

STRUCTURE OF ALGEBRAS SATISFYING IDENTITY $x^2x^3 = 2\omega(x)^2x^3 - \omega(x)^3x^2$

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Received Aug. 17, 2025

ABSTRACT. The aim of this paper is to study commutative baric algebra (A, ω) with an idempotent and satisfying the identity $x^2x^3 = 2\omega(x)^2x^3 - \omega(x)^3x^2$. We give its Peirce decomposition and the product among Peirce subsets according to the given idempotent. We also give a necessary and sufficient conditions for this algebra to be a normal algebra in one hand, and other hand to be an exclusive algebra. We provide the necessary and sufficient condition for exclusive algebra to be a second-order Bernstein algebra and we prove that the normal algebra is a second-order Bernstein algebra. Finally, we show that if $(\ker \omega)^2$ is one-dimensional, then the given algebra is second-order Bernstein algebra.

2020 Mathematics Subject Classification. 17A30; 17A60.

Key words and phrases. non-associative algebra satisfying a polynomial identity; normal algebra; exclusive algebra.

1. INTRODUCTION

A commutative algebra A over a field K is called n th-order Bernstein algebra if there exists a nontrivial algebras homomorphism $\omega : A \longrightarrow K$ such that

$$x^{[2+n]} - \omega(x)^{2^n} x^{[1+n]} = 0 \text{ for all } x \in A,$$

where $x^{[n]}$ is defined inductively by $x^{[1]} = x$, $x^{[1+k]} = x^{[k]}x^{[k]}$ for $k \in \mathbb{N}$. For $n = 1$, we say that A is a Bernstein algebra and for $n = 2$, we say that it is a second-order Bernstein algebra. Bernstein algebras have their origins in genetics [4]. In 1975, Holgate was the first to translate Bernstein problem into the language of non associative algebra [7]. The principal power of an element x of an algebra is defined by $x^1 = x$ and $x^{i+i} = x^i x$ for a nonzero integer i . In the Bernstein algebra, the following identity holds

$$2x^i x^j = \omega(x)^i x^j + \omega(x)^j x^i \text{ for all } i, j \geq 2 \text{ and } x \in A.$$

The Studying of this identity is difficult in the general cases. However, specifics cases have been investigated in depth, in the recent years (see for instance [2], [3], [8] and [14]). Tenkodogo et al.

study in [10], the structure of commutative algebra satisfying the identity

$$x^2x^3 = (1 - \alpha)\omega(x)^2x^3 + \alpha\omega(x)^3x^2 \text{ for } \alpha \in [0, 1].$$

Our objective is to study a commutative algebra with an idempotent and satisfying this previous identity for $\alpha = -1$, i.e a commutative algebra satisfying the identity

$$x^2x^3 = 2\omega(x)^2x^3 - \omega(x)^3x^2 \text{ for all } x \in A, \quad (1)$$

over a commutative field K of characteristic different from 2, 3. In section 2, we give a Peirce decomposition of this algebra and the relations among Peirce subsets. We also study the link between this algebra and well know class of non associative algebra such as Bernstein algebras and Jordan algebras. In section 3, we investigate normal and exclusive algebra verifying the identity (1). The last section is devoted to study a baric algebra (A, ω) satisfying the identity (1) such that $(\ker \omega)^2$ is one-dimensional.

2. STRUCTURE OF ALGEBRA SATISFYING THE IDENTITY (1)

The class of commutative algebra satisfying the identity (1) with an idempotent is not empty as the example below shows:

Example 2.1. Let A be 4-dimensional algebra over the field K and with the base (e, u, z_1, z_2) which multiplication table is given by $e^2 = e$, $eu = \frac{1}{2}u$, $ez_1 = z_2$, $u^2 = \alpha z_2$, $uz_1 = \beta z_2$. The others products are vanish and $\alpha, \beta \in K$. Let $x = be + \gamma_0u + \gamma_1z_1 + \gamma_2z_2$, we have $x^2 = b^2e + b\gamma_0u + (\alpha\gamma_0^2 + 2b\gamma_1 + 2\gamma_0\gamma_1\beta)z_2$; $x^3 = b^3e + b^2\gamma_0u + (b\alpha\gamma_0^2 + b^2\gamma_1 + b\beta\gamma_0\gamma_1)z_2$ and $x^2x^3 = b^5e + b^4\gamma_0u + b^3\gamma_0^2\alpha z_2 = 2\omega(x)^2x^3 - \omega(x)^3x^2$.

In the remainder of this paper, (A, ω) denotes a baric algebra over a field K and equipped with an idempotent e .

