Asia Pac. J. Math. 2025 **12**:93

THE HOMOGENEOUS q-SHIFT OPERATOR FOR POLYNOMIALS WITH DOUBLE q-BINOMIAL COEFFICIENTS

NEDAA Q. MOHAMMED[1], HUSAM L. SAAD[2,*]

[1]Department of Educational and Psychological Sciences, University of Basrah, Iraq

[2]Department of Mathematics, University of Basrah, Iraq

[*]Corresponding author: hus6274@hotmail.com

Received Aug. 4, 2025

Abstract. Within this document, a homogeneous q-shift operator is constructed and we form a polynomial with double q-binomial coefficients $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. The generating function, the Rogers type, Mehler's formula, and mixed generating functions and all of its extension are inferred for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. When we take a few special values for a polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ are provided for the purpose of find the polynomials' identities for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$. Also we develop the generating functions of a class mixed. Using generating functions for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$, a transformation formula is derived.

2020 Mathematics Subject Classification. 05A30; 33D45.

Key words and phrases. homogeneous q-shift operator; generating function; Rogers formula; Mehler's formula; double q-binomial coefficients; mixed generating function.

1. Introduction

This section, the q-series symbols and definitions used in [15] shall be adhered to. Our hypothesis is that $|q| < 1$.

The q-shifted factorial for $c \in \mathbb{C}$ is defined as [13–15].

$$(c;q)_0 = 1, \quad (c;q)_m = \prod_{k=0}^{m-1}(1-cq^k), \quad (c;q)_\infty = \prod_{k=0}^{\infty}(1-cq^k),$$

The following relation is true:

$$(c_1,c_2,\ldots,c_r;q)_g = (c_1;q)_g(c_2;q)_g\cdots(c_r;q)_g,$$

where $g \in \mathbb{Z}$ or ∞.

DOI: 10.28924/APJM/12-93

The fundamental series ${}_r\phi_s$ can be presented as follows [6,15,25]:

$$ {}_r\phi_s\left(\begin{array}{c}\beta_1,\ldots,\beta_r\\ \alpha_1,\ldots,\alpha_s\end{array};q,x\right)=\sum_{n=0}^{\infty}\frac{(\beta_1,\ldots,\beta_r;q)_n}{(q,\alpha_1,\ldots,\alpha_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n, $$

where $q\neq 0$ when $r>s+1$.

Note that

$$ {}_{r+1}\phi_r\left(\begin{array}{c}\beta_1,\ldots,\beta_{r+1}\\ \alpha_1,\ldots,\alpha_r\end{array};q,x\right)=\sum_{i=0}^{\infty}\frac{(\beta_1,\ldots,\beta_{r+1};q)_i}{(q,\alpha_1,\ldots,\alpha_r;q)_i}x^i,\quad |x|<1. $$

The q-binomial coefficients is defined as follows [15,19,32]:

$$ \begin{bmatrix}m\\ l\end{bmatrix}=\frac{(q;q)_m}{(q;q)_L\,(q;q)_{m-l}}\quad for\quad 0\leqslant l\leqslant m, $$

where $m,l\in\mathbb{N}$.

The following relationships apply to the generalized coefficients of q-binomial [10] :

$$ \begin{bmatrix}\gamma\\ k\end{bmatrix}_q=\frac{(q^{-\gamma};q)_k}{(q;q)_k}(-1)^k q^{\gamma k-\binom{k}{2}}. \tag{1.1} $$

$$ \begin{bmatrix}\gamma\\ k\end{bmatrix}_{-q}=\frac{(-q^{-\gamma};q)_k}{(-q;q)_k}q^{\gamma k-\binom{k}{2}}, \tag{1.2} $$

where $\gamma\in\mathbb{C}$.

The following identity is called Cauchy [2,7,8]:

$$ \sum_{j=0}^{\infty}\frac{(b;q)_j}{(q;q)_j}\,y^j=\frac{(by;q)_\infty}{(y;q)_\infty},\quad |y|<1. \tag{1.3} $$

Cauchy identity with $b=0$ provides Euler's identity [15]:

$$ \sum_{j=0}^{\infty}\frac{y^j}{(q;q)_j}=\frac{1}{(y;q)_\infty},\quad |y|<1. $$

The q-Chu-Vandermonde's identity is [15,21,29]:

$$ {}_2\phi_1(q^{-m},d;e;q,q)=\frac{(e/d;q)_m}{(e;q)_m}\,d^m. \tag{1.4} $$

This study will make use of the following identities [18,22,23]:

$$ (aq^{-m};q)_m=(q/a;q)_m(-a)^m q^{-m-\binom{m}{2}}. \tag{1.5} $$

$$ (q^{-m};q)_l=\frac{(q;q)_m}{(q;q)_{m-l}}(-1)^l q^{\binom{l}{2}-ml}. \tag{1.6} $$

The definition of Hahn polynomials is as follows [5,17,28]:

$$ \phi_u^{(\beta)}(x|q)=\sum_{i=0}^{u}\begin{bmatrix}u\\ u\end{bmatrix}_q(\beta;q)_i\,x^i. $$

The generating function that follows was introduced by Srivastava and Agarwal [33] in 1989:

$$\sum_{m=0}^{\infty} \phi_m^{(\beta)}(x|q)(\lambda;q)_m \frac{s^m}{(q;q)_m} = \frac{(\lambda s;q)_\infty}{(s;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}\lambda,\beta \\ \lambda s\end{matrix};q,xs\right), \quad \max\{|s|,|xs|\}<1.$$

The following is a definition of the Cauchy polynomials [3,30,31]:

$$P_m(s,t) = (s-t)(s-qt)\cdots(s-q^{m-1}t) = (t/s;q)_m\, s^m,$$

whose generating function is [1,4,27]:

$$\sum_{m=0}^{\infty} P_m(x,y)\frac{t^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.7}$$

Goulden and Jackson gave the following identity [16]:

$$P_m(x,y) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} y^j x^{m-j}. \tag{1.8}$$

The canonical definition of the q-differential operator, also referred to as the q-derivative, is [12,24]:

$$D_q f(c) = \frac{f(c)-f(qc)}{c}.$$

Regarding the operator D_q, the q-Leibniz rule, is shown below [12,26]:

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}.$$

The key property of the operator D_q is [10]:

$$D_q^m\left\{\frac{(bx;q)_\infty}{(by;q)_\infty}\right\} = y^m \frac{(x/y;q)_m}{(bx;q)_m}\frac{(bx;q)_\infty}{(by;q)_\infty}, \quad |by|<1.$$

The q-exponential operator was defined by Chen and Liu [12] as

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}.$$

Using functions in two variables, the homogeneous q-difference operator D_{st} was defined by Chen et al. [11] in 2003 as:

$$D_{st}\{f(s,t)\} = \frac{f(s,q^{-1}t)-f(qs,t)}{s-q^{-1}t}.$$

Additionally, they erected the operator which is homogeneous q-shift like this:

$$\mathbb{E}(D_{st}) = \sum_{i=0}^{\infty} \frac{D_{st}^i}{(q;q)_i}.$$

Proposition 1.1. [11]. *We have*

$$D_{st}^k\{P_m(s,t)\} = \frac{(q;q)_m}{(q;q)_{m-k}} P_{m-k}(s,t).$$

$$D_{st}^k\left\{\frac{(tl;q)_\infty}{(sl;q)_\infty}\right\} = l^k \frac{(tl;q)_\infty}{(sl;q)_\infty}.$$

