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EXTENDING THE MEDIAN PROBLEM TO SIGNED PERMUTATIONS VIA GENERALIZED KENDALL-τ DISTANCE

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Abstract. The "median of permutation" problem involves determining a permutation that is the "closest" to a given set of permutations under the Kendall-τ distance metric and is a central challenge in rank aggregation. In this article, we extend this framework to the hyperoctahedral group of type $\mathcal{B}_n$ of signed permutations by introducing a generalized Kendall-τ distance metric capturing both positional and sign disagreements. This enables the formulation of median problems in contexts where directionality is inherent, such as gene regulatory networks (GRN). We show that any subset of $\mathcal{B}_n$ closed under total negation has $\mathcal{B}_n$ as its median set, and that unsigned subsets closed under reversal yield the symmetric group $\mathfrak{S}_n$. To support efficient distance computation, we construct a weighted distance graph $\mathcal{G}_n$ whose edges represent elementary operations. Our findings provide new theoretical insights into signed rank aggregation and offer a foundation for combinatorial optimization beyond the classical setting of $\mathfrak{S}_n$.

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1. Introduction

The Kendall-τ distance, which quantifies the number of pairwise disagreements in the relative ordering of two permutations, has long been a topic of interest in combinatorial optimization and ranking theory. The problem of finding medians of a set of permutations under the Kendall-τ distance [15, 20], which counts the number of pairwise order disagreements between permutations, is a central challenge in rank aggregation and consensus-building across various disciplines. This problem, often referred to as the Kemeny Score Problem [14], has significant applications in social choice theory, decision-making, and data aggregation. Initially formulated in Kemeny's seminal work on ranking problems, the task involves determining a consensus order of n candidates based on rankings provided by m voters in a way that minimises the overall Kendall-τ distance.

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The Kemeny Score Problem is NP-complete when the number of input rankings m is an even number at least 4 [4,12] and remains NP-hard for $m \geq 7$ when m is odd [2]. However, the complexity of the problem for smaller odd values such as $m = 3$ or $m = 5$ remains an open question.

From the late twentieth century to recent years, various approaches have been developed to cope with this computational challenge, including approximation algorithms, fixed parameter tractable (FPT) strategies [7,10], and even a polynomial-time approximation scheme (PTAS) [16]. Comparative evaluations of these methods are available in [1,19]. Complementing algorithmic efforts, several theoretical techniques have been proposed to reduce the search space, thereby simplifying median-finding. For example, [6] introduced constraints based on pairwise orderings and adjacency in candidate medians. Later, [3] introduced the idea of non-dirty candidates, elements consistently ranked above or below others in a significant fraction of input permutations, which allow for decomposition of the problem into smaller, independent subproblems. However, such candidates are rarely encountered in randomly generated instances. Further refinements in data reduction with less restrictive were proposed in [17], using the combinatorial properties of "almost adjacent" elements in median sets.

Moreover, modern efforts have explored novel paradigms such as quantum optimization for solving the Kemeny ranking aggregation problem. For instance, Combarro et al. [8] formulated the problem using multiple Quadratic Unconstrained Binary Optimization (QUBO)-based encodings and evaluated their effectiveness using quantum approximate optimization algorithms and quantum annealing. These explorations, while still constrained by current hardware limitations, underscore the importance of efficient formulations in preparing for scalable quantum solutions in the near future. Additionally, Rico et al. [18] proposed exact algorithms based on necessary conditions for a ranking to be optimal under the Kemeny method, significantly reducing computation time for instances with up to 14 alternatives. These advancements reinforce the centrality of the Kemeny problem in computational social choice and its evolving relevance in high-performance and hybrid computing contexts.

An intriguing angle on this problem is offered by the notion of automedian sets, which are subsets of permutations that remain invariant under the median operation. Such sets inherently satisfy a centrality property under the Kendall-τ distance and thus provide promising candidates for efficient median computation in polynomial time. Recent studies have examined how automedian sets behave under operations such as the direct sum and shuffle product, enabling constructive strategies for building such sets in larger permutation groups [11]. Two notable constructions identified in [13] include sets formed by a permutation and its cyclic shifts and sets with a shared S_k-kernel, potentially with some fixed or common elements. These examples reveal that symmetry and regular structure often underpin the automedian property. Furthermore, new variants based on direct sum operations have emerged, along with parallel algorithms aimed at efficient median detection in separable permutation sets.

This problem has gained significant attention in computational biology, particularly in the study of genome rearrangement and gene expression patterns. In recent work, Cunha et al. [9] have conducted a comprehensive parameterized complexity analysis of the median and closest permutation problems under various genome rearrangement metrics. They have explored the computational complexity and structural aspects of permutation medians under various distance functions—such as swap, breakpoint, transposition, and block-interchange distances. They demonstrate that even when restricted to only three input permutations, most variants of the problem remain NP-hard. While their analysis focuses on unsigned permutations, it underscores the inherent intractability of consensus problems over permutation spaces and motivates the need to study the analogous median problem in the signed setting.

However, in many real-world contexts, rankings are inherently signed—each element not only has a relative position but also an associated sign indicating activation or repression, presence or absence, or positive or negative sentiment. This naturally leads to considering rankings as signed permutations. In the context of Gene Regulatory Networks (GRNs), understanding the relative activity and influence of genes under varying experimental or biological conditions is of central importance. These conditions often yield signed rankings of genes, where each gene is not only ranked by importance but also annotated with a direction—upregulation (activation) or downregulation (repression). Aggregating such signed rankings across multiple datasets or conditions enables the identification of consensus regulatory behavior, providing insight into core regulatory mechanisms.

In this work, we propose a novel approach for this aggregation task: we extend the classical Kemeny framework to the hyperoctahedral group of type $\mathcal{B}_n$, the signed permutation group, which generalizes the symmetric group $\mathfrak{S}_n$, thereby enabling the computational study of median sets of signed rankings that faithfully reflect both gene ordering and regulatory direction. The additional structure introduced by signed permutations necessitates a refinement of classical notions such as the Kendall-τ distance and order disagreements. By defining Type I and Type II disagreements and incorporating sign differences, we establish a new generalized Kendall-τ distance (see Definition 5) suitable for $\mathcal{B}_n$. By leveraging the combinatorial structure of the hyperoctahedral group $\mathcal{B}_n$, our framework offers a principled and mathematically grounded method to summarize signed gene rankings across experiments. This work not only contributes a new angle to rank aggregation in computational biology but also broadens the applicability of median-based methods beyond the classical symmetric group $\mathfrak{S}_n$.

The computational complexity of the median problem under the generalized Kendall-τ distance in $\mathcal{B}_n$ also presents a fundamental theoretical challenge. While the problem is computationally tractable for small input sizes, it appears to be NP-hard in general as the number of input signed permutations increases. This aligns with known results for the classical Kendall-τ distance in $\mathcal{S}_n$ and suggests that a similar complexity-theoretic barrier may exist in the signed case. However, a formal classification of the problem's computational complexity remains an open direction for future research.

Our primary contributions are the following. We formulate a Generalized Kendall-τ distance metric on the hyperoctahedral group of type $\mathcal{B}_n$ and extend the median concept to this group. We prove that the median set of a subset $\mathcal{A} \subseteq \mathcal{B}_n$ equals the entire group, i.e., $\mathcal{M}(\mathcal{A}) = \mathcal{B}_n$ if it is closed with regard to the total negation operation '$-$'. Similarly, if a subset consists solely of unsigned permutations and is closed under reversal, then its median set coincides with the symmetric group: $\mathcal{M}(\mathcal{A}) = \mathfrak{S}_n$. We also introduce the notion of a distance graph $\mathcal{G}_n(V, E, \omega)$ over $\mathcal{B}_n$, where edges correspond to elementary operations such as adjacent transpositions and sign flips. This graph framework enables efficient computation of pairwise distances and medians, particularly for smaller instances.

The paper is organized as follows. Section 2 lays out the theoretical framework, including definitions of order disagreements, sign differences, and the generalized Kendall-τ distance in $\mathcal{B}_n$. Section 3 introduces and characterizes universal-median sets, which are subsets of $\mathcal{B}_n$ whose median set equals the entire group. Section 4 presents the construction of the distance graph $\mathcal{G}_n$ and its application to median computation. Section 5 focuses on $\mathfrak{S}_n$-median sets, where the median set aligns with the symmetric group.

2. Generalized Kendall-τ distance on $\mathcal{B}_n$

Throughout this article, n is a positive integer. A **permutation** π is a bijection of $[n] = \{1, 2, \ldots, n\}$ onto itself. The set of all permutations of $[n]$ under composition operation forms a group, called the **symmetric group** $\mathfrak{S}_\mathbf{n}$. The order of $\mathfrak{S}_n$ is $n!$. By convention, we stick to the order of $\mathcal{S}_0$ as 1. We follow the one-line notation to write a permutation.