2.1. Structure theorem.

Proposition 2.1. The baric algebra (A, ω) satisfies the identity (1) if and only if for all $x \in \ker(\omega)$ the following relations hold:

- (i) $2e(e(ex)) = e(ex)$;
- (ii) $e(ex^2 + 2(ex)x) + 2(ex)(ex + 2e(ex)) = ex^2 + 4(ex)x - x^2$;
- (iii) $ex^3 + x^2(ex + 2e(ex)) + 2(ex)(ex^2 + 2(ex)x) = 2x^3$;
- (iv) $x^2(ex^2 + 2x(ex)) + 2(ex)x^3 = 0$;
- (v) $x^2x^3 = 0$.

Proof. Suppose that (A, ω) satisfies the identity (1) and let $y = \alpha e + x \in A$ such that $\alpha \in K$ and $x \in \ker(\omega)$. We have

$$\begin{aligned}
y^2 &= \alpha^2 e + 2\alpha ex + x^2 \\
y^3 &= \alpha^3 e + \alpha^2 [2e(ex) + ex] + \alpha [ex^2 + 2(ex)x] + x^3 \\
y^2 y^3 &= \alpha^5 e + \alpha^4 [2e(e(ex)) + 3e(ex)] + \alpha^3 [e(ex^2 + 2(ex)x) + 2(ex)(2e(ex) + ex) + ex^2] + \\
&\quad \alpha^2 [ex^3 + 2(ex)(ex^2 + 2(ex)x) + x^2(ex + 2e(ex))] + \alpha [2x^3(ex) + x^2(ex^2 + 2x(ex))] + x^2 x^3
\end{aligned}$$

By identifying the coefficients of α^i for $0 \leq i \leq 4$, we obtain the relations (i) to (v). Conversely, suppose that the assertions (i) to (v) are satisfied and let $y = \alpha e + x \in A$ such that $\alpha \in K$ and $x \in \ker(\omega)$. By direct calculation, we obtain $0 = y^2 y^3 - 2\omega(y)^2 y^3 + \omega(y)^3 y^2$. It follows that the algebra (A, ω) satisfies the identity (1). $\square$

A partial linearization of the relation (ii) of the Proposition 2.1 gives us, for all $x, y \in \ker(\omega)$,

$$e[e(xy) + (ex)y + (ey)x] + (ey)[ex + 2e(ex)] + (ex)[ey + 2e(ey)] = e(xy) + 2(ey)x + 2(ex)y - xy. \quad (2)$$

Set $V_e = \ker(L_e)$ where L_e denotes the right multiplication operator by the idempotent e .

Theorem 2.1. Suppose that the baric algebra (A, ω) satisfies the identity (1). Then $A = Ke \oplus U_e \oplus Z_e$, where $U_e = \{x \in \ker(\omega) \mid ex = \frac{1}{2}x\}$ and $Z_e = \{x \in \ker(\omega) \mid e(ex) = 0\}$. Furthermore, we have $U_e^2 \subseteq Z_e$ and $Z_e^2 = 0$.

Proof. We have $2e(e(ex)) = e(ex)$ for all $x \in \ker(\omega)$. Using the same reasoning as [5, Proof of Proposition 2], we get $A = Ke \oplus U_e \oplus Z_e$, where $U_e = \{x \in \ker(\omega) \mid ex = \frac{1}{2}x\}$ and $Z_e = \{x \in \ker(\omega) \mid e(ex) = 0\}$.

- Let $x, y \in U_e$. Identity (2) gives us $e(e(xy)) = 0$. It follows that $U_e^2 \subseteq Z_e$.
- Let $x, y \in V_e$. According to the identity (2), we have $e(e(xy)) - e(xy) + xy = 0$, i.e. $(L_e^2 - L_e + Id_{\ker(\omega)})(xy) = 0$. Since $\text{pgc}(X^3 - X^2, X^2 - X + 1) = 1$, we deduce that $xy = 0$ and $V_e^2 = 0$.
- Let $x \in V_e$ and $y \in Z_e$. We have $ey \in V_e$ and using the relation $V_e^2 = 0$ and the identity (2), we get $e(e(xy)) - e(xy) + xy = 0$. It follows that $xy = 0$ and $V_e Z_e = 0$.
- Let $x, y \in Z_e$; we have $ex, ey \in V_e$. From the relations $V_e^2 = V_e Z_e = 0$ and the identity (2), it follows that $e(e(xy)) - e(xy) + xy = 0$. So, $Z_e^2 = 0$.