In 2021, Jia et al. [20] presented the subsequent polynomials:

$$L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}\beta\\l\end{bmatrix}_{-q}q^{\tau(\tilde{j},\tilde{k})+\binom{l}{2}}(r;q)_l o^l u^{n-l}, \tag{1.9}$$

where

$$\tau(\tilde{j},\tilde{k})=\tilde{j}\binom{l}{2}-\tilde{k}\binom{l+1}{2},\quad \tilde{j},\tilde{k}\in\mathbb{R}. \tag{1.10}$$

The findings of Jia et al. [20] are

Theorem 1.2. [20]. *We have*

The generating function for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$:

$$\sum_{n=0}^{\infty}L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)\frac{t^n}{(q;q)_n}=\frac{1}{(ut;q)_\infty}\sum_{l=0}^{\infty}\frac{(-q^{-\beta},r;q)_l}{(q^2;q^2)_l}q^{\tau(\tilde{j},\tilde{k})+l\beta}(ot)^l,\quad |ut|<1. \tag{1.11}$$

The mixed generating function for the polynomials $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ *is:*

$$\sum_{n=0}^{\infty}L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)\frac{P_n(t,s)}{(q;q)_n}=\frac{(su;q)_\infty}{(ut;q)_\infty}\sum_{i=0}^{\infty}\begin{bmatrix}\beta\\i\end{bmatrix}_{-q}\frac{(r,s/t;q)_i}{(q,su;q)_i}q^{\tau(\tilde{j},\tilde{k})+\binom{i}{2}}(ot)^i, \tag{1.12}$$

where $|ut|<1$.

An extension of the polynomials $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ was presented in 2021 by Cao et al. [9] as:

$$L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}\beta\\l\end{bmatrix}_{-q}q^{\tau(\tilde{j},\tilde{k})+\binom{l}{2}}(r;q)_l P_{n-l}(u,m)o^l. \tag{1.13}$$

The findings of Cao et al. [9] are as follows:

Theorem 1.3. [9]. *We have*

The generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ *is*

$$\sum_{n=0}^{\infty}L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)\frac{t^n}{(q;q)_n}=\frac{(mt;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}\frac{(-q^{-\beta},r;q)_l}{(q^2;q^2)_l}q^{\tau(\tilde{j},\tilde{k})+l\beta}(ot)^l, \tag{1.14}$$

where $|ut|<1$.

For generalised q-polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$, *the Rogers formula is*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}L_{n+k}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)\frac{t^n}{(q;q)_n}\frac{z^k}{(q;q)_k}\frac{(mz;q)_\infty}{(t/z,uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k q^k}{(q,mz,qz/t;q)_k}\\ =\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta},r;q)_l}{(q^2;q^2)_l}(ozq^k)^l,\quad \max\{|t/z|,|uz|\}<1.\end{aligned} \tag{1.15}$$

Mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ *is*

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)\frac{t^n}{(q;q)_n}=\frac{(mt;q)_\infty}{(ut;q)_\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{i}\frac{(\alpha;q)_i x^i}{(q;q)_i}\frac{(q^{-i},ut;q)_n q^n}{(mt,q;q)_n}\\ \times\frac{(-q^{-\beta};q)_l(r;q)_l(otq^n)^l}{(q^2;q^2)_l}q^{l\beta+\tau(\tilde{j},\tilde{k})},\quad |ut|<1.\end{aligned} \tag{1.16}$$

In 2022, Cao et al. [10] showed how the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ can be extended as:

$$L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}\beta\\l\end{bmatrix}_{-q}q^{\tau(\tilde{j},\tilde{k})+\binom{l}{2}}\frac{(r,s;q)_l}{(p;q)_k}o^lP_{n-k}(u,m)o^l. \tag{1.17}$$

Cao et al. [10] lead to the following outcomes:

Theorem 1.4. [10] . *We have*

The generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ *is*

$$\sum_{n=0}^{\infty}L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}=\frac{(mt;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}\frac{(-q^{-\beta},r,s;q)_l}{(q^2;q^2)_l(p;q)_l}q^{l\beta+\tau(\tilde{j},\tilde{k})}(ot)^l, \tag{1.18}$$

where $|ut|<1$.

The Rogers formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ *is*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}L_{n+k}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}\frac{z^k}{(q;q)_k}&=\frac{(mz;q)_\infty}{(t/z,uz;q)_\infty}\\&\times\sum_{k=0}^{\infty}\frac{(uz;q)_k}{(q,mz,qz/t;q)_k}q^k\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta},r,s;q)_l}{(q^2;q^2)_l(p;q)_l}(ozq^k)^l,\quad\max\{|t/z|,|uz|\}<1.\end{aligned} \tag{1.19}$$

The mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ *is*

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(X|q)L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}&=\frac{(mt;q)_\infty}{(ut;q)_\infty}\\&\times\sum_{K=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{K}\frac{(\alpha;q)_KX^K}{(q;q)_K}\frac{(q^{-K},ut;q)_nq^n}{(mt,q;q)_n}\frac{(-q^{-\beta};q)_l(r,s;q)_l(otq^n)^l}{(q^2;q^2)_l(p;q)_l}q^{l\beta+\tau(\tilde{j},\tilde{k})},\quad|ut|<1.\end{aligned} \tag{1.20}$$

The findings above for Jia et al. [20], Cao et al. [9] and Cao et al. [10] are obtained by the method of q-difference equations.

The rest of the paper is organized as follows: The homogeneous q-shift operator $\mathbb{O}(\beta,r,s,p;q,oD_{um})$ will be outlined in section 2, and we'll figure out a few of its identities. To get the generating function and its extension, we will use the operator $\mathbb{O}(\beta,r,s,p;q,oD_{um})$. We will use the operator $\mathbb{O}(\beta,r,s,p;q,oD_{um})$ to find the generating function and its extension for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ in Section 3. The operator technique in forth Section will be used to drive an extension of Rogers' for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. An extension of Mehler's formula for polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ are presented in Section 5. In section 6, two mixed generating functions are constructed for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. Finally, We will use the homogeneous q-shift operator $\mathbb{O}(\beta,r,s,p;q,oD_{um})$ in section 7 to obtain the transformational identity for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$.