Definition 1 (Hyperoctahedral group of type $\mathcal{B}_n$)**.** [5] *The hyperoctahedral group of type $\mathcal{B}_n$ is the group of signed permutations on n elements, representing the symmetry group of the n-dimensional hypercube. Formally, it consists of all bijection*

$$\sigma : \{\pm 1, \pm 2, \ldots, \pm n\} \to \{\pm 1, \pm 2, \ldots, \pm n\}$$

such that $\sigma(-i) = -\sigma(i)$ for all i.

Note that for any positive integer n, we have $|\mathcal{B}_n| = 2^n n!$ and $|\mathcal{B}_0| = 1$. For a signed permutation σ, we use the following window-like notation: $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$.

In order to generalize the Kendall-τ distance for signed permutations in the hyperoctahedral group of type $\mathcal{B}_n$, we first introduce the notion of order disagreements between two signed permutations. These disagreements are classified into two types based on the relative ordering and signs of their entries in the respective inverses of the permutations. We begin by defining the order disagreement of Type I.

Definition 2 (Order disagreement of Type I)**.** *The order disagreement of Type I between pairs of elements of two signed permutations in $\mathcal{B}_n$ is defined as follows:*

For $\pi = \pi_1\pi_2\cdots\pi_n,\ \sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{B}_n$,

$$O_1(\pi,\sigma) = \Big\{(i,j) \mid i<j \text{ and } \Big[(|\pi_i^{-1}| < |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| > |\sigma_j^{-1}|) \text{ or } (|\pi_i^{-1}| > |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| < |\sigma_j^{-1}|)\Big]\Big\}.$$

Similarly, we define another type of order disagreement which considers not only the relative positions but also the relative signs of the elements in the inverses of the permutations.

Definition 3 (Order disagreement of Type II)**.** *The order disagreement of Type II between pairs of elements of two signed permutations in $\mathcal{B}_n$ is defined as follows: For $\pi = \pi_1\pi_2\cdots\pi_n,\ \sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{B}_n$,*

$$\begin{aligned} O_2(\pi,\sigma) = \Big\{(i,j) \mid i<j \text{ and } &\Big[(|\pi_i^{-1}| < |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| < |\sigma_j^{-1}|) \\ &\quad \text{or } (|\pi_i^{-1}| > |\pi_j^{-1}| \text{ and } |\sigma_i^{-1}| > |\sigma_j^{-1}|)\Big] \\ \text{and } &\Big[(\pi_i^{-1} < \pi_j^{-1} \text{ and } \sigma_i^{-1} > \sigma_j^{-1}) \\ &\quad \text{or } (\pi_i^{-1} > \pi_j^{-1} \text{ and } \sigma_i^{-1} < \sigma_j^{-1})\Big]\Big\}. \end{aligned}$$

Remark 1. *Each pair (i,j) contributes to at most one type of order disagreement (Type 1 or Type 2). That is, if a pair contributes to one type, it does not contribute to the other. It is also possible that a pair does not contribute to either type.*

Apart from order disagreements, another crucial aspect when comparing signed permutations is the difference in signs at corresponding positions. This is formalized in the following definition.

Definition 4 (Sign difference)**.** *The sign difference between the elements of signed permutations in $\mathcal{B}_n$ is defined as, for $\pi = \pi_1\pi_2\cdots\pi_n,\ \sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{B}_n$,*

$$\operatorname{sgn}(\pi,\sigma) = \{i \in \mathbb{N} \mid \operatorname{sgn}(\pi_i^{-1})\operatorname{sgn}(\sigma_i^{-1}) < 0\}$$

where $\operatorname{sgn}(\alpha)$ denotes the sign of $\alpha \in \mathbb{Z}\setminus\{0\}$, which is $+1$ for positive elements and -1 for negative elements.

Using these notions of order disagreements and sign differences, we now define the generalized Kendall-τ distance for signed permutations in $\mathcal{B}_n$.

Definition 5 (Generalized Kendall-τ distance on $\mathcal{B}_n$)**.** *For $\pi,\ \sigma \in \mathcal{B}_n$, the generalized Kendall-τ distance, denoted as $\overline{d_{KT}}$, is defined as the sum of the cardinalities (counting multiplicities) of the sets of order disagreements of Type I, order disagreements of Type II, and the difference in sign between π and σ. i.e.,*

$$\overline{d_{KT}}(\pi,\sigma) = 2|O_1(\pi,\sigma)| + 4|O_2(\pi,\sigma)| + |\operatorname{sgn}(\pi,\sigma)|$$

To illustrate the above definitions, we present a simple example demonstrating the computation of the generalized Kendall-τ distance.

Example 1. *Consider $\pi = 1\ -3\ 2$, $\sigma = -1\ 2\ 3 \in \mathcal{B}_n$. To compute the generalized Kendall-τ distance between π and σ, consider $\pi^{-1} = 1\ 3\ -2$ and $\sigma^{-1} = -1\ 2\ 3$ to be the inverses of π and σ, respectively. Clearly, the pair $(2,3)$ contributes to the order disagreement of Type I and the pair $(1,3)$ contributes to the order disagreement of Type II. In addition, the pair $(1,2)$ does not contribute to any of the order disagreement types. It is easy to check $sgn(\pi,\sigma) = \{1,3\}$. Thus we have*

$$\overline{d_{KT}}(\pi,\sigma) = 2(1) + 4(1) + 2 = 8$$

We now establish that the generalized Kendall-τ distance satisfies the axioms of a metric on the hyperoctahedral group of type $\mathcal{B}_n$.

Theorem 1. *The generalized Kendall-τ distance, $\overline{d_{KT}}$ forms a metric on $\mathcal{B}_n$, i.e., the function*

$$\overline{d_{KT}} : \mathcal{B}_n \times \mathcal{B}_n \to \mathbb{R}$$

satisfying the following axioms for all signed permutations $\pi, \sigma, \alpha \in \mathcal{B}_n$:

- **(i):** $\overline{d_{KT}}(\pi,\sigma) \geq 0$ *for* $\pi \neq \sigma$ *(Non-negativity)*
- **(ii):** $\overline{d_{KT}}(\pi,\sigma) = 0$ *if and only if* $\pi = \sigma$
- **(iii):** $\overline{d_{KT}}(\pi,\sigma) = \overline{d_{KT}}(\sigma,\pi)$ *(Symmetry)*
- **(iv):** $\overline{d_{KT}}(\pi,\sigma) \leq \overline{d_{KT}}(\pi,\alpha) + \overline{d_{KT}}(\alpha,\sigma)$. *(Triangle inequality)*

Proof. **(i)** (Non-negativity). We need to show that $\overline{d_{KT}}(\pi,\sigma) \geq 0$ for all $\pi, \sigma \in \mathcal{B}_n$.

Since $O_1(\pi,\sigma)$, $O_2(\pi,\sigma)$, and $\mathrm{sgn}(\pi,\sigma)$ are sets that count the pairs of elements of some sort as defined above, and the cardinality of these sets are always non-negative, we have

$$|O_1(\pi,\sigma)| \geq 0, \quad |O_2(\pi,\sigma)| \geq 0, \quad |\,\mathrm{sgn}(\pi,\sigma)| \geq 0.$$

Thus the generalized Kendall-τ distance is

$$\overline{d_{KT}}(\pi,\sigma) = 2|O_1(\pi,\sigma)| + 4|O_2(\pi,\sigma)| + |\,\mathrm{sgn}(\pi,\sigma)| \geq 0.$$

Therefore, $\overline{d_{KT}}(\pi,\sigma) \geq 0$ for all $\pi, \sigma \in \mathcal{B}_n$.

(ii) We need to show that $\overline{d_{KT}}(\pi,\sigma) = 0$ if and only if $\pi = \sigma$.

If $\pi = \sigma$, then the elements of π and σ are identical at every position. Therefore, there are no order disagreement of Type I or Type II, and there are no sign differences:

$$O_1(\pi,\sigma) = O_2(\pi,\sigma) = \mathrm{sgn}(\pi,\sigma) = \varnothing.$$

Thus

$$\overline{d_{KT}}(\pi,\sigma) = 2|O_1(\pi,\sigma)| + 4|O_2(\pi,\sigma)| + |\,\mathrm{sgn}(\pi,\sigma)| = 0.$$

Hence, if $\pi = \sigma$, then $\overline{d_{KT}}(\pi, \sigma) = 0$.