$\square$

A partial linearization of the relations (iii) to (v) of the Proposition 2.1 give us respectively, for all $x, y \in \ker(\omega)$,

$$\begin{aligned}
&e[yx^2 + 2x(xy)] + 2(xy)[ex + 2e(ex)] + x^2[ey + 2e(ey)] + \\
&2(ey)[ex^2 + 2(ex)x] + 4(ex)[e(xy) + (ey)x + (ex)y] = 2yx^2 + 4x(xy)
\end{aligned} \quad (3)$$

$$(xy)[ex^2 + 2x(ex)] + x^2[e(xy) + (ey)x + (ex)y] + x^3(ey) + (ex)(yx^2 + 2x(xy)) = 0 \quad (4)$$

$$2x^3(xy) + x^2[yx^2 + 2x(xy)] = 0. \quad (5)$$

Lemma 2.1. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1).

For all $u \in U_e$ and $z \in Z_e$, we have

- (i) $2e(u(uz)) + 2u[e(uz) + (ez)u] = u(uz)$;
- (ii) $e(z(zu)) + 2(zu)(ez) + 2(ez)[e(zu) + (ez)u] = 2z(zu)$;
- (iii) $3(uz)u^2 + 2(uz)(eu^2) + 2u^2[e(uz) + (ez)u] + 2u^3(ez) + 2u(u(uz)) = 0$;
- (iv) $(ez)(z(zu)) = 0$;
- (v) $u^3(uz) + u^2(u(uz)) = 0$.

Proof. In the identities (3) to (5), set $x = u$; $y = z$ on the one hand and $x = z$; $y = u$ on the other. $\square$

Lemma 2.2. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1).

For all $u \in U_e$ and $z \in Z_e$, we have

- (i) $e(uz) + u(ez) \in U_e$
- (ii) $eu^3 + u(eu^2) = 0$;
- (iii) $u^4 = 0$;
- (iv) $(uz)^2 + u(z(uz)) + 2(uz)[e(uz) + u(ez)] + 2(ez)[u(uz) + z(uz)] = 0$;
- (v) $u^2u^3 = 0$;
- (vi) $u^2(z(uz)) + 2(uz)[u(uz) + z(uz)] = 0$.

Proof. Set $x = u + z$ in the Proposition (2.1) with $u \in U_e$ and $z \in Z_e$. Taking into account the Lemma 2.1, we get the result. $\square$

2.2. Link with power-associative and Bernstein algebras.

Definition 2.1. An algebra A is called:

- (i) Jordan if it is commutative and $(x^2y)x - x^2(xy) = 0$, for all $x, y \in A$.
- (ii) Power-associative if $x^i x^j = x^{i+j}$ for all $x \in A$ and for all integer $i, j \geq 1$.

Theorem 2.2. [1] Let K be a field of characteristic $\neq 2, 3, 5$. An algebra A is Power-associative if and only if $x^2x^2 = x^4$, for all $x \in A$.

The Lemma below is useful to characterize a non nil power-associative algebra and a Bernstein algebra satisfying identity (1).

Lemma 2.3. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1) such that $V_e = Z_e$ and $U_e Z_e \subset U_e$. For all $u, u_1, u_2 \in U_e$ and $z, z_1 \in Z_e$, we have

- (i) $z(uz) = 0$;

- (ii) $u^3 = 0$;
- (iii) $u(u(uz)) = 0$;
- (iv) $(uz)^2 = 0$;
- (v) $z_1(uz) + z(uz_1) = 0$;
- (vi) $(uz)(u_1z) = (uz)(uz_1) = 0$;
- (vii) $u_2u_1^2 + 2u_1(u_1u_2) = 0$.

Proof. The assumptions (i) to (iv) follows from Lemma 2.1 and Lemma 2.2. While that of (v) to (vii) come from partial linearization of the equalities $z(uz) = 0$; $(uz)^2 = 0$ and $u^3 = 0$ respectively. $\square$

Proposition 2.2. *Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1). The following statements are equivalents:*

- (i) *A is a Bernstein algebra.*
- (ii) *$Z_e = \ker(L_e)$, $U_e Z_e \subseteq U_e$ and $u(uz) = 0$ for all $u \in U_e$ and $z \in Z_e$.*
- (iii) *A is a Jordan algebra.*
- (iv) *A is power-associative.*
- (v) *$x^3 - \omega(x)x^2 = 0$ for all $x \in A$.*

Proof. Suppose that the baric algebra (A, ω) satisfies the identity (1).

(i) $\implies$ (ii) The relations hold in all Bernstein algebra.

(ii) $\implies$ (i) For $x \in A$, we prove by direct calculation that $x^2x^2 = \omega(x)^2x^2$. So, A is a Bernstein algebra.