2. Operator Identities for the Generalised Homogeneous q-Shift Operator

We start this section by presenting the homogeneous q-shift operator in the manner described below:

$$\mathbb{O}(\beta, r, s, p; q, oD_{um}) = \sum_{l=0}^{\infty} q^{l\beta+\tau(\tilde{j},\tilde{k})} \frac{(-q^{-\beta}, r, s; q)_l}{(q^2; q^2)_l (p; q)_l} (oD_{um})^l. \tag{2.1}$$

$\mathbb{O}(oD_{um})$ will be used to represent the operator $\mathbb{O}(\beta, r, s, p; q, oD_{um})$. We use the following notations for simplicity:

$$O(\beta, r, s, p; q, o) = \sum_{l=0}^{\infty} q^{l\beta+\tau(\tilde{j},\tilde{k})} \frac{(-q^{-\beta}, r, s; q)_l}{(q^2; q^2)_l (p; q)_l} o^l.$$

$$M(\beta, r; q, o) = \sum_{l=0}^{\infty} q^{l\beta+\tau(\tilde{j},\tilde{k})} \frac{(-q^{-\beta}, r; q)_l}{(q^2; q^2)_l} o^l.$$

Proving the following lemma is easy:

Lemma 2.1. *Assuming that the operator* $\mathbb{O}(oD_{um})$ *is defined as in* (2.1), *then*

$$\mathbb{O}(\beta, r, s, p; q, oD_{um}) \{P_n(u, m)\} = L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p). \tag{2.2}$$

$$\mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{(mt; q)_\infty}{(ut; q)_\infty} \right\} = \frac{(mt; q)_\infty}{(ut; q)_\infty} O(\beta, r, s; q, ot), \quad |ut| < 1. \tag{2.3}$$

Lemma 2.2. *Given the operator* $\mathbb{O}(\beta, r, s, p; q, oD_{um})$ *defined in* (2.1), *then*

$$\mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{P_i(u, m)(mt; q)_\infty}{(mt; q)_i (ut; q)_\infty} \right\}$$
$$= \frac{(mt; q)_\infty}{(ut; q)_\infty} t^{-i} \sum_{c=0}^{i} \frac{(q^{-i}, ut; q)_c}{(q, mt; q)_c} q^c O(\beta, r, s, p; q, otq^c), \quad |ut| < 1. \tag{2.4}$$

Proof. We shall discover that by employing q-Chu-Vandermonde sum (1.4).

$$\begin{aligned} \frac{P_i(u, m)(mt; q)_\infty}{(mt; q)_i (ut; q)_\infty} &= \frac{(mt; q)_\infty}{(ut; q)_\infty} \frac{(m/u; q)_i}{(mt; q)_i} u^i \\ &= \frac{(mt; q)_\infty}{(ut; q)_\infty} t^{-i} \,{}_2\phi_1(q^{-i}, ut; mt; q, q) \\ &= t^{-i} \sum_{c=0}^{i} \frac{(q^{-i}; q)_c}{(q; q)_c} \frac{(q^c mt; q)_\infty}{(q^c ut; q)_\infty} q^c. \end{aligned} \tag{2.5}$$

Utilising (2.5), we arrive at

$$\begin{aligned} &\mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{P_i(u, m)(mt; q)_\infty}{(mt; q)_i (ut; q)_\infty} \right\} \\ &= t^{-i} \sum_{c=0}^{i} \frac{(q^{-i}; q)_c}{(q; q)_c} q^c \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{(q^c mt; q)_\infty}{(q^c ut; q)_\infty} \right\} \\ &= t^{-i} \sum_{c=0}^{i} \frac{(q^{-i}; q)_c}{(q; q)_c} q^c \frac{(q^c mt; q)_\infty}{(q^c ut; q)_\infty} O(\beta, r, s, p; q, otq^c). \quad \text{(by using (2.3))} \end{aligned}$$

□

3. The Generating Function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$

For the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$, we shall derive the generating function and its extension using the operator representation (2.2). We will provide some distinctive values for the parameters in the generating function and its extension for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ in order to obtain the generating function and its extension for both $L_n^{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

Theorem 3.1. (Generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$). *We have*

$$\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n} = \frac{(mt;q)_\infty}{(ut;q)_\infty} O(\beta,r,s,p;q,ot), \quad |ut|<1. \tag{3.1}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_m^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \mathbb{O}(\beta,r,s,p,q,oD_{um})\{P_n(u,m)\}\frac{t^n}{(q;q)_n} \\
&= \mathbb{O}(\beta,r,s,p,q,oD_{um})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\right\} \\
&= \mathbb{O}(\beta,r,s,p,q,oD_{um})\left\{\frac{(mt;q)_\infty}{(ut;q)_\infty}\right\} \\
&= \frac{(mt;q)_\infty}{(ut;q)_\infty} O(\beta,r,s,p;q,ot).
\end{aligned}$$

□

- Setting $m=s=p=0$ in equation (3.1), we get the generating function for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ (equation (1.11)).
- Letting $s=p=0$ in equation (3.1), we regain the generating function for ${}_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ (equation (1.14)).

Theorem 3.2. (An extension of the generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$). *We have*

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n} &= \frac{(mt;q)_\infty}{(ut;q)_\infty}\, t^{-k} \\
&\times \sum_{j=0}^{k} \frac{(q^{-k},ut;q)_j}{(q,mt;q)_j}\, q^j O(\beta,r,s,p;q,otq^j), \quad |ut|<1.
\end{aligned} \tag{3.2}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{P_{n+k}(u,m)\right\}\frac{t^n}{(q;q)_n} \quad \text{(by using (2.2))}
\end{aligned}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ P_k(u, m) \sum_{n=0}^{\infty} P_n(u, mq^k) \frac{t^n}{(q;q)_n} \right\}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ P_k(u, m) \frac{(mtq^k;q)_\infty}{(ut;q)_\infty} \right\} \quad \text{(by using (1.7))}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{P_k(u, m)}{(mt;q)_k} \frac{(mt;q)_\infty}{(ut;q)_\infty} \right\}$$

$$= \frac{(mt;q)_\infty}{(ut;q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, ut;q)_j}{(q, mt;q)_j} q^j O(\beta, r, s, p; q, otq^j). \quad \text{(by using (2.4))}$$

□

- Setting $k = 0$ in equation (3.2), we obtain the generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$ (equation (3.1)).
- If $s = p = 0$ in equation (3.2), we find an extension to the generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r)$.

Corollary 3.3. *We have*

$$\sum_{n=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r) \frac{t^n}{(q;q)_n} = \frac{(mt;q)_\infty}{(ut;q)_\infty} t^{-k}$$
$$\times \sum_{j=0}^{k} \frac{(q^{-k}, ut;q)_j}{(q, mt;q)_j} q^j M(\beta, r; q, otq^j), \quad |ut| < 1.$$

- If $m = s = p = 0$ in equation (3.2), we get an extension to the generating function for $L_{(\tilde{j},\tilde{k})}(\beta, u, o, r)$.

Corollary 3.4. *We have*

$$\sum_{n=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta, u, o, r) \frac{t^n}{(q;q)_n}$$
$$= \frac{t^{-k}}{(ut;q)_\infty} \sum_{j=0}^{k} \frac{(q^{-k}, ut;q)_j}{(q;q)_j} q^j M(\beta, r; q, otq^j), \quad |ut| < 1.$$

4. Rogers Formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$

We will present an operator approach to Rogers' formula and its extension for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$ in this section.