Conversely, suppose $\overline{d_{KT}}(\pi, \sigma) = 0$. Then

$$\overline{d_{KT}}(\pi, \sigma) = 2|O_1(\pi, \sigma)| + 4|O_2(\pi, \sigma)| + |\operatorname{sgn}(\pi, \sigma)| = 0.$$

$$\Rightarrow |O_1(\pi, \sigma)| = |O_2(\pi, \sigma)| = |\operatorname{sgn}(\pi, \sigma)| = 0.$$

since $O_1(\pi, \sigma),\ O_2(\pi, \sigma),$ and $\operatorname{sgn}(\pi, \sigma)$ are non-negative. This implies that there are no Type I or Type II order disagreements and there are no sign differences. Therefore, the relative order of the elements of π and σ must be identical, and the signs of the elements must match. The only way this can happen is if $\pi = \sigma$. Thus, $\overline{d_{KT}}(\pi, \sigma) = 0$ implies that $\pi = \sigma$.

Therefore, $\overline{d_{KT}}(\pi, \sigma) = 0$ if and only if $\pi = \sigma$.

(iii) (Symmetry). We need to show that $\overline{d_{KT}}(\pi, \sigma) = \overline{d_{KT}}(\sigma, \pi)$ for all $\pi, \sigma \in \mathcal{B}_n$.

By the definition of $O_1(\pi, \sigma)$, $O_2(\pi, \sigma)$, and $\operatorname{sgn}(\pi, \sigma)$, these sets depend only on the relative ordering and signs of the elements of π and σ, not on the order in which they are compared. Specifically:

$$O_1(\pi, \sigma) = O_1(\sigma, \pi), \quad O_2(\pi, \sigma) = O_2(\sigma, \pi), \quad \operatorname{sgn}(\pi, \sigma) = \operatorname{sgn}(\sigma, \pi).$$

$$\begin{aligned}\text{Thus, } \overline{d_{KT}}(\pi, \sigma) &= 2|O_1(\pi, \sigma)| + 4|O_2(\pi, \sigma)| + |\operatorname{sgn}(\pi, \sigma)| \\ &= 2|O_1(\sigma, \pi)| + 4|O_2(\sigma, \pi)| + |\operatorname{sgn}(\sigma, \pi)| = \overline{d_{KT}}(\sigma, \pi).\end{aligned}$$

Therefore, $\overline{d_{KT}}$ is symmetric.

(iv) (Triangle Inequality). We need to prove that for all $\pi, \sigma, \rho \in \mathcal{B}_n$,

$$\overline{d_{KT}}(\pi, \sigma) \leq \overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\rho, \sigma). \tag{1}$$

We aim to show that for all $\pi, \sigma, \rho \in B_n$:

$$|O_1(\pi, \sigma)| \leq |O_1(\pi, \rho)| + |O_1(\rho, \sigma)|,$$

$$|O_2(\pi, \sigma)| \leq |O_2(\pi, \rho)| + |O_2(\rho, \sigma)|,$$

$$|\operatorname{sgn}(\pi, \sigma)| \leq |\operatorname{sgn}(\pi, \rho)| + |\operatorname{sgn}(\rho, \sigma)|$$

so that

$$\begin{aligned}2|O_1(\pi, \sigma)| + 4|O_2(\pi, \sigma)| + |\operatorname{sgn}(\pi, \sigma)| &\leq 2|O_1(\pi, \rho)| + 4|O_2(\pi, \rho)| \\ &+ |\operatorname{sgn}(\pi, \rho)| + 2|O_1(\rho, \sigma)| + 4|O_2(\rho, \sigma)| + |\operatorname{sgn}(\rho, \sigma)|,\end{aligned}$$

which will prove the inequality (1).

Claim 1: $|O_1(\pi, \sigma)| \leq |O_1(\pi, \rho)| + |O_1(\rho, \sigma)| \quad \forall \pi, \sigma, \rho \in B_n.$

Consider any pair $i < j$ of positions in the set $[n]$. Consider the signed permutations in B_n:

$$\pi = \pi_1 \pi_2 \cdots \pi_n, \quad \sigma = \sigma_1 \sigma_2 \cdots \sigma_n, \quad \rho = \rho_1 \rho_2 \cdots \rho_n$$

and their inverses:

$$\pi^{-1} = \alpha_1\alpha_2\cdots\alpha_n, \quad \sigma^{-1} = \beta_1\beta_2\cdots\beta_n, \quad \rho^{-1} = \gamma_1\gamma_2\cdots\gamma_n.$$

If (i,j) does not count as a disagreement for $O_1(\pi,\sigma)$, then either ($|\alpha_i| < |\alpha_j|$ and $|\beta_i| < |\beta_j|$) or ($|\alpha_i| > |\alpha_j|$ and $|\beta_i| > |\beta_j|$).

Suppose $|\alpha_i| < |\alpha_j|$ and $|\beta_i| < |\beta_j|$. The possibilities in ρ^{-1} are: $|\gamma_i| < |\gamma_j|$ or $|\gamma_i| > |\gamma_j|$. If $|\gamma_i| < |\gamma_j|$, then the pair (i,j) does not belong to $O_1(\pi,\sigma), O_1(\pi,\rho)$ and $O_1(\rho,\sigma)$. If $|\gamma_i| > |\gamma_j|$, then (i,j) contributes to each of $|O_1(\pi,\rho)|$ and $|O_1(\rho,\sigma)|$, but contributes 0 to $|O_1(\pi,\sigma)|$.

Similarly, if we suppose $|\alpha_i| > |\alpha_j|$ and $|\beta_i| > |\beta_j|$, then ($|\gamma_i| > |\gamma_j|$) or ($|\gamma_i| < |\gamma_j|$). If $|\gamma_i| > |\gamma_j|$, then (i,j) contributes nothing to all three terms, and if $|\gamma_i| < |\gamma_j|$, we have (i,j) contributing 0 to $|O_1(\pi,\sigma)|$ and (i,j) belongs to both $O_1(\pi,\rho)$ and $O_1(\rho,\sigma)$.

If (i,j) counts as a disagreement for $O_1(\pi,\sigma)$, then either ($|\alpha_i| < |\alpha_j|$ and $|\beta_i| > |\beta_j|$) or ($|\alpha_i| > |\alpha_j|$ and $|\beta_i| < |\beta_j|$).

Suppose $|\alpha_i| < |\alpha_j|$ and $|\beta_i| > |\beta_j|$. Then in ρ^{-1}, $|\gamma_i| < |\gamma_j|$ or $|\gamma_i| > |\gamma_j|$. If $|\gamma_i| < |\gamma_j|$, then the pair (i,j) belongs to $O_1(\pi,\sigma)$, and $O_1(\rho,\sigma)$ but does not belong to $O_1(\pi,\rho)$. If $|\gamma_i| > \gamma_j|$, then (i,j) is in $O_1(\pi,\sigma)$, and $O_1(\pi,\rho)$ but not in $O_1(\rho,\sigma)$.

Similarly, if we consider $|\alpha_i| > |\alpha_j|$ and $|\beta_i| < |\beta_j|$, then $|\gamma_i| > |\gamma_j|$ or $|\gamma_i| < |\gamma_j|$. In either case, there is an order disagreement of Type I between π and ρ, contributing to $O_1(\pi,\rho)$, or between ρ and σ, contributing to $O_1(\rho,\sigma)$.

Thus, considering all the cases and summing over all pairs (i,j), we conclude that:

$$|O_1(\pi,\sigma)| \leq |O_1(\pi,\rho)| + |O_1(\rho,\sigma)|.$$

Claim 2: $|O_2(\pi,\sigma)| \leq |O_2(\pi,\rho)| + |O_2(\rho,\sigma)| \quad \forall \pi,\sigma,\rho \in B_n$.

This inequality is similar to Claim 1, but now it applies to Type II disagreements, which involve both relative order inversions and sign changes. The proof follows an analogous reasoning to that of Claim 1.

Claim 3: $|\operatorname{sgn}(\pi,\sigma)| \leq |\operatorname{sgn}(\pi,\rho)| + |\operatorname{sgn}(\rho,\sigma)| \quad \forall \pi,\sigma,\rho \in B_n$.

On the contrary, suppose $\forall \pi,\sigma,\rho \in \mathcal{B}_n$

$$|\operatorname{sgn}(\pi,\sigma)| > |\operatorname{sgn}(\pi,\rho)| + |\operatorname{sgn}(\rho,\sigma)| \tag{2}$$

It is easy to see that $|\operatorname{sgn}(\pi,\sigma)|$ is at most n. Let us break down and check for all possibilities of the value of $|\operatorname{sgn}(\pi,\sigma)|$.

If $|\operatorname{sgn}(\pi,\sigma)| = n$, then $\operatorname{sgn}(\pi_i) \neq \operatorname{sgn}(\sigma_i)$ for all i, therefore, each position i contributes 1 to $|\operatorname{sgn}(\pi,\sigma)|$. Consider $\rho = \rho_1\rho_2\cdots\rho_n$. Here, for each i, ρ_i can be positive or negative. That is, for each i, either $\operatorname{sgn}(\rho_i) = \operatorname{sgn}(\pi_i)$ or $\operatorname{sgn}(\rho_i) = \operatorname{sgn}(\sigma_i)$.