(ii) $\implies$ (iii) Let $x = \alpha e + u_1 + z_1$, $y = \beta e + u_2 + z_2 \in A$ with $\alpha, \beta \in K$, $u_1, u_2 \in U_e$, $z_1, z_2 \in Z_e$. We have:

$$\begin{aligned}
 xy &= \alpha\beta e + \frac{1}{2}(\alpha u_2 + \beta u_1) + u_1u_2 + u_1z_2 + u_2z_1 \\
 x^2 &= \alpha^2 e + \alpha u_1 + u_1^2 + 2u_1z_1 \\
 x^2(xy) &= \alpha^3\beta e + \frac{1}{4}\alpha^3u_2 + \frac{3}{4}\alpha^2\beta u_1 + \frac{1}{2}\alpha^2(u_1u_2 + u_1z_2 + u_2z_1) + \frac{1}{2}\alpha\beta(u_1^2 + 2u_1z_1) \\
 &\quad + \alpha\left[\frac{1}{2}u_2u_1^2 + u_1(u_1u_2)\right] + u_1^2(u_2z_1) + 2(u_1z_1)(u_1u_2) \\
 x^2y &= \alpha^2\beta e + \frac{1}{2}\alpha^2u_2 + \frac{1}{2}\alpha\beta u_1 + \alpha(u_1u_2 + u_1z_2) + \beta u_1z_1 + u_2u_1^2 + 2u_2(u_1z_1) + 2z_2(u_1z_1) \\
 (x^2y)x &= \alpha^3\beta e + \frac{1}{4}\alpha^3u_2 + \frac{3}{4}\alpha^2\beta u_1 + \frac{1}{2}\alpha^2(u_1u_2 + u_1z_2 + u_2z_1) + \frac{1}{2}\alpha\beta(u_1^2 + 2u_1z_1) \\
 &\quad + \alpha\left[\frac{1}{2}u_2u_1^2 + u_1(u_1u_2)\right] + u_1(u_2u_1^2) + 2u_1(u_2(u_1z_1) + 2u_1(z_2(u_1z_1))) \\
 &\quad + z_1(u_2u_1^2) + 2z_1(z_2(u_1z_1)).
 \end{aligned}$$

Using equalities $z_1(uz) + z(uz_1) = 0$ and $u_1(uz) + u(u_1z) = 0$ of the Lemma 2.3 we get $u_1(u_2u_1^2) = -u_2u_1^3 = 0$, $u_1(z_2(u_1z_1)) = -(u_1z_1)(u_1z_2) = 0$, $z_1(z_2(u_1z_1)) = -z_2(z_1(u_1z_1)) = 0$. In the assumption (vii) of the Lemma 2.3 replace u_2 by u_2z_1 , we then obtain $0 = u_1^2(u_2z_1) + 2u_1(u_1(u_2z_1)) = z_1(u_2u_1^2) + 2u_1(u_2(u_1z_1))$. We also have $u_1^2(u_2z_1) + 2(u_1z_1)(u_1u_2) = -z_1(u_2u_1^2) - z_1[2u_1(u_1u_2)] = -z_1[u_2u_1^2 + 2u_1(u_1u_2)] = 0$. According to these equalities, we deduce that $x^2(xy) = (x^2y)x$ and A is a Jordan algebra.

(iii) $\implies$ (iv) This is clear because a commutative Jordan algebra is Power-associative.

(iv) $\implies$ (v) Let $x = e + u + z \in A$; we show that $0 = x^2x^2 - x^4 = 2u(uz)$, i.e. $u(uz) = 0$ for all $u \in U_e$ and $z \in Z_e$. We deduce that $(u + z)^3 = (u^2 + 2uz)(u + z) = 0$ for all $u \in U_e$ and $z \in Z_e$. It follows that $x^3 - \omega(x)x^2 = 0$ for all $x \in A$ [9, Théorème 2.3].

(v) $\implies$ (i) See [13, Theorem 9.12] or [12, Theorem]. $\square$

3. EXCLUSIVE AND NORMAL ALGEBRAS SATISFYING IDENTITY (1)

Definition 3.1. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1). We said that A is an exclusive algebra if $U_e^2 = 0$ for some idempotent e where $A = Ke \oplus U_e \oplus Z_e$.

We will show that this notion does not depend on the given idempotent.

Proposition 3.1. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra with an idempotent e such that $U_e^2 = Z_e^2 = 0$. Then, the algebra A satisfies the identity (1) if and only if, the following statement hold for all $u \in U_e$, $z \in Z_e$:

- (i) $u(uz) \in U_e$;
- (ii) $e(uz) + u(ez) \in U_e$;
- (iii) $e(z(uz)) + 2(uz)(ez) + 2(ez)[e(uz) + u(ez)] = 2z(uz)$;
- (iv) $(ez)(z(uz)) = 0$;
- (v) $(uz)^2 + u(z(uz)) + 2(uz)[e(uz) + u(ez)] + 2(ez)(u(uz)) = 0$;
- (vi) $(uz)[u(uz) + z(uz)] = 0$.