Theorem 4.1. (Roger's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$). *We have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p) \frac{t^n}{(q;q)_n} \frac{z^k}{(q;q)_k} = \frac{(mz;q)_\infty}{(t/z, uz;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(uz;q)_k}{(q, mz, qz/t;q)_k} q^k O(\beta, r, s, p; q, ozq^k), \quad \max\{|t/z|, |uz|\} < 1. \tag{4.1}$$

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} L_{n+k}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}\frac{z^k}{(q;q)_k}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\,P_{n+k}(u,m)\right\}\frac{t^n}{(q;q)_n}\frac{z^k}{(q;q)_k}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{n=0}^{\infty}P_n(u,m)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(u,q^nm)\frac{z^k}{(q;q)_k}\right\}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{n=0}^{\infty}P_n(u,m)\frac{t^n}{(q;q)_n}\frac{(q^nmz;q)_\infty}{(uz;q)_\infty}\right\}\quad\text{(by using (1.7))}\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\frac{P_n(u,m)}{(mz;q)_n}\frac{(mz;q)_\infty}{(uz;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\,z^{-n}\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},uz;q)_k}{(q,mz;q)_k}\,q^kO(\beta,r,s,p;q,ozq^k)\quad\text{(by using (2.4))}\\
&=\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/z)^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},uz;q)_k}{(q,mz;q)_k}\,q^kO(\beta,r,s,p;q,ozq^k)\\
&=\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k}{(q,mz;q)_k}\,q^kO(\beta,r,s,p;q,ozq^k)\sum_{n=k}^{\infty}\frac{(t/z)^n(-1)^kq^{\binom{k}{2}-nk}}{(q;q)_{n-k}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by using (1.6))}\\
&=\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k}{(q,mz;q)_k}\,q^kO(\beta,r,s,p;q,ozq^k)\sum_{n=0}^{\infty}\frac{(t/z)^{n+k}(-1)^kq^{\binom{k}{2}-(n+k)k}}{(q;q)_n}\\
&=\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k(t/z)^k(-1)^kq^{-\binom{k}{2}}}{(q,mz;q)_k}O(\beta,r,s,p;q,ozq^k)\sum_{n=0}^{\infty}\frac{(q^{-k}t/z)^n}{(q;q)_n}\\
&=\frac{(mz;q)_\infty}{(uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k(t/z)^k(-1)^kq^{-\binom{k}{2}}}{(q,mz;q)_k(q^{-k}t/z;q)_\infty}O(\beta,r,s,p;q,ozq^k)\\
&=\frac{(mz;q)_\infty}{(uz,t/z;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k(t/z)^k(-1)^kq^{-\binom{k}{2}}}{(q,mz;q)_k(q^{-k}t/z;q)_k}O(\beta,r,s,p;q,ozq^k)\\
&=\frac{(mz;q)_\infty}{(t/z,uz;q)_\infty}\sum_{k=0}^{\infty}\frac{(uz;q)_k\,q^k}{(q,mz,qz/t;q)_k}O(\beta,r,s,p;q,ozq^k).\quad\text{(by using (1.5))}
\end{aligned}
$$

□

- If $m=s=p=0$ in equation (4.1), we obtain Rogers formula for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$.

Corollary 4.2. *We have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}L_{n+k}(\beta,u,o,r)\frac{t^n}{(q;q)_n}\frac{z^k}{(q;q)_k}$$

$$= \frac{1}{(t/z, uz; q)_\infty} \sum_{k=0}^{\infty} \frac{(uz;q)_k q^k}{(q, qz/t;q)_k} M(\beta, r; q, ozq^k), \quad \max\{|t/z|, |uz|\} < 1.$$

- If $s = p = 0$ in equation (4.1), we find Rogers formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r)$ (equation (1.15)).

Theorem 4.3. (An extention to Roger's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$. *We have*

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} L_{a+b+c}^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{w^c}{(q;q)_c}$$
$$= \frac{(mw;q)_\infty}{(t/w, uw, \tau/t; q)_\infty} \sum_{c=0}^{\infty} \frac{(uw;q)_c q^c}{(q, mw, qw/t; q)_c} O(\beta, r, s, p; q, owq^c), \tag{4.2}$$

provided that $\max\{|t/w|, |uw|, |\tau/t|\} < 1$.

Proof.

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} L_{a+b+c}^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{w^c}{(q;q)_c}$$
$$= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \mathbb{O}(\beta, r, s, p; q, oD_{um}) \{P_{a+b+c}(u,m)\} \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{w^c}{(q;q)_c}$$
$$= \mathbb{O}(\beta, r, s, p; q, oD_{um}) \Bigg\{ \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} P_{a+b}(u,m) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \times \sum_{c=0}^{\infty} P_c(u, q^{a+b}m) \frac{w^c}{(q;q)_c} \Bigg\}$$
$$= \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} P_{a+b}(u,m) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{(q^{a+b}mw, q)_\infty}{(uw;q)_\infty} \right\} \quad \text{(by using (1.7))}$$
$$= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \mathbb{O}(\beta, r, s, p; q, oD_{um}) \left\{ \frac{P_{a+b}(u,m)}{(mw;q)_{a+b}} \frac{(mw;q)_\infty}{(uw;q)_\infty} \right\}$$
$$= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} w^{-(a+b)} \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{c=0}^{a+b} \frac{(q^{-(a+b)}, uw, q)_c q^c}{(q, mw;q)_c} \times O(\beta, r, s, p; q, owq^c) \quad \text{(by using (2.4))}$$
$$= \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{(t/w)^a}{(q;q)_{a+b}} \frac{(\tau/w)^b}{(q;q)_b} \sum_{c=0}^{a+b} \frac{(q^{-(a+b)}, uw, q)_c q^c}{(q, mw;q)_c} O(\beta, r, s, p; q, owq^c)$$
$$= \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{(t/w)^a (\tau/w)^b}{(q;q)_c} \sum_{c=0}^{a+b} \frac{(uw,q)_c q^c}{(mw;q)_c} \frac{(-1)^c q^{\binom{c}{2}} q^{-(a+b)c}}{(q;q)_{a+b-c}(q;q)_c} \times O(\beta, r, s, p; q, owq^c) \quad \text{(by using (1.6))}$$
$$= \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{a+b=c}^{\infty} \frac{(t/w)^a (\tau/w)^b}{(q;q)_b} \frac{(uw,q)_c q^c}{(mw;q)_c} \frac{(-1)^c q^{\binom{c}{2}} q^{-(a+b)c}}{(q;q)_{a+b-c}(q;q)_c}$$

$$\times\ O(\beta, r, s, p; q, owq^c)$$

$$= \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{b=0}^{\infty}\sum_{I=0}^{\infty}\sum_{c=0}^{\infty} \frac{(\tau/t)^b (t/w)^{c+I}}{(q;q)_b (q;q)_I} \frac{(-1)^c q^{\binom{c}{2}} q^{-(c+I)c+c}}{(q;q)_c} \frac{(uw,q)_c}{(mw;q)_c}$$