Furthermore, if $\operatorname{sgn}(\rho_i) = \operatorname{sgn}(\pi_i)$, then $\operatorname{sgn}(\rho_i) \neq \operatorname{sgn}(\sigma_i)$, which means that the element at position i contributes 1 to $|\operatorname{sgn}(\rho, \sigma)|$ and 0 to $|\operatorname{sgn}(\pi, \rho)|$ and if $\operatorname{sgn}(\rho_i) = \operatorname{sgn}(\sigma_i)$, then $\operatorname{sgn}(\rho_i) \neq \operatorname{sgn}(\pi_i)$, contributing 1 to $|\operatorname{sgn}(\pi, \rho)|$ and 0 to $|\operatorname{sgn}(\rho, \sigma)|$. In either case, we will have the contribution of 1 (from position i) to the Right Hand Side (RHS) of (2).

This scenario implies that

$$|\operatorname{sgn}(\pi, \sigma)| = |\operatorname{sgn}(\pi, \rho)| + |\operatorname{sgn}(\rho, \sigma)|,$$

which is a contradiction to our assumption in (2).

If $|\operatorname{sgn}(\pi, \sigma)| < n$, then there exists at least one position i at which $\operatorname{sgn}(\pi_i) = \operatorname{sgn}(\sigma_i)$, contributing 0 on the Left Hand Side (LHS) of (2). Now, ρ_i can be such that $\operatorname{sgn}(\rho_i) = \operatorname{sgn}(\pi_i) = \operatorname{sgn}(\sigma_i)$ or $\operatorname{sgn}(\rho_i) \neq \operatorname{sgn}(\pi_i) = \operatorname{sgn}(\sigma_i)$, contributing 0 and 1 to both the terms of the RHS of (2) by the former and latter case, respectively. We continue with all such i's with this property.

Thus, in the former case, we will get the contribution 0 on both LHS and RHS of (2) at i's, ensuring LHS = RHS again, a contradiction to (2). In the latter case, at position $i's$, it contributes 0 to the LHS of (2) and contributes $1 + 1 = 2$ to the RHS of (2), showing clearly that

$$|\operatorname{sgn}(\pi, \sigma)| < |\operatorname{sgn}(\pi, \rho)| + |\operatorname{sgn}(\rho, \sigma)|,$$

which is a contradiction. So we must have

$$|\operatorname{sgn}(\pi, \sigma)| \leq |\operatorname{sgn}(\pi, \rho)| + |\operatorname{sgn}(\rho, \sigma)| \quad \forall \pi, \sigma, \rho \in B_n.$$

Summing these inequalities, we get:

$$\begin{aligned}
\overline{d_{KT}}(\pi, \sigma) &= 2|O_1(\pi, \sigma)| + 4|O_2(\pi, \sigma)| + |\operatorname{sgn}(\pi, \sigma)| \\
&\leq 2|O_1(\pi, \rho)| + 4|O_2(\pi, \rho)| + |\operatorname{sgn}(\pi, \rho)| \\
&\quad + 2|O_1(\rho, \sigma)| + 4|O_2(\rho, \sigma)| + |\operatorname{sgn}(\rho, \sigma)|
\end{aligned}$$

Thus, the triangle inequality holds:

$$\overline{d_{KT}}(\pi, \sigma) \leq \overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\rho, \sigma).$$

□

Let m be the number of elements in $\mathcal{B}_n$. Consider $(\pi^1, \pi^2, \ldots, \pi^m)$ denote an ordered sequence of signed permutations in $\mathcal{B}_n$. We define a $m \times m$ table, called the distance table of $\mathcal{B}_n$, where the entry in the (i, j)-cell of the distance table represents the generalized Kendall-τ distance $\overline{d_{KT}}(\pi^i, \pi^j)$ between the signed permutations π^i and π^j, for $1 \leq i, j \leq m$.

For a signed permutation $\pi \in \mathcal{B}_n$, the column sum of π is defined as:

$$\sum_{\sigma \in \mathcal{B}_n} \overline{d_{KT}}(\pi, \sigma),$$

which represents the sum of distances from π to all elements in $\mathcal{B}_n$. Similarly, the row sum of π is defined as:

$$\sum_{\sigma\in\mathcal{B}_n} \overline{d_{KT}}(\sigma,\pi),$$

which represents the sum of the distances from all elements in $\mathcal{B}_n$ to π.

For a subset $\mathcal{A} \subseteq \mathcal{B}_n$, we define the column sum of $\mathcal{A}$ as:

$$\sum_{\pi\in\mathcal{A}}\sum_{\sigma\in\mathcal{B}_n} \overline{d_{KT}}(\pi,\sigma),$$

and similarly, the row sum of $\mathcal{A}$ is:

$$\sum_{\pi\in\mathcal{A}}\sum_{\sigma\in\mathcal{B}_n} \overline{d_{KT}}(\sigma,\pi).$$

These sums can also be obtained directly from the distance table by summing the relevant rows and columns.

Since $\overline{d_{KT}}$ is symmetric, it follows that the distance table is also symmetric; specifically, the entry in the (i,j)-cell equals the entry in the (j,i)-cell. Consequently, for any signed permutation $\pi \in \mathcal{B}_n$ and subset $\mathcal{A} \subseteq \mathcal{B}_n$, the row sum and column sum of π (respectively, of $\mathcal{A}$) are identical. Additionally, the entries along the main diagonal of the distance table are zero, as the distance from a signed permutation to itself is zero.

In the following, we provide the distance table for $\mathcal{B}_2$. See Table 1 below. Let the elements of $\mathcal{B}_2$ be ordered as $\pi^1 = 1\ 2$, $\pi^2 = -1\ 2$, $\pi^3 = 1\ -2$, $\pi^4 = -1\ -2$, $\pi^5 = 2\ 1$, $\pi^6 = -2\ 1$, $\pi^7 = 2\ -1$, and $\pi^8 = -2\ -1$.

TABLE 1. The distance table for $\mathcal{B}_2$

	π^1	π^2	π^3	π^4	π^5	π^6	π^7	π^8
π^1	0	1	5	6	2	3	3	4
π^2	1	0	6	5	3	4	2	3
π^3	5	6	0	1	3	2	4	3
π^4	6	5	1	0	4	3	3	2
π^5	2	3	3	4	0	1	5	6
π^6	3	4	2	3	1	0	6	5
π^7	3	2	4	3	5	6	0	1
π^8	4	3	3	2	6	5	1	0

3. Universal-median set of signed permutations

In this section, we extend the concept of medians to the context of signed permutations under the generalized Kendall-τ distance, introducing the notion of a universal-median set and key operations that play a role in its structural properties.

Given any set of signed permutations $\mathcal{A} \subseteq \mathcal{B}_n$ and a signed permutation $\pi \in \mathcal{B}_n$, we have

$$\overline{d_{KT}}(\pi, \mathcal{A}) = \sum_{\sigma \in \mathcal{A}} \overline{d_{KT}}(\pi, \sigma).$$

We begin by formally defining the median of a set of signed permutations $\mathcal{A}$ in $\mathcal{B}_n$ under the generalized Kendall-τ distance.

Definition 6 (Medians). *Given $\mathcal{A} \subseteq \mathcal{B}_n$, a median of $\mathcal{A}$ under the generalized Kendall-τ distance is a signed permutation $\pi^* \in \mathcal{B}_n$ such that $\overline{d_{KT}}(\pi^*, \mathcal{A}) \leq \overline{d_{KT}}(\pi, \mathcal{A}),\ \forall \pi \in \mathcal{B}_n$.*

Define $\mathcal{M}(\mathcal{A})$ as the set of all medians of $\mathcal{A}$. i.e.,

$$\mathcal{M}(\mathcal{A}) = \{\sigma \in \mathcal{B}_n \mid \overline{d_{KT}}(\sigma, \mathcal{A}) \leq \overline{d_{KT}}(\pi, \mathcal{A}), \forall \pi \in \mathcal{B}_n\}$$

Having established the notion of a median, it is natural to ask whether there exist special subsets of signed permutations for which every element of the group is a median. This leads us to the definition of a universal-median set.

Definition 7 (Universal-median set). *A set $\mathcal{A} \subseteq \mathcal{B}_n$ is said to be a Universal-median set if $\mathcal{M}(\mathcal{A}) = \mathcal{B}_n$.*

To study the structure of median sets in $\mathcal{B}_n$, we now introduce two elementary operations on signed permutations, namely, transposition and negation.