Proof. Suppose that the algebra A satisfies the identity (1). The relations (i) to (vi) follow from the Lemma 2.1 and Lemma 2.2. Reciprocally, suppose that the relations (i) to (vi) are satisfied. Let $x = \alpha e + u + z \in A$ with $\alpha \in K$, $u \in U_e$, $z \in Z_e$. We have:

$$\begin{aligned}
 x^2 &= \alpha^2 e + \alpha(u + 2ez) + 2uz \\
 x^3 &= \alpha^3 e + \alpha^2(u + ez) + \alpha[2e(uz) + 2u(ez) + uz] + 2[u(uz) + z(uz)] \\
 x^2x^3 &= \alpha^5 e + \alpha^4 u + 4\alpha^3[e(uz) + u(ez)] + 4\alpha^2[z(uz) + u(uz)] \\
 &= 2\omega(x)^2x^3 - \omega(x)^3x^2.
 \end{aligned}$$

$\square$

Proposition 3.2. *Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1) such that $U_e^2 = 0$. Then, the algebra (A, ω) is a second-order Bernstein algebra if and only if $(uz)(ez) \in U_e$ and $(uz)^2 \in U_e$ for all $u \in U_e$ and $z \in Z_e$.*

Proof. The result follows from the Proposition 3.1 and from [6, Theorem 1]. $\square$

Theorem 3.1. *Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1) such that $U_e^2 = 0$. Then, the set $\mathcal{I}$ of idempotent element of A is given by*

$$\mathcal{I} = \{e + u \mid u \in U_e\}.$$

Proof. An element $e + u$ with $u \in U_e$ is an idempotent. Conversely, let $e' = e + u + z$ with $u \in U_e$, $z \in Z_e$ an idempotent of A . We have $e + u + z = e'^2 = e + u + 2ez + 2uz$, i.e. $z = 2(ez + uz)$. By multiplying this last equality by e and u respectively, we get $ez = 2e(uz)$ and $uz = 2u(ez) + 2u(uz)$. According to the Proposition 3.1, it follows that $z = 2[e(uz) + u(ez)] + 2u(uz) \in U_e \cap Z_e = 0$ and we deduce that $e' = e + u$. $\square$

Proposition 3.3. *Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1) such that $U_e^2 = 0$. Then $U_e = U_{e'}$ for every idempotent element e' of A . Consequently, $\dim U_e$ is an invariant of A .*

Proof. According to the Theorem 3.1, there is some $u' \in U_e$ such that $e' = e + u' \in U_e$. Let $u \in U_e$, we have $e'u = (e + u')u = \frac{1}{2}u$ and we deduce that $U_e \subset U_{e'}$. Conversely, let $x = u + z \in U_{e'}$ with $u \in U_e$ and $z \in Z_e$. We have $\frac{1}{2}(u + z) = e'x = (e + u')(u + z) = \frac{1}{2}u + ez + u'z$, i.e. $z = 2(ez + u'z)$. Using the same reasoning as in the proof of Theorem 3.1, we get $z = 0$. So $x = u \in U_e$ and $U_{e'} \subset U_e$. It follows that $U_e = U_{e'}$ and $\dim U_e$ is an invariant of the algebra A . $\square$

Proposition 3.4. *Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1) such that $U_e^2 = 0$. If $e' = e + u'$, $u' \in U_e$ is another idempotent of A , then $Z_{e'} = \{z - 4[e(u'z) + u'(ez)] - 4u'(u'z) \mid z \in Z_e\}$. Moreover, $\dim Z_e$ is an invariant of A .*

Proof. For $z \in Z_e$, we have $(e + u')((e + u')(z - 4[e(u'z) + u'(ez)] - 4u'(u'z))) = (e + u')(ez - 2[e(u'z) + u'(ez)] - 2u'(u'z) + u'z) = -[e(u'z) + u'(ez)] - u'(u'z) + e(u'z) + u'(ez) + u'(u'z) = 0$. So $\{z - 4[e(u'z) + u'(ez)] - 4u'(u'z) \mid z \in Z_e\} \subset Z_{e'}$. Conversely, let $x = u + z \in Z_{e'}$ such that $u \in U_e$ and $z \in Z_e$. We have $0 = e'(e'x) = \frac{1}{4}u + e(u'z) + u'(ez) + u'(u'z)$, i.e. $u = -4[u'(ez) + e(u'z)] - 4u'(u'z)$ and $x = z - 4[u'(ez) + e(u'z)] - 4u'(u'z)$. It follows that $Z_{e'} \subset \{z - 4[e(u'z) + u'(ez)] - 4u'(u'z) \mid z \in Z_e\}$. Since, $[e(u'z) + u'(ez)] + u'(u'z) \in U_e$, the linear operator $\varphi : Z_e \longrightarrow Z_{e'}$, $z \longmapsto z - 4[u'(ez) + e(u'z)] - 4u'(u'z)$ is an isomorphism. We deduce that $\dim Z_e$ is an invariant of A . $\square$

Remark 3.1. $\dim U_e V_e$ is another invariant of the baric algebra (A, ω) satisfying the identity (1) and admitting an idempotent element e such that $U_e^2 = 0$. Indeed, $(\ker \omega)^2 = U_e V_e$.

