates:

$$\|Lx\|_F \leq C\|x\|_E$$

$$\|x\|_E \leq C\|Lx\|_F$$

From the first estimate, we conclude that the operator $L : E \to F$ is continuous. The second estimate implies that L has a continuous inverse, ensuring that the range $R(L)$ is a closed subset of F. In other words, L defines a linear homeomorphism from E onto its closed range $R(L)$, which establishes the uniqueness of the solution. To complete the proof, it remains to show that $R(L)$ is dense in F, thereby confirming the existence of a solution.

The a priori estimation method is a powerful tool for analyzing a wide range of applied problems in physics. It is built on strong theoretical foundations and has been developed within an elegant abstract framework, making it highly effective in studying the well-posedness of differential equations and boundary value problems.

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