Definition 8 (Transposition operation). *A unary operation on $\pi = \pi_1 \cdots \pi_n \in \mathcal{B}_n$, called "transposition operation" denoted by t_i, for any $1 \leq i \leq n-1$, is given by*

$$t_i : \mathcal{B}_n \to \mathcal{B}_n \text{ defined as}$$

$$t_i(\pi) = \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+2} \cdots \pi_n.$$

Definition 9 (Negation operation). *A unary operation on $\pi = \pi_1 \cdots \pi_n \in \mathcal{B}_n$, called "negation operation at i" denoted by η_i, for any $1 \leq i \leq n$, is given by*

$$\eta_i : \mathcal{B}_n \to \mathcal{B}_n \text{ defined as}$$

$$\eta_i(\pi) = \pi_1 \cdots \pi_{i-1} - \pi_i\ \pi_{i+1} \cdots \pi_n.$$

Let us introduce some notation to avoid confusion and for better presentation.

For $1 \le i \le n$, denote $\widehat{\pi}_i$ as the signed permutation $\eta_i(\pi)$, image of $\pi = \pi_1 \cdots \pi_r \cdots \pi_n$ under the negation operation η_i, and denote $\pi^{(r)}$ to be the signed permutation obtained from π by replacing π_i with $-\pi_i$ for any r number of i's.

Having introduced these definitions and notations, we now establish a fundamental lemma that quantifies the effect of a negation operation.

Lemma 1. *Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be an element of $\mathcal{B}_n$ and consider $\widehat{\pi}_r = \eta_r(\pi) = \pi_1 \cdots \pi_{r-1} \ -\pi_r \ \pi_{r+1} \cdots \pi_n$. Then*

$$\overline{d_{KT}}(\pi, \widehat{\pi}_r) = 1 + 4(r-1).$$

Proof. Consider π and $\widehat{\pi}_r$ as follows:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}, \quad \widehat{\pi}_r = \begin{pmatrix} 1 & \cdots & r-1 & r & r+1 & \cdots & n \\ \pi_1 & \cdots & \pi_{r-1} & -\pi_r & \pi_{r+1} & \cdots & \pi_n \end{pmatrix}$$

Assume $\pi_r = k$ (which can be either positive or negative). Then the inverses of π and $\widehat{\pi}_r$ are:

$$\pi^{-1} = \beta_1 \cdots \beta_k \cdots \beta_n, \quad \widehat{\pi}_r^{\,-1} = \beta_1 \cdots \beta_{k-1} \ -\beta_k \ \beta_{k+1} \cdots \beta_n.$$

Here, $\beta_k = r$ or $-r$ depending on the sign of π_r.

Clearly, the number of sign differences between π and $\widehat{\pi}_r$ is

$$|\operatorname{sgn}(\pi, \widehat{\pi}_r)| = |\{r\}| = 1 \quad (\text{since } \operatorname{sgn}(\beta_k) \neq \operatorname{sgn}(-\beta_k)).$$

Thus, this contributes 1 to the total distance d.

Since the absolute values of the elements in π^{-1} and $\widehat{\pi}_r^{\,-1}$ are the same, it follows that:

$$O_1(\pi, \widehat{\pi}_r) = \varnothing,$$

meaning there are no pairs of Type I disagreement (i.e., no pairs (i, j) where $|\beta_i| < |\beta_j|$ in π^{-1} but $|\beta_i| > |\beta_j|$ in $\widehat{\pi}_r^{\,-1}$, or vice versa for all i, j).

Observe that no pairs of elements in π and $\widehat{\pi}_r$ are Type II order disagreement pairs and none contributes to the distance d, since all elements in π^{-1} and $\widehat{\pi}_r^{\,-1}$ are the same except for the pairs of elements containing β_k in π^{-1} and $-\beta_k$ in $\widehat{\pi}_r^{\,-1}$, which has to be carefully checked.

Without loss of generality, let $\beta_k = r$. For Type II pairs, we can consider two cases:

- **Case 1:** $\beta_k < \beta_j$ and $-\beta_k > \beta_j$ (i.e., $r < \beta_j$ and $-r > \beta_j$ for all $j \neq k$). This implies:

  $$r < \beta_j < -r,$$

  which cannot happen as r is not less than $-r$.

- **Case 2:** $\beta_k > \beta_j$ and $-\beta_k < \beta_j$ (i.e., $r > \beta_j$ and $-r < \beta_j$ for all $j \neq k$). This implies:

$$-r < \beta_j < r.$$

So, β_j can be any of the numbers between $-(r-1)$ and $r-1$, but it cannot simultaneously be both $-(r-1)$ and $r-1$ as β_j is an element of a signed permutation. Thus, there are $r-1$ such pairs that satisfy case 2. Therefore,

$$|O_2(\pi, \widehat{\pi_r})| = r - 1.$$

Similarly, if $\beta_k = -r$, the same argument applies, yielding $|O_2(\pi, \widehat{\pi_r})| = r - 1$.

Hence, the total distance is:

$$\overline{d_{KT}}(\pi, \widehat{\pi_r}) = 2|O_1(\pi, \widehat{\pi_r})| + 4|O_2(\pi, \widehat{\pi_r})| + |\operatorname{sgn}(\pi, \widehat{\pi_r})| = 0 + 4(r-1) + 1.$$

This proves the lemma. □

Building on this lemma, we generalize the result to multiple negation operations and establish an additive property of the generalized Kendall–τ distance when multiple positions of a signed permutation are negated. This result will be crucial in characterizing the behavior of certain subsets under median operations.

Theorem 2. *Let $\overline{d_{KT}}(\pi, \sigma)$ be the generalized Kendall-τ distance between $\pi = \pi_1\pi_2\cdots\pi_n$ and $\sigma = \pi^{(r)} = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i = \pi_i$ for all i except for certain positions $i_1, i_2, \ldots, i_r$ such that $\sigma_{i_k} = -\pi_{i_k}$ for $1 \leq i_1 < i_2 < \ldots < i_r \leq n$. Then*

$$\begin{aligned}
\overline{d_{KT}}(\pi, \sigma) &= \sum_{k=1}^{r} \overline{d_{KT}}(\pi_1\cdots\pi_n, \sigma_1\cdots\sigma_{i_k}\cdots\sigma_n) \\
\text{i.e., } \overline{d_{KT}}(\pi, \pi^{(r)}) &= \overline{d_{KT}}(\pi, \widehat{\pi_{i_1}}) + \cdots + \overline{d_{KT}}(\pi, \widehat{\pi_{i_r}})
\end{aligned}$$

Proof. We prove this theorem by induction on r. For the base case $r = 1$, we have already proved in Lemma 1.

Let us now take $r = 2$ and consider π and σ as follows:

$$\pi = \pi_1\ \pi_2\cdots\pi_n, \qquad \sigma = \pi_1\cdots\pi_{i_1-1}\ -\pi_{i_1}\ \pi_{i_1+1}\cdots\pi_{i_2-1}\ -\pi_{i_2}\ \pi_{i_2+1}\cdots\pi_n$$

where $1 \leq i_1 < i_2 \leq n$.

$$\begin{aligned}
\text{To prove that } \overline{d_{KT}}(\pi, \sigma) &= \overline{d_{KT}}(\pi, \pi_1\cdots\pi_{i_1-1}\ -\pi_{i_1}\ \pi_{i_1+1}\cdots\pi_n) \\
&\quad + \overline{d_{KT}}(\pi, \pi_1\cdots\pi_{i_2-1}\ -\pi_{i_2}\ \pi_{i_2+1}\cdots\pi_n) \\
\text{i.e., } \overline{d_{KT}}(\pi, \sigma) &= 1 + 4(i_1 - 1) + 1 + 4(i_2 - 1).
\end{aligned}$$

Assume $\pi_{i_1} = k_1$ and $\pi_{i_2} = k_2$ (both of which can be either positive or negative). Without loss of generality, let $k_1 < k_2$. Then the inverses of π and σ are:

$$\pi^{-1} = \beta_1 \cdots \beta_{k_1} \cdots \beta_{k_2} \cdots \beta_n,$$

$$\sigma^{-1} = \beta_1 \cdots \beta_{k_1 - 1} \; - \beta_{k_1} \; \beta_{k_1+1} \cdots \beta_{k_2-1} \; - \beta_{k_2} \; \beta_{k_2+1} \cdots \beta_n.$$

Here, $\beta_{k_1} = i_1$ or $-i_1$ and $\beta_{k_2} = i_2$ or $-i_2$ depending on the sign of π_{i_1} and π_{i_2} respectively.

Clearly, the cardinality of the set of sign differences between π and σ is

$$|\operatorname{sgn}(\pi, \sigma)| = |\{i_1, i_2\}| = 2.$$

Thus, this contributes 2 to the total distance $\overline{d_{KT}}(\pi, \sigma)$.