Definition 3.2. Let $A = Ke \oplus U_e \oplus Z_e$ be a Peirce decomposition of the baric algebra (A, ω) verifying the identity (1). We said that the algebra A is normal if it admits an idempotent e such that $(\ker \omega)^2 \subset Z_e$.

This notion does not depend on the given idempotent too, as we will see.

Proposition 3.5. Let (A, ω) be a baric algebra with an idempotent element e such that $A = Ke \oplus U_e \oplus Z_e$ where $U_e = \{x \in \ker \omega \mid ex = \frac{1}{2}x\}$, $Z_e = \{x \in \ker \omega \mid e(ex) = 0\}$, $Z_e^2 = 0$ and $(\ker \omega)^2 \subset Z_e$. Then (A, ω) satisfies the identity (1) if and only if for every $u \in U_e$ and $z \in Z_e$, the following identities hold:

- (i) $e(uz) + u(ez) = 0$;
- (ii) $u(uz) = 0$;
- (iii) $eu^3 + u(eu^2) = 0$;
- (iv) $u^4 = 0$.

Proof. Suppose that the algebra A satisfies the identity (1). The relations (i) to (iv) follow from the Lemma 2.1 and Lemma 2.2. Conversely, suppose that the relations (i) to (iv) are satisfied. Let $x = \alpha e + u + z \in A$ with $\alpha \in K$, $u \in U_e$, $z \in Z_e$. We have:

$$\begin{aligned} x^2 &= \alpha^2 e + \alpha(u + 2ez) + u^2 + 2uz \\ x^3 &= \alpha^3 e + \alpha^2(u + ez) + \alpha(eu^2 + u^2 + uz) + u^3 \\ x^2 x^3 &= \alpha^5 e + \alpha^4 u + \alpha^3(2eu^2 + u^2) + 2\alpha^2 u^3 \\ &= 2\omega(x)^2 x^3 - \omega(x)^3 x^2. \end{aligned}$$

□

Proposition 3.6. Let (A, ω) be a normal algebra satisfying the identity (1) and with a nonzero idempotent. Then, the algebra A is a second-order Bernstein algebra.

Proof. The result follows from the Proposition 3.5 and from [6, Theorem 5].

□

Theorem 3.2. Let (A, ω) be a normal algebra satisfying the identity (1) and with a nonzero idempotent. Then, the set $\mathcal{I}$ of idempotent element of A is given by

$$\mathcal{I} = \{e + u + u^2 + 2u^3 + 2eu^2 \mid u \in U_e\}.$$

Moreover, for another idempotent $e' = e + u + u^2 + 2u^3 + 2eu^2$, we have

$$U_{e'} = \{u' + 2(uu' + 2e(uu')) + 2(u^2u' + 2u(uu')) \mid u' \in U_e\} \text{ and } Z_{e'} = Z_e.$$

Proof. The theorem follows from The Proposition 3.6 and from [6, Theorem 5, Theorem 6 and Proposition 5]

□

4. SYMMETRIC BILINEAR FORMS

Assume that the baric algebra (A, ω) satisfies the identity (1) and let N be the kernel of ω such that $\dim(N^2) = 1$. The multiplication in N is given by

$$xy = \alpha(x, y)n \text{ for all } x, y \in N,$$

where n is a fixed nonzero element of N and $\alpha(x, y)$ is a symmetric bilinear form.

Proposition 4.1. *We have $n^2 = 0$ and $x^2x^2 = 0$ for all $x \in N$.*

Proof. We have $0 = n^2n^3 = \alpha(n, n)^4n$, i.e. $\alpha(n, n) = 0$ and $n^2 = 0$. Let $x \in N$, we have $(x^2)^2 = (\alpha(x, x)n)^2 = \alpha(x, x)^2n^2 = 0$ and we get the proposition. $\square$

Remark 4.1. $N^{[3]} = N^2N^2 = Kn^2 = 0$.