$$\times\ O(\beta, r, s, p; q, owq^c)$$

$$= \frac{(mw;q)_\infty}{(uw;q)_\infty} \sum_{b=0}^{\infty} \frac{(\tau/t)^b}{(q;q)_b} \sum_{c=0}^{\infty} \frac{(-1)^c q^{-\binom{c}{2}} (t/w)^c}{(q;q)_c} \frac{(uw,q)_c}{(mw;q)_c} O(\beta, r, s, p; q, owq^c)$$

$$\times \sum_{I=0}^{\infty} \frac{(q^{-c}t/w)^I}{(q;q)_I}$$

$$= \frac{(mw;q)_\infty}{(uw, \tau/t;q)_\infty} \sum_{c=0}^{\infty} \frac{(-1)^c q^{-\binom{c}{2}} (t/w)^c}{(q;q)_c} \frac{(uw,q)_c}{(mw;q)_c} O(\beta, r, s, p; q, owq^c)) \frac{1}{(q^{-c}t/w;q)_\infty}$$

$$= \frac{(mw;q)_\infty}{(uw, \tau/t, t/w;q)_\infty} \sum_{c=0}^{\infty} \frac{(-1)^c q^{-\binom{k}{2}} (t/w)^k}{(q;q)_k} \frac{(xw,q)_k}{(yw, q^{-k}t/w;q)_k} O(\beta, r, s, p; q, owq^c)$$

$$= \frac{(mw;q)_\infty}{(t/w, uw, \tau/t;q)_\infty} \sum_{c=0}^{\infty} \frac{(uw;q)_c q^c}{(q, mw, qw/t;q)_c} O(\beta, r, s, p; q, owq^c). \quad \text{(by using (1.5))}$$

□

- Setting $b = 0$ in equation (4.2), we obtain Rogers formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p)$ (equation (4.1)).
- Setting $m = s = p = 0$ in equation (4.2), we obtain an extension to Rogers formula for $L_{(\tilde{j},\tilde{k})}(\beta, u, o, r)$.

Corollary 4.4. *We have*

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} L_{n+m+k}(\beta, u, o, r) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{w^c}{(q;q)_c}$$
$$= \frac{1}{(uw, \tau/t, t/w;q)_\infty} \sum_{c=0}^{\infty} \frac{(uw;q)_c q^c}{(q, qw/t;q)_c} M(\beta, r; q, owq^c),$$

where $\max\{|uw|, |\tau/t|, |t/w|\} < 1$.

- If we setting $s = p = 0$ in equation (4.2), we obtain an extension to Rogers formula for $L_n^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r)$.

Corollary 4.5. *We have*

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} L_{a+b+c}^{(\tilde{j},\tilde{k})}(\beta, u, m, o, r, s, p) \frac{t^a}{(q;q)_{a+b}} \frac{\tau^b}{(q;q)_b} \frac{w^c}{(q;q)_c} = \frac{(mw;q)_\infty}{(uw, \tau/t, t/w;q)_\infty}$$
$$\times \sum_{c=0}^{\infty} \frac{(uw;q)_c q^c}{(q, mw, qw/t;q)_c} M(\beta, r; q, owq^c), \quad \max\{|uw|, |\tau/t|, |t/w|\} < 1.$$

5. Mehler Formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$

Mehler's formula using the operator technique and its application to polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ will be illustrated in this section. For the polynomials $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$, we will construct Mehler's formula and its extension by replacing variables in the Mehler's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ with special values.

Theorem 5.1. (Mehler formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}\\
&\quad=\frac{(mt;q)_\infty}{(ut;q)_\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_k(w,g)\frac{h^k}{(q;q)_k}\\
&\quad\quad\times\sum_{j=0}^{k}\frac{(q^{-k},ut;q)_j}{(q,mt;q)_j}\,q^jO(\beta,r,s,p;q,otq^j),\quad |ut|<1.
\end{aligned}\tag{5.1}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}\\
&=\sum_{n=0}^{\infty}\mathbb{O}(\beta,r,s,p;q,oD_{um})\{P_n(u,m)\}L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}\quad\text{(by using (2.2))}\\
&=O(\beta,r,s,p;q,oD_{um})\Bigg\{\sum_{n=0}^{\infty}P_n(u,m)\frac{t^n}{(q;q)_n}\\
&\qquad\times\sum_{k=0}^{n}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}h^kP_{n-k}(w,g)\frac{t^n}{(q;q)_n}\Bigg\}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\Bigg\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}\frac{t^nh^k}{(q;q)_k(q;q)_{n-k}}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_{n-k}(w,g)P_n(u,m)\Bigg\}\\
&\qquad\text{(by using (1.17))}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\Bigg\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_k(w,g)\frac{(ht)^k}{(q;q)_k}\\
&\qquad P_k(u,m)P_n(u,q^km)\frac{t^n}{(q;q)_n}\Bigg\}\quad\text{(by using (1.7))}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_k(w,g)p_k(u,m)\sum_{n=0}^{\infty}P_n(u,q^km)\frac{t^n}{(q;q)_n}\right\}\\
&=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{n}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,m;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_k(w,g)\frac{(ht)^k}{(q;q)_k}P_k(u,m)\frac{(q^kmt;q)_\infty}{(ut;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r,s;q)_k}{(p;q)_k}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}P_k(w,g)\frac{(ht)^k}{(q;q)_k}O(\beta,r,s,p;q,owq^k)\left\{\frac{P_k(u,m)(mt;q)_\infty}{(mt;q)_k(ut;q)_\infty}\right\}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} \frac{(r,s;q)_k}{(p;q)_k} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} P_k(w,g) \frac{(ht)^k}{(q;q)_k} t^{-k} \sum_{j=0}^{k} \{ \frac{(q^{-k},ut;q)_j}{(q,mt;q)_j} \} q^j O(\beta,r,s,p;q,otq^j)$$

$$= \frac{(mt;q)_\infty}{(ut;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} \frac{(r,s;q)_k}{(p;q)_k} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} P_k(w,g) \frac{h^k}{(q;q)_k} \sum_{j=0}^{k} \frac{(q^{-k},ut;q)_j q^j}{(q,mt;q)_j} q^j O(\beta,r,s,p;q,otq^j).$$

□

- As a spacial case of equation (5.1), we let $m = s = p = g = 0$. Then we get Mehler's formula for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$.