Since the absolute values of the elements in π^{-1} and σ^{-1} are the same, it follows that:

$$O_1(\pi, \sigma) = \varnothing,$$

meaning there are no pairs of Type I disagreement (i.e., no pairs where $|\beta_i| < |\beta_j|$ in π^{-1} but $|\beta_i| > |\beta_j|$ in σ^{-1}, or vice versa, for any i, j).

Observe that no pairs of elements in π and σ are Type II order disagreement pairs and none contributes to distance $\overline{d_{KT}}(\pi, \sigma)$, since all elements in π^{-1} and σ^{-1} are the same except for pairs of elements containing $\beta_{k_1}, -\beta_{k_1}$ in π^{-1} and $\beta_{k_2}, -\beta_{k_2}$ in σ^{-1}. We now check all these instances.

- **Case 1:** Firstly we investigate those pairs containing β_{k_1}. So Type II order disagreement pairs is possible if for $j \neq k_2$

$$\beta_{k_1} > \beta_j \text{ and } - \beta_{k_1} < \beta_j \text{ (or) } \beta_{k_1} < \beta_j \text{ and } - \beta_{k_1} > \beta_j$$

$$\Rightarrow -\beta_{k_1} < \beta_j < \beta_{k_1} \text{ (or) } \beta_{k_1} < \beta_j < -\beta_{k_1}.$$

  Either of these hold if $\beta_{k_1} = i_1$ (or) $\beta_{k_1} = -i_1$ respectively. In both of these cases, we have $-i_1 < \beta_j < i_1$. So β_j can be any of the numbers from $-(i_1 - 1)$ to $(i_1 - 1)$(except 0) but since β_j is an element of signed permutation, it cannot be both $-l$ and l, for any $l, \; -(i_1 - 1) \leq l \leq i_1 - 1$. Thus, there are $i_1 - 1$ such pairs contributing to this case.

- **Case 2:** Now let us check for the pairs containing β_{k_2}. So Type II order disagreement pairs is possible if for $j \neq k_1$

$$\beta_{k_2} > \beta_j \text{ and } - \beta_{k_2} < \beta_j \text{ (or) } \beta_{k_2} < \beta_j \text{ and } - \beta_{k_2} > \beta_j$$

$$\Rightarrow -\beta_{k_2} < \beta_j < \beta_{k_2} \text{ (or) } \beta_{k_2} < \beta_j < -\beta_{k_2}.$$

  Similarly we have $-i_2 < \beta_j < i_2$ but since $j \neq k_1$, we cannot have $\beta_j = \beta_{k_1} = \pm i_1$. Thus, there are $i_2 - 2$ such pairs contributing to this case.

- **Case 3:** Finally we check the pairs containing β_{k_1} and β_{k_2}, in which case

$$\beta_{k_1} > \beta_{k_2} \text{ and } -\beta_{k_1} < -\beta_{k_2} \text{ (or) } \beta_{k_1} < \beta_{k_2} \text{ and } -\beta_{k_1} > -\beta_{k_2}$$

$$\Rightarrow \beta_{k_1} > \beta_{k_2} \text{ (or) } \beta_{k_1} < \beta_{k_2}.$$

Either of these always hold. Hence, there is one pair of Type II disagreements between β_{k_1} and β_{k_2}.

Thus, $|O_2(\pi, \sigma)| = (i_1 - 1) + (i_2 - 2) + 1 = (i_1 - 1) + (i_2 - 1)$.

Therefore, combining all contributions, we get the total generalized Kendall-τ distance as:

$$\overline{d_{KT}}(\pi, \sigma) = 2 + 4(i_1 - 1) + 4(i_2 - 1).$$

Thus, for $r = 2$, the identity holds:

$$\overline{d_{KT}}(\pi, \sigma) = \overline{d_{KT}}(\pi, \pi_1 \cdots \pi_{i_1 - 1} - \pi_{i_1} \cdots \pi_n) + \overline{d_{KT}}(\pi, \pi_1 \cdots \pi_{i_2 - 1} - \pi_{i_2} \cdots \pi_n).$$

Inductive Step: Now, assume that the formula holds for $r = m \geq 1$. That is, suppose for any m-tuple of positions $i_1, i_2, \ldots, i_m$, where $\sigma_{i_j} = -\pi_{i_j}$ for $j = 1, \ldots, m$, we have

$$\overline{d_{KT}}(\pi, \pi^{(m)}) = \overline{d_{KT}}(\pi, \widehat{\pi_{i_1}}) + \cdots + \overline{d_{KT}}(\pi, \widehat{\pi_{i_r}}) = \sum_{j=i}^{m} (1 + 4(i_j - 1)). \tag{3}$$

Now consider the case $r = m + 1$. Let $\pi^{(m+1)}$ be a signed permutation where $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}$ are the same as in the inductive hypothesis, and in addition, $\sigma_{i_{m+1}} = -\pi_{i_{m+1}}$ for some $i_{m+1} > i_m$. Then the distance between π and $\pi^{(m+1)}$ is

$$\overline{d_{KT}}(\pi, \pi^{(m+1)}) = \overline{d_{KT}}(\pi, \pi^{(m)}) + \overline{d_{KT}}(\pi^{(m)}, \pi^{(m+1)}).$$

By the inductive hypothesis (3), the first term in the right hand side of the above equation becomes

$$\overline{d_{KT}}(\pi, \pi^{(m)}) = \sum_{j=i}^{m} (1 + 4(i_j - 1)),$$

and using the Lemma 1 for the second term, in which the additional distance between $\pi^{(m)}$ and $\pi^{(m+1)}$ comes solely from the sign change at position i_{m+1}, which contributes $1 + 4(i_{m+1} - 1)$ to the distance. Therefore

$$\overline{d_{KT}}(\pi, \pi^{(m+1)}) = \sum_{j=i}^{m+1} (1 + 4(i_j - 1)).$$

Hence, $\overline{d_{KT}}(\pi, \pi^{(m)}) = \overline{d_{KT}}(\pi, \widehat{\pi_{i_1}}) + \cdots + \overline{d_{KT}}(\pi, \widehat{\pi_{i_m}})$.

Thus, this completes the proof.

□

Corollary 1. *Let $\overline{d_{KT}}$ be the generalized Kendall-τ distance between $\pi = \pi_1\pi_2\cdots\pi_n$ and $\pi^- = -\pi_1 - \pi_2 \cdots - \pi_n$. Then*

$\overline{d_{KT}}(\pi, \pi^-) = \overline{d_{KT}}(\pi_1\cdots\pi_n, -\pi_1\ \pi_2\cdots\pi_n) + \cdots + \overline{d_{KT}}(\pi_1\cdots\pi_n, \pi_1\cdots\pi_{n-1} - \pi_n)$.

Furthermore, this simplifies to $\overline{d_{KT}}(\pi, \pi^-) = n(2n-1)$.

Proof. The proof is straightforward from the above Theorem 2 by setting $r = n$ and with a simple calculation we get $\overline{d_{KT}}(\pi, \pi^-) = n(2n-1)$. □

Definition 10 (Total negation operation)**.** *A unary operation '$-$' on the elements of $\mathcal{B}_n$ is defined to be the map*

$$- : \mathcal{B}_n \to \mathcal{B}_n \text{ such that}$$

$$-(\pi_1\pi_2\cdots\pi_n) = -\pi_1 - \pi_2 \cdots - \pi_n$$

for $\pi = \pi_1\pi_2\cdots\pi_n$. The image of $\pi = \pi_1\pi_2\cdots\pi_n$ with respect to this operation '$-$' is denoted by π^- and we call the operation "$-$" a total negation operation.

We now give the remarkable result of this section which characterize the Universal-median set. The following lemma will be useful to prove this result.

Lemma 2. *For any $\pi, \sigma \in \mathcal{B}_n$, we have*

$$\overline{d_{KT}}(\sigma, \pi) + \overline{d_{KT}}(\sigma, \pi^-) = n(2n-1). \tag{4}$$

Proof. By Corollary 1, we know that for any $\pi \in \mathcal{B}_n$,

$$\overline{d_{KT}}(\pi, \pi^-) = n(2n-1).$$

Setting $\sigma = \pi$ in (4), we recover the equation

$$\overline{d_{KT}}(\pi, \pi) + \overline{d_{KT}}(\pi, \pi^-) = 0 + n(2n-1).$$

Similarly, setting $\sigma = \pi^-$ yields

$$\overline{d_{KT}}(\pi^-, \pi) + \overline{d_{KT}}(\pi^-, \pi^-) = n(2n-1) + 0.$$

Both cases trivially satisfy (4).

Now consider the case where $\sigma \in \mathcal{B}_n \setminus \{\pi, \pi^-\}$. To verify the claim, we analyze the contributions from Type I and Type II order disagreements between the signed permutations.