Definition 4.1. Let $(\mathcal{E}, \omega)$ be a baric algebra with an idempotent e such that $\mathcal{E} = Ke \oplus U_e \oplus Z_e$ where $U_e = \{x \in \ker \omega \mid ex = \frac{1}{2}x\}$ and $Z_e = \{x \in \ker \omega \mid e(ex) = 0\}$. If $N^2 \subseteq U_e$, we say that the algebra $\mathcal{E}$ is of type (I). Otherwise it is of type (II).

Proposition 4.2. *The following assertions are equivalent.*

- (i) *The algebra A is of type (I) and satisfies the identity (1).*
- (ii) *The algebra A is a baric algebra with weight function ω and with an idempotent e such that*
 - (a) *$A = Ke \oplus U_e \oplus Z_e$ where $eu = \frac{1}{2}u$, for all $u \in U_e$, $e(ez) = 0$ for all $z \in Z_e$;*
 - (b) *$N^2 = Kn \subset U_e$ and $U_e^2 = Z_e^2 = 0$ where $N = \ker \omega$;*
 - (c) *$6(ez)(uz) + 4(ez)(u(ez)) = 3z(uz)$ for all $u \in U_e$ and $z \in Z_e$;*
 - (d) *$(ez)(z(uz)) = 0$ for all $u \in U_e$ and $z \in Z_e$.*

Proof. We suppose that, the algebra A satisfies the identity (1) and it is of type (I). So, it admits an idempotent element e such that its Peirce decomposition is given by $A = Ke \oplus U_e \oplus Z_e$ where $U_e = \{x \in \ker \omega \mid ex = \frac{1}{2}x\}$; $Z_e = \{x \in \ker \omega \mid e(ex) = 0\}$; $U_e^2 \subset Z_e$ and $Z_e^2 = 0$. Since, A is of type (I), we have $N^2 = Kn \subset U_e$. We also have $U_e^2 \subset Z_e \cap U_e = 0$, i.e. $U_e^2 = 0$. From above and according to the Proposition 3.1, it follows that $6(ez)(uz) + 4(ez)(u(ez)) = 3z(uz)$ and $(ez)(z(uz)) = 0$ for all $u \in U_e$ and $z \in Z_e$. Conversely, suppose that the assertion (ii) is satisfying. It follows that the algebra A is of type (I). Let $x = \alpha e + u + z \in A$ where $u \in U_e$ and $z \in Z_e$. We have $x^2 = \alpha^2e + \alpha(u + 2ez) + 2uz$, $x^3 = \alpha^3e + \alpha^2(u + ez) + 2\alpha[u(ez) + uz] + 2z(uz)$ and $x^2x^3 = \alpha^5e + \alpha^4u + 2\alpha^3(uz + 2u(ez)) + \alpha^2(z(uz) + 6(ez)(uz) + 4(ez)(u(ez))) = \alpha^5e + \alpha^4u + 2\alpha^3(uz + 2u(ez)) + 4\alpha^2z(uz)$. We deduce that $x^2x^3 = 2\omega(x)^2x^3 - \omega(x)^3x^2$. $\square$

Corollary 4.1. *Let (A, ω) be a baric algebra of type (I) and satisfying the identity (1). Then, A is a second-order Bernstein algebra.*

Proof. The corollary follows from the Proposition 4.2 and from [11, Theorem 2.3] $\square$

Now, (A, ω) denoted an algebra satisfying the identity (1) with an idempotent e such that $N^2 = Kn$ and $n \notin U_e$. Since $U_e^2 \subset Z_e$, we have either $U_e^2 = 0$ or $n \in Z_e$. For $n \in Z_e$, the Proposition 3.6 tells us that the algebra A is a second-order Bernstein algebra.

Theorem 4.1. *Let (A, ω) be a baric algebra with an idempotent e and $A = Ke \oplus U_e \oplus Z_e$ where $U_e = \{x \ker \omega \mid ex = \frac{1}{2}x\}$, $Z_e = \{x \ker \omega \mid e(ex) = 0\}$; $N = \ker \omega$; $N^2 = Kn$ for some nonzero $n \in N \setminus U_e$ and $U_e^2 = 0$. Then, the algebra A satisfies the identity (1) if and only if, the following statement are satisfied for every $u \in U_e$ and $z \in Z_e$:*

- (i) $Z_e^2 = U_e(eZ_e) = 0$;
- (ii) $U_e Z_e = U_e + \ker(L_e)$ where L_e is a right multiplication operator by e ;
- (iii) $u(uz) = 0$;
- (iv) $e(z(uz)) + 2(uz)(ez) + 2(ez)(e(uz)) = 2z(uz)$;
- (v) $(ez)(z(uz)) = 0$;
- (vi) $u(z(uz)) = 0$;
- (vii) $(uz)(e(uz)) = 0$.