Corollary 5.2. *We have*

$$\sum_{n=0}^{\infty} L_{(\tilde{j},\tilde{k})}(\beta,u,o,r) L_{(\tilde{d},\tilde{e})}(\beta,w,h,r) \frac{t^n}{(q;q)_n} = \frac{1}{(ut;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} \frac{(r;q)_k (wh)^k}{(q;q)_k} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} \sum_{j=0}^{k} \frac{(q^{-k},ut;q)_j}{(q;q)_j} q^j M(\beta,r;q,otq^J), \quad |ut|<1.$$

- Substituting $s = p = 0$ in equation (5.1), we get Mehler's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

Corollary 5.3. *We have*

$$\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r) L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r) \frac{t^n}{(q;q)_n} = \frac{(mt;q)_\infty}{(ut;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} \frac{P_k(w,g)(r;q)_k h^k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(q;q)_k} \sum_{j=0}^{k} \frac{(q^{-k},ut;q)_j}{(q,mt;q)_j} q^j M(\beta,r;q,otq^j), \quad |ut|<1.$$

Theorem 5.4. (An extension Mehler's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$).

$$\sum_{n=0}^{\infty} L_{n+m}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p) L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p) \frac{t^n}{(q;q)_n}$$
$$= \frac{(mt;q)_\infty}{(ut;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) \frac{(ht)^k}{(q;q)_k} \frac{(r,s;q)_k}{(p;q)_k} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} t^{-(m+k)} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)},ut;q)_j}{(q,mt;q)_j} q^j$$
$$\times\ O(\beta,r,s,p;q,otq^j), \quad |ut|<1. \tag{5.2}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty} L_{n+m}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p) L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p) \frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} \mathbb{O}(\beta,r,s,p;q,oD_{um}) \{P_{m+n}(u,m)\} L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r,s,p) \frac{t^n}{(q;q)_n}$$
$$= \mathbb{O}(\beta,r,s,p;q,oD_{um}) \left\{ \sum_{n=0}^{\infty} P_{m+n}(u,g) \frac{(t)^n}{(q;q)_n} \right\}$$

$$\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} \frac{(r,s;q)_k}{(p;q)_k} P_{n-k}(w,g) h^k \quad \text{(by using (1.17))}$$

$$= \mathbb{O}(\beta,r,s,p;q,oD_{um}) \Big\{ \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \frac{f^k t^n}{(q;q)_k (q;q)_{n-k}}$$

$$\times \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_{n-k}(w,g) P_{m+n}(u,m) \Big\}$$

$$= \mathbb{O}(\beta,r,s,p;q,oD_{um}) \Big\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \frac{h^k t^{n+k}}{(q;q)_k (q;q)_n}$$

$$\times \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) P_{m+n+k}(u,m) \Big\}$$

$$= \mathbb{O}(\beta,r,s,p;q,oD_{um}) \Big\{ \sum_{k=0}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) P_{m+k}(u,m) \frac{(ht)^k}{(q;q)_k}$$

$$\times \sum_{n=0}^{\infty} P_n(x, q^{m+k} m) \frac{t^n}{(q;q)_n} \Big\}$$

$$= \mathbb{O}(\beta,r,s,p;q,oD_{um}) \Big\{ \sum_{k=0}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) P_{m+k}(u,m) \frac{(ht)^k}{(q;q)_k}$$

$$\times \frac{(q^{m+k} yt;q)_\infty}{(xt;q)_\infty} \Big\} \quad \text{(by using (1.7))}$$

$$= \sum_{k=0}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) \frac{(ht)^k}{(q;q)_k} \mathbb{O}(\beta,r,s,p;q,oD_{um}) \Big\{ \frac{P_{m+k}(u,m)}{(mt;q)_{m+k}} \frac{(mt;q)_\infty}{(ut;q)_\infty} \Big\}$$

$$= \sum_{k=0}^{\infty} \frac{(r,s;q)_k q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(p;q)_k} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) \frac{(ht)^k}{(q;q)_k} t^{-(m+k)} \frac{(mt;q)_\infty}{(ut;q)_\infty}$$

$$\times \sum_{j=0}^{m+k} \frac{(q^{-(m+k)}, ut;q)_j}{(q,mt;q)_j} q^j O(\beta,r,s,p;q,otq^j) \quad \text{(by using (2.4))}$$

$$= \frac{(mt;q)_\infty}{(ut;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \beta \\ k \end{bmatrix}_{-q} P_k(w,g) \frac{(ht)^k}{(q;q)_k} \frac{(r,s;q)_k}{(p;q)_k} q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}} t^{-(m+k)} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)}, ut;q)_j}{(q,mt;q)_j} q^j$$

$$\times\ O(\beta,r,s,p;q,otq^j).$$

□

- We restore Mehler's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ by substituting $m=0$ in equation (5.2).
- An extension to Mehler's formula for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ is obtained if we put $m=s=p=g=0$ in equation (5.2):

Corollary 5.5. *We have*

$$\sum_{n=0}^{\infty} L_{n+m}^{(\tilde{j},\tilde{k})}(\beta,u,o,r) L_n^{(\tilde{d},\tilde{e})}(\beta,w,h,r) \frac{t^n}{(q;q)_n} = \frac{1}{(ut;q)_\infty}$$

$$\times\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(r;q)_k(wht)^kq^{\binom{k}{2}}t^{-(m+k)}}{(q;q)_k}\sum_{j=0}^{m+k}\frac{(q^{-(m+k)},ut;q)_j}{(q;q)_j}q^jM(\beta,r;q,otq^j),$$

where $|ut|<1$.

- By setting $s=p=0$ in equation (5.2), Mehler's formula for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ is obtained.

Corollary 5.6. *We have*

$$\sum_{n=0}^{\infty}L_{n+m}^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)L_n^{(\tilde{d},\tilde{e})}(\beta,w,g,h,r)\frac{t^n}{(q;q)_n}=\frac{(mt;q)_\infty}{(ut;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\begin{bmatrix}\beta\\k\end{bmatrix}_{-q}\frac{(u;q)_kP_k(w,g)(ht)^kt^{-(m+k)}q^{\tau(\tilde{d},\tilde{e})+\binom{k}{2}}}{(q;q)_k}\sum_{j=0}^{m+k}\frac{(q^{-(m+k)},wt;q)_jq^j}{(q;q)_j}$$
$$\times\ M(\beta,r;q,otq^j),\quad |wt|<1.$$

6. Mixed Generating Functions for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$

The polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ has two mixed generating functions they are stated and proved in this section. We obtain another mixed generating function for both the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ by providing some certain values for the parameters in the two mixed generating functions for the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$. Also, we get Srivastava-Agarwal type generating function for the polynomiasls $L_n^{(\tilde{j},\tilde{k})}(\beta,u,o,r)$, $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$, and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$.

Theorem 6.1. (Mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,w,g,h,r,s,p)$). *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)L_n^{(\tilde{j},\tilde{k})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}=\frac{(gt,\alpha x;q)_\infty}{(wt,x;q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(wt,\alpha;q)_n}{(q,gt,q/x;q)_n}q^n\ O(\beta,r,s,p;q,otq^n),\quad \max\{|wt|,|x|\}<1. \tag{6.1}$$

Proof.