Let the inverses of π, π^- and σ be $\pi^{-1} = \alpha_1\cdots\alpha_n, (\pi^-)^{-1} = -\alpha_1 \cdots - \alpha_n$ and $\sigma^{-1} = \beta_1\cdots\beta_n$, respectively.

For an arbitrary pair of positions (i, j), assume without loss of generality that $|\beta_i| < |\beta_j|$ in σ^{-1}.

Case 1: If $|\alpha_i| > |\alpha_j|$ in π^{-1}, then $|-\alpha_i| > |-\alpha_j|$ in $(\pi^-)^{-1}$. This pair (i, j) belongs to both $O_1(\sigma, \pi)$ and $O_1(\sigma, \pi^-)$.

Case 2: If $|\alpha_i| < |\alpha_j|$ in π^{-1}, then $|-\alpha_i| < |-\alpha_j|$ in $(\pi^-)^{-1}$. In this case, (i,j) does not belong to both $O_1(\sigma,\pi)$ and $O_1(\sigma,\pi^-)$. Thus, we focus on Type II order disagreements.

Suppose $\beta_i < \beta_j$ in σ^{-1}. We divide this case into two subcases.

(i): If $\alpha_i < \alpha_j$ and $-\alpha_i > -\alpha_j$, then the pair (i,j) belongs to $O_2(\sigma,\pi^-)$ but not to $O_2(\sigma,\pi)$.

(ii): If $\alpha_i > \alpha_j$ and $-\alpha_i < -\alpha_j$, then the pair (i,j) belongs to $O_2(\sigma,\pi)$ but not to $O_2(\sigma,\pi^-)$.

The same conclusions hold if $\beta_i > \beta_j$. In any case, each pair (i,j) belongs to both $O_1(\sigma,\pi)$ and $O_1(\sigma,\pi^-)$ or belongs to either $O_2(\sigma,\pi)$ or $O_2(\sigma,\pi^-)$ but not both. There are $n(n-1)/2$ such distinct pairs, and each pair contributes 4 to the total distance $\overline{d_{KT}}(\sigma,\pi) + \overline{d_{KT}}(\sigma,\pi^-)$.

Finally, we have $|\operatorname{sgn}(\sigma,\pi)| + |\operatorname{sgn}(\sigma,\pi^-)| = n$ since each position i gives a contribution to either $|\operatorname{sgn}(\sigma,\pi)|$ or $|\operatorname{sgn}(\sigma,\pi^-)|$. That is, in an arbitrary position i, if the signs of β_i and α_i are the same, then the signs of β_i and $-\alpha_i$ have to be different, and vice versa.

Thus, summing all contributions, we obtain

$$\overline{d_{KT}}(\sigma,\pi) + \overline{d_{KT}}(\sigma,\pi^-) = 4\frac{n(n-1)}{2} + n = n(2n-1).$$

This completes the proof. □

Theorem 3. *If $\mathcal{A}$ is a subset of signed permutations of $\mathcal{B}_n$ with cardinality m and is closed under operation '$-$', i.e., for all $\pi \in \mathcal{A}$, we have $\pi^- \in \mathcal{A}$, then the set of medians equals $\mathcal{B}_n$, i.e.,*

$$\mathcal{M}(\mathcal{A}) = \mathcal{B}_n.$$

Moreover, for any $\pi \in \mathcal{A}$, the generalized Kendall-τ distance between π and $\mathcal{A}$ satisfies

$$\overline{d_{KT}}(\pi,\mathcal{A}) = \overline{d_{KT}}(\pi^-,\mathcal{A}) = \frac{m}{2}n(2n-1).$$

Proof. Let $\mathcal{A} = \{\pi^1, \pi^2, \ldots, \pi^m\}$ be a set of m elements in $\mathcal{B}_n$. Since $\mathcal{A}$ is closed with respect to the operation $-$, it follows that if $\pi \in \mathcal{A}$, then $\pi^- \in \mathcal{A}$. Observe that for every $i \in \{1,2,\ldots,m\}$,we can find $j \in \{1,2,\ldots,m\} \setminus \{i\}$ such that $\pi^j = (\pi^i)^-$. We arrange these elements in pairs such that $(\pi^i)^-$ is followed by π^i, and rename this to get $\sigma^1, \sigma^2, \cdots, \sigma^m$, where $\sigma^2 = (\sigma^1)^-,\ \sigma^4 = (\sigma^3)^-$, and so on. That is,

$$\mathcal{A} = \{\sigma^1, (\sigma^1)^-, \sigma^2, (\sigma^2)^-, \ldots, \sigma^{m/2}, (\sigma^{m/2})^-\},$$

We want to show that $\mathcal{M}(\mathcal{A}) = \mathcal{B}_n$. To do so, we need to find the signed permutation π^* in $\mathcal{B}_n$ that minimizes the total generalized Kendall-τ distance from $\mathcal{A}$. This distance is given by

$$\overline{d_{KT}}(\pi^*,\mathcal{A}) = \overline{d_{KT}}(\pi^*,\sigma^1) + \overline{d_{KT}}(\pi^*,(\sigma^1)^-) + \cdots + \overline{d_{KT}}(\pi^*,\sigma^{m/2}) + \overline{d_{KT}}(\pi^*,(\sigma^{m/2})^-).$$

By Lemma 2, for any signed permutation π^*, the total distance can be calculated as:

$$\overline{d_{KT}}(\pi^*,\mathcal{A}) = n(2n-1) + \cdots + n(2n-1) \quad \left(\text{summed for } \frac{m}{2} \text{ pairs}\right).$$

Thus, we find

$$\overline{d_{KT}}(\pi^*, \mathcal{A}) = \frac{m}{2} n(2n-1).$$

Hence, we conclude that the subset $\mathcal{A}$ of $\mathcal{B}_n$ is a universal-median set, i.e.,

$$\mathcal{M}(\mathcal{A}) = \mathcal{B}_n.$$

□

Corollary 2. *For π in $\mathcal{B}_n$, the column sum (resp., row sum) of π is $\frac{|\mathcal{B}_n|}{2} n(2n-1)$.*

Proof. The proof follows easily by setting $\mathcal{A} = \mathcal{B}_n$ in Theorem 3. □

Remark 2. *For any element $\pi \in \mathcal{B}_n$ and any non-empty subset $\mathcal{A} \subseteq \mathcal{B}_n$, we have*

$$\overline{d_{KT}}(\pi, \mathcal{A}) + \overline{d_{KT}}(\pi, \mathcal{A}^c) = \frac{|\mathcal{B}_n|}{2} n(2n-1).$$

4. Distance graph $\mathcal{G}_n$ of $\mathcal{B}_n$

We define the distance graph $\mathcal{G}_n(V, E, \omega)$ for $\mathcal{B}_n$, where

- the set of vertices V are the signed permutations in $\mathcal{B}_n$ written in one-line notation, i.e., $V = \mathcal{B}_n$,
- $e = (\pi, \sigma) \in E$ iff σ is obtained from π by using the 'first negation operation η_1' (refer Definition 9) or the 'transposition operation t_i' (refer Definition 8) and vice versa.
- the weight function $\omega : E \to \mathbb{N}$ defined as

$$\omega(e) = \omega((\pi, \sigma)) = \begin{cases} 1, & \text{if } \sigma = \eta_1(\pi) \\ 2, & \text{if } \sigma = t_i(\pi). \end{cases}$$

The graph $\mathcal{G}_n$ is a weighted rooted connected graph with identity-signed permutation as the root. This graph clearly does not contain loops since $\pi \neq \eta_1(\pi)$ and $\pi \neq t_i(\pi)$ for any signed permutation $\pi \in \mathcal{B}_n$.

In the context of distance graph $\mathcal{G}_n$, the generalized Kendall-τ distance (see Definition 5) between any two signed permutations $\pi, \sigma \in \mathcal{B}_n$ is defined as the minimum total weight of a path that connects π and σ in the graph $\mathcal{G}_n$.

The distance graph $\mathcal{G}_n$ can be constructed by following simple steps.

Step 1: Begin with the identity element $e = 12 \cdots n$ in $\mathcal{B}_n$ as the root of the graph.

Step 2: In each step, apply the operations η_1 and t_i to every vertex generated in the previous step.

For a given node $\pi = \pi_1 \pi_2 \cdots \pi_n$:

- Connect π to the element $-\pi_1 \pi_2 \cdots \pi_n$ with an edge of weight 1.
- Connect π to the elements $\pi_1 \cdots \pi_{i+1} \pi_i \cdots \pi_n$ with an edge of weight 2, for every $1 \leq i \leq n-1$.

Step 3: Repeat the process iteratively until all elements of $\mathcal{B}_n$ are included in the graph.