Proof. Suppose that the algebra A satisfies the identity (1). It follows that $Z_e^2 = 0$ and since $U_e^2 = 0$, the identities of the Proposition 3.1 are verified. We have $U_e \ni e(uz) + u(ez) = \alpha(u, z)en + \alpha(u, ez)n$. From [11, Lemma 4.2], it follows that $u(ez) = 0$ and $e(uz) \in U_e$. We deduce that $U_e(eZ_e) = 0$ and $U_e Z_e \subset U_e + \ker(L_e)$. So we get statement (i) and (ii). The relation $U_e \ni u(uz) = \alpha(u, z)\alpha(u, n)n$ leads to $0 = \alpha(u, z)$ or $0 = \alpha(u, n)$, i.e. $u(uz) = 0$ and we obtain the assertion (iii). Taking account the relations $u(ez) = u(uz) = (uz)^2 = 0$ and the identities of the Proposition 3.1, we obtain the statements (iv), (v) and the relation $u(z(uz)) + 2(uz)(e(uz)) = 0$. We also have $u(z(uz)) = \alpha(u, z)\alpha(z, n)\alpha(u, n)n = \alpha(z, n)[\alpha(u, z)\alpha(u, n)n] = \alpha(z, n)u(uz) = 0$ and we get the assertion (vi). By the equality $0 = u(z(uz)) + 2(uz)(e(uz)) = 2(uz)(e(uz))$, we have the statement (vii) since the characteristic of K is different from 2. Conversely, suppose that the statements (i) to (vii) are satisfied and let $x = \alpha e + u + z$ with $u \in U_e$ and $z \in Z_e$. We have $x^2 = \alpha^2 e + \alpha(u + 2ez) + 2uz$; $x^3 = \alpha^3 e + \alpha^2(u + ez) + \alpha(uz + 2e(uz)) + 2z(uz)$ and $x^2 x^3 = \alpha^5 e + \alpha^4 u + 4\alpha^3 e(uz) + 4\alpha^2 z(uz) = 2\omega(x)^2 x^3 - \omega(x)^3 x^2$. $\square$

Corollary 4.2. *Let (A, ω) be an algebra satisfying the identity (1) and of type (II). Then either*

- (i) $U_e Z_e = 0$ or $N^2 \subset Z_e$, in which cases A is a second-order Bernstein algebra.
- (ii) $A^2 = Ke \oplus U_e \oplus Z_1$, where $Z_1 = (ez + N^2) \cap Z_e$, $eZ_1 = 0$, $U_e^2 = Z_1^2 = 0$ and $U_e Z_1 \subset U_e \oplus \ker(L_e)$.

Proof. Suppose first that $U_e Z_e = 0$ or $N^2 \subseteq Z_e$, it follows that A is a normal algebra. We deduce that A is a second-order Bernstein algebra. Now, suppose that $U_e Z_e \neq 0$ and $N^2 \not\subseteq Z_e$. Then $N^2 = Kn \subseteq U_e \oplus \ker(L_e)$ according to (ii) of the Proposition (4.1). Set $n = n_1 + n_2$ with $n_1 \in U_e$ and $n_2 \in \ker(L_e)$.

We have

$$A^2 = Ke \oplus U_e \oplus eZ_e + N^2 = Ke \oplus U_e \oplus (eZ_e + Kn_z) = Ke \oplus U_e \oplus Z_1.$$

So $eZ_1 = U_e^2 = Z_1^2 = 0$ and $U_e Z_1 = U_e(eZ_e + Kn_z) = U_e n_z \subseteq U_e \oplus \ker(L_e)$. $\square$

Proposition 4.3. *Let (A, ω) be an algebra satisfying the identity (1) and of type (II) such that $U_e Z_e \neq 0$ and $N^2 \not\subseteq Z_e$. Then, A is a second-order Bernstein algebra.*

Proof. We have $U_e^2 = Z_e^2 = 0$, so there are $u_0 \in U_e$ and $z_0 \in Z_e$ such that $u_0 z_0 = n \neq 0$. Set $n = n_1 + n_2$ where $n_1 \in U_e$ and $n_2 \in Z_e$. According to the relation (i) of the Theorem 4.1, we have $(uz)(ez) = \alpha(u, z)n(ez) = (n_1 + n_2)(ez) = n_1(ez) = 0$ for all $u \in U_e$ and $z \in Z_e$. We also have $(uz)^2 = 0$ for all $u \in U_e$ and $z \in Z_e$. According to the Proposition 3.2, we deduce that A is a second-order Bernstein algebra. $\square$

Conflicts of Interest. The author declare that there are no conflicts of interest regarding the publication of this paper.

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