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)L_n^{(\tilde{j},\tilde{k})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(\alpha;q)_kx^kL_n^{(\tilde{j},\tilde{k})}(\beta,w,g,h,r,s,p)\frac{t^n}{(q;q)_n}\quad\text{(by using (1.9))}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(\alpha;q)_kx^k\mathbb{O}(\beta,r,s,p;q,oD_{wg})\left\{P_n(w,g)\right\}\frac{t^n}{(q;q)_n}\quad\text{(by using (2.2))}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{wg})\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(\alpha;q)_kx^kP_n(w,g)\frac{t^n}{(q;q)_n}\right\}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{wg})\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(\alpha;q)_kx^kP_n(w,g)t^n}{(q;q)_k(q;q)_{n-k}}\right\}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{wg})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_k x^k P_{n+k}(w,g)t^{n+k}}{(q;q)_k(q;q)_n}\right\}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{wg})\left\{\sum_{k=0}^{\infty}P_k(w,g)\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}P_n(w,q^k g)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{O}(\beta, r, s, p; q, oD_{wg})\left\{\sum_{k=0}^{\infty}P_k(w,g)\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\frac{(q^k gt;q)_\infty}{(wt;q)_\infty}\right\} \quad \text{(by using (1.7))}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\mathbb{O}(\beta, r, s, p; q, oD_{wg})\left\{\frac{P_k(w,g)}{(gt;q)_k}\frac{(gt;q)_\infty}{(wt;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\frac{(gt;q)_\infty}{(wt;q)_\infty}t^{-k}\sum_{n=0}^{k}\frac{(q^{-k},wt;q)_n}{(q,gt;q)_n}q^n O(\beta,r,s,p;q,otq^n)$$

$$\text{(by using (2.4))}$$

$$= \frac{(gt;q)_\infty}{(wt;q)_\infty}\sum_{k=0}^{\infty}\frac{(\alpha;q)_k}{(q;q)_k}x^k\sum_{n=0}^{k}\frac{(q^{-k},wt;q)_n}{(q,gt;q)_n}q^n O(\beta,r,s,p;q,otq^n))$$

$$= \frac{(gt;q)_\infty}{(wt;q)_\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{k}\frac{(\alpha;q)_k x^k}{(q;q)_k}\frac{(q^{-k},wt;q)_n q^n}{(gt,q;q)_n}\frac{(-q^{-\beta};q)_l(r,s;q)_l(otq^n)^l}{(q^2;q^2)_l(p;q)_l}q^{l\beta+\tau(\tilde{j},\tilde{i})} \tag{6.2}$$

$$= \frac{(gt;q)_\infty}{(wt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(wt;q)_n}{(q,gt;q)_n}q^n O(\beta,r,s,p;q,otq^n)\sum_{k=n}^{\infty}\frac{(\alpha;q)_k x^k q^{-nk}}{(q;q)_{k-n}}$$

$$= \frac{(gt;q)_\infty}{(wt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(wt,\alpha;q)_n}{(q,gt;q)_n}x^n O(\beta,r,s,p;q,otq^n)\sum_{k=0}^{\infty}\frac{(q^n\alpha;q)_k}{(q;q)_k}(q^{-n}x)^k$$

$$= \frac{(gt;q)_\infty}{(wt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(ut,\alpha;q)_n}{(q,gt;q)_n}x^n O(\beta,r,s,p;q,otq^n)\frac{(\alpha x;q)_\infty}{(q^{-n}x;q)_\infty}$$

$$= \frac{(gt,\alpha x;q)_\infty}{(wt,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(wt,\alpha;q)_n}{(q,gt;q)_n(q^{-n}x;q)_n}x^n O(\beta,r,s,p;q,otq^n)$$

$$= \frac{(gt,\alpha x;q)_\infty}{(wt,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(wt,\alpha;q)_n}{(q,gt,q/x;q)_n}q^n O(\beta,r,s,p;q,otq^n). \quad \text{(by using (1.5))}$$

□

- Not that equation (6.2) is the same as equation (1.20).
- Allowing $g = s = p = 0$ in equation (6.1), we obtain another mixed generating function for $L_{(\tilde{j},\tilde{k})}(\beta, w, h, r)$.

Corollary 6.2. *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)L_{(\tilde{j},\tilde{k})}(\beta,w,h,r)\frac{t^n}{(q;q)_n} = \frac{(\alpha x;q)_\infty}{(wt,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,wt;q)_n}{(q,q/x;q)_n}q^n M(\beta,r;q,otq^n),$$

where $\max\{|wt|,|x|\} < 1$.

- Letting $s=p=0$ in equation (6.2), we obtain mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ (equation (1.16)).

Theorem 6.3. (Another mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$). *We have*

$$\sum_{n=0}^{\infty}P_n(w,g)L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}$$
$$=\frac{(g/w,mwt;q)_\infty}{(uwt;q)_\infty}\sum_{j=0}^{\infty}\frac{(uwt;q)_j(g/w)^j}{(q,mwt;q)_j}O(\beta,r,s,p;q,owtq^j),\quad |uwt|<1. \tag{6.3}$$

Proof.

$$\sum_{n=0}^{\infty}P_n(w,g)L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}P_n(w,g)\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{P_n(u,g)\right\}\frac{t^n}{(q;q)_n}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{n=0}^{\infty}P_n(w,g)P_n(u,m)\frac{t^n}{(q;q)_n}\right\}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^kq^{\binom{k}{2}}g^kw^{n-k}P_n(u,m)\frac{t^n}{(q;q)_n}\right\}$$
(by using (1.8))
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}g^k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{P_n(u,m)w^{n-k}t^n}{(q;q)_{n-k}}\right\}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}g^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_{n+k}(u,m)w^nt^{n+k}}{(q;q)_n}\right\}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(gt)^kP_k(u,m)}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_n(u,q^km)}{(q;q)_n}(wt)^n\right\}$$
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(gt)^kP_k(u,m)}{(q;q)_k}\frac{(q^kmwt;q)_\infty}{(uwt;q)_\infty}\right\}$$
(by using (1.7))
$$=\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(gt)^k}{(q;q)_k}\frac{P_k(u,m)}{(mwt;q)_k}\frac{(mwt;q)_\infty}{(uwt;q)_\infty}\right\}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(gt)^k}{(q;q)_k}\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\frac{P_k(u,m)}{(mwt;q)_k}\frac{(mwt;q)_\infty}{(uwt;q)_\infty}\right\}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(gt)^k}{(q;q)_k}(wt)^{-k}\frac{(mwt;q)_\infty}{(uwt;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},uwt;q)_jq^j}{(q,mwt;q)_j}$$
$$\times\ O(\beta,r,s,p;q,owtq^j)\quad\text{(by using (2.4))}$$

$$=\frac{(mwt;q)_\infty}{(uwt;q)_\infty}\sum_{j=0}^{\infty}\frac{(uwt;q)_j(-1)^jq^{\binom{j}{2}}q^j}{(q,mwt;q)_j}O(\beta,r,s,p;q,owtq^j)$$
$$\times\sum_{k=j}^{\infty}\frac{(-1)^kq^{\binom{k}{2}}(g/w)^kq^{-kj}}{(q;q)_{k-j}}$$
$$=\frac{(g/w,mwt;q)_\infty}{(uwt;q)_\infty}\sum_{j=0}^{\infty}\frac{(uwt;q)_j(g/w)^j}{(q,mwt;q)_j}O(\beta,r,s,p;q,owtq^j).$$

□

- Setting $m=s=p=0$, $t=1$, and then $w=t$, $g=s$ in equation (6.3), we regain the mixed generating function for $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$ (equation (1.12)).