As an example, we provide the distance graph for $\mathcal{B}_2$ and $\mathcal{B}_3$. See Figure 1 and Figure 2, respectively.

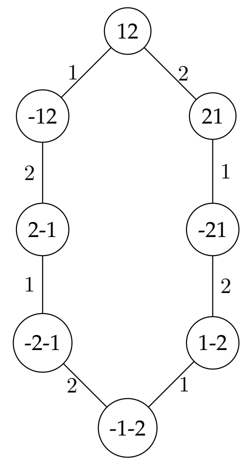

FIGURE 1. Distance graph for $\mathcal{B}_2$

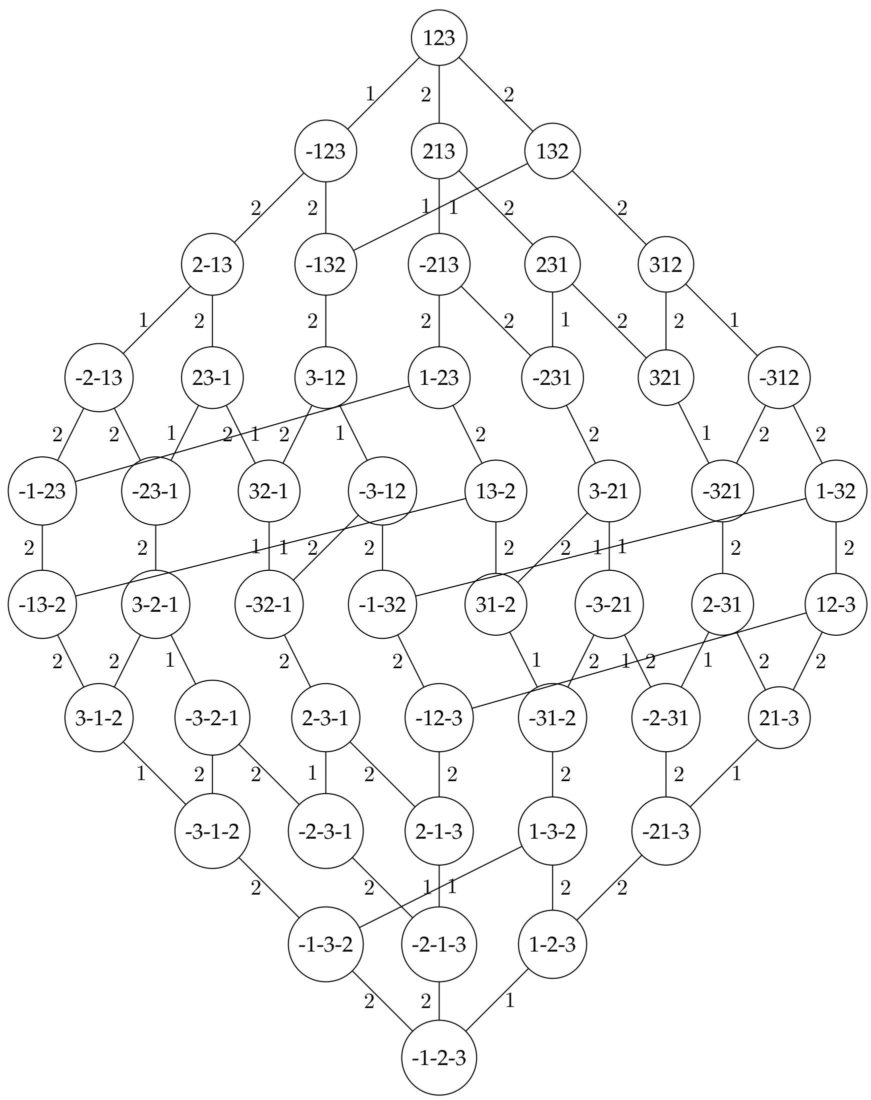

FIGURE 2. Distance graph for $\mathcal{B}_3$

Now we shall see some properties relating the generalized Kendall-τ distance with reversal of a signed permutation.

Definition 11. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any element in $\mathcal{B}_n$. The reversal of a signed permutation π is a signed permutation denoted as π^{rev} and defined by $\pi^{rev} = \pi_n\pi_{n-1}\cdots\pi_1$.*

Lemma 3. *For any element π in $\mathcal{B}_n$, we explicitly give the generalized Kendall-τ distance between π and its reversal, i.e.,*

$$\overline{d_{KT}}(\pi, \pi^{rev}) = n(n-1).$$

Proof. To compute the generalized Kendall-τ distance $\overline{d_{KT}}(\pi, \pi^{rev})$ for any $\pi \in \mathcal{B}_n$, we consider the reversal operation on π and calculate the total minimum distance required to transform π into π^{rev} using the distance graph $\mathcal{G}_n$ of $\mathcal{B}_n$.

The transformation from $\pi = \pi_1\pi_2\cdots\pi_n$ to its reversal $\pi^{rev} = \pi_n\pi_{n-1}\cdots\pi_1$ can be systematically achieved by sequentially relocating each element of π to its final position in π^{rev}. We construct a specific path in $\mathcal{G}_n$ to achieve this transformation and calculate the total distance along this path.

We begin by swapping π_1 with π_2, π_1 with π_3, and so on until π_1 is placed in position n. This intermediate step transforms π into $\pi_2\pi_3\cdots\pi_n\pi_1$. The number of swaps required is $n-1$, and each swap contributes a distance of 2. Thus, the total distance for this step is $2(n-1)$. This is evident from the following path in $\mathcal{G}_n$:

$$\pi \to \pi_2\pi_1\pi_3\cdots\pi_n \to \pi_2\pi_3\pi_1\cdots\pi_n \to \cdots \pi_2\pi_3\cdots\pi_n\pi_1$$

Next, we repeat this process for π_2, swapping it with π_3, π_4, and so on, until it reaches the position $n-1$. This transforms $\pi_2\pi_3\cdots\pi_n\pi_1$ into $\pi_3\pi_4\cdots\pi_n\pi_2\pi_1$. The number of swaps required in this step is $n-2$, and the total distance contributed by these swaps is $2(n-2)$. In $\mathcal{G}_n$, this is given by the path:

$$\pi_2\pi_3\cdots\pi_n\pi_1 \to \pi_3\pi_2\pi_4\cdots\pi_n\pi_1 \to \cdots \to \pi_3\pi_4\cdots\pi_n\pi_2\pi_1$$

Continuing in this manner, we iteratively relocate each element to its final position in π^{rev}. The total distance for the whole process is summarized as follows:

$$2(n-1) + 2(n-2) + \cdots + 2.$$

Using the formula for the sum of the first m natural numbers, the total distance is computed as:

$$\begin{aligned}\overline{d_{KT}}(\pi, \pi^{rev}) &= 2\sum_{k=1}^{n-1} k \\ &= 2 \cdot \frac{(n-1)n}{2} \\ &= n(n-1).\end{aligned}$$

Note that there may exist multiple paths in $\mathcal{G}_n$ connecting π to π^{rev}, but all of these paths yield the same total distance $\overline{d_{KT}}(\pi, \pi^{rev}) = n(n-1)$ due to the nature of the generalized Kendall-τ metric. This completes the proof. □

Note that $(\pi^-)^{rev} = (\pi^{rev})^-$, thus we have the following:

Theorem 4. *For any element $\pi \in \mathcal{B}_n$, the generalized Kendall-τ distance*

$$\overline{d_{KT}}(\pi, (\pi^-)^{rev}) = \overline{d_{KT}}(\pi, (\pi^{rev})^-) = n^2.$$

Proof. To compute the generalized Kendall-τ distance $\overline{d_{KT}}(\pi, (\pi^-)^{rev})$ for any $\pi \in \mathcal{B}_n$, we systematically trace the path in the distance graph $\mathcal{G}_n$ of $\mathcal{B}_n$ that connects π to $(\pi^-)^{rev}$ and calculate the minimum total distance.

The transformation from π to $(\pi^-)^{rev}$ involves reversing the order of elements and changing the signs of all entries. To achieve this transformation, we proceed iteratively, ensuring that at each step, the appropriate distance is accounted for.

We first describe the path that transforms $\pi_1\pi_2\cdots\pi_n$ to the intermediate state $\pi_2\pi_3\cdots\pi_n - \pi_1$. This is done in two stages:

(1) Apply the operation η_1, which flips the sign of the first element, transforming π into $-\pi_1\pi_2\cdots\pi_n$. This operation contributes a distance of 1.

(2) Sequentially reposition π_1 to the last position by applying transpositions t_i, $i = 1, 2, \ldots,$ $n-1$. Each transposition contributes a distance of 2, and since $n-1$ transpositions are required, the total distance for this step is $2(n-1)$.

Therefore, the