Proof.

$$\sum_{n=0}^{\infty}L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)\frac{P_n(t,s)}{(q;q)_n}$$
$$=\frac{(s/t;q)_\infty}{(ut;q)_\infty}\sum_{j=0}^{\infty}\frac{(ut;q)_j(s/t)^j}{(q;q)_j}\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta},r;q)_l}{(q^2;q^2)_l}(otq^j)^l$$
$$=\frac{(s/t;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta},r;q)_l}{(q^2;q^2)_l}(ot)^l\sum_{j=0}^{\infty}\frac{(ut;q)_j}{(q;q)_j}(sq^l/t)^j$$
$$=\frac{(s/t;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta},r,s/t;q)_l}{(q^2;q^2)_l(us;q)_l}(ot)^l\frac{(us;q)_\infty}{(s/t;q)_\infty}\quad\text{(by using (1.3))}$$
$$=\frac{(us;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}q^{l\beta+\tau(\tilde{j},\tilde{k})}\frac{(-q^{-\beta};q)_l}{(-q;q)_l}\frac{(r,s/t;q)_l}{(q,us;q)_l}(ot)^l$$
$$=\frac{(us;q)_\infty}{(ut;q)_\infty}\sum_{l=0}^{\infty}\begin{bmatrix}\beta\\ l\end{bmatrix}_{-q}q^{\binom{l}{2}+\tau(\tilde{j},\tilde{k})}\frac{(r,s/t;q)_l}{(q,us;q)_l}(ot)^l.\quad\text{(by using (1.2))}$$

□

- Setting $s=p=0$ in equation (6.3) yields another mixed generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

Corollary 6.4. *We have*

$$\sum_{n=0}^{\infty}P_n(w,g)L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)\frac{t^n}{(q;q)_n}$$
$$=\frac{(g/w,mwt;q)_\infty}{(uwt;q)_\infty}\sum_{j=0}^{\infty}\frac{(uwt;q)_j(g/w)^j}{(q,mwt;q)_j}M(\beta,r;q,owtq^j),\quad |uwt|<1.$$

- Letting $m=s=p=0$ in equation (6.3), produces Srivastava-Agarwal type generating function for $L_n(\beta,u,o,r)$.

Corollary 6.5. *We have*

$$\sum_{n=0}^{\infty} L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)(\lambda;q)_n \frac{t^n}{(q;q)_n}$$
$$= \frac{(\lambda;q)_\infty}{(ut;q)_\infty} \sum_{j=0}^{\infty} \frac{(ut;q)_j}{(q;q)_j} \lambda^j O(\beta,r;q,otq^j), \quad |ut|<1.$$

- Setting $s=p=0$ in equation (6.3), we get Srivastava-Agarwal type generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

Corollary 6.6. *We have*

$$\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)(\lambda;q)_n \frac{t^n}{(q;q)_n}$$
$$= \frac{(\lambda,mt;q)_\infty}{(ut;q)_\infty} \sum_{j=0}^{\infty} \frac{(ut;q)_j}{(q,mt;q)_j} (q\lambda)^j O(\beta,r;q,otq^j), \quad |ut|<1.$$

- If we set $g/w=\lambda$ and letting $wt\to t$ in equation (6.3), we obtain Srivastava-Agarwal type generating function for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$.

Corollary 6.7. *We have*

$$\sum_{n=0}^{\infty} L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)(\lambda;q)_n \frac{t^n}{(q;q)_n}$$
$$= \frac{(\lambda,m;q)_\infty}{(ut;q)_\infty} \sum_{j=0}^{\infty} \frac{(ut;q)_j}{(q,mt;q)_j} \lambda^j O(\beta,r,s,p;q,otq^j), \quad |ut|<1.$$

7. A Transformational Identity Involving Generating Functions for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$

In this section, the transformational identity for $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ is obtained by applying the homogeneous q-shift operator $\mathbb{O}(\beta,r,s,p;q,oD_{um})$.

Theorem 7.1. *Assume the following relation is satisfied by the coefficients $E(h)$ and $G(h)$:*

$$\sum_{h=0}^{\infty} E(h)P_h(u,m) = \sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty}. \tag{7.1}$$

Then

$$\sum_{h=0}^{\infty} E(h)_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$$
$$= \sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty} O(\beta,r,s,p;q,otq^h), \tag{7.2}$$

where each of the series in (7.1) and (7.2) are absolutely convergent.

Proof. Let $g(u,m,c)$ represent (7.2)'s right-hand side. We discover that by employing (7.1) and (2.2).

$$
\begin{aligned}
g(u,m,c) &= \sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty} O(\beta,r,s,p;q,otq^h) \\
&= \sum_{h=0}^{\infty} G(h)\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty}\right\} \\
&= \mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty}\right\} \\
&= \mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{\sum_{h=0}^{\infty} E(h)P_h(u,m)\right\} \\
&= \sum_{h=0}^{\infty} E(h)\mathbb{O}(\beta,r,s,p;q,oD_{um})\left\{P_h(u,m)\right\} \\
&= \sum_{h=0}^{\infty} E(h)_h^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p).
\end{aligned} \tag{7.3}
$$

It is exactly (7.2)'s left-hand side. Theorem 7.1 has been fully proved. □

- Letting $m=s=p=0$ in equations (7.1) and (7.2), we get the transformational formula for the polynomials $L_{(\tilde{j},\tilde{k})}(\beta,u,o,r)$.

Corollary 7.2. *Assume that the following connection is satisfied by the coefficients $E(h)$ and $G(h)$:*

$$\sum_{h=0}^{\infty} E(h)u^h = \sum_{h=0}^{\infty} \frac{G(h)}{(q^h ut;q)_\infty}. \tag{7.4}$$

Then

$$\sum_{h=0}^{\infty} E(h)L_{(\tilde{j},\tilde{k})}(\beta,u,o,r) = \sum_{h=0}^{\infty} \frac{G(h)}{(q^h ut;q)_\infty} O(\beta,r;q,otq^h), \tag{7.5}$$

where each of the series in (7.4) and (7.5) are absolutely convergent.

- Setting $s=p=0$ in equations (7.1)and (7.2), we find the transformational identity for the polynomials $_h^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

Corollary 7.3. *Assume that the coefficients $E(h)$ and $G(h)$ meet the following criteria:*

$$\sum_{h=0}^{\infty} E(h)P_h(u,m) = \sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty} \tag{7.6}$$

Then

$$\sum_{h=0}^{\infty} E(h)_h^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r) = \sum_{h=0}^{\infty} G(h)\frac{(q^h mt;q)_\infty}{(q^h ut;q)_\infty} O(\beta,r;q,otq^h), \tag{7.7}$$

where each of the series in (7.6) and (7.7) are absolutely convergent.

8. Conclusions

(1) The polynomials $L_n(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ are special cases of the polynomials $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$.

(2) The polynomials identities of $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ are a generalization for polynomials identities for both polynomials $L_n(\beta,u,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$.

(3) We obtained numerous results for the polynomials $L_n(\beta,u,o,r)$, $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r)$ and $L_n^{(\tilde{j},\tilde{k})}(\beta,u,m,o,r,s,p)$ which are described by Jia et al. [20] and Cao et al. [9] and [10], respectively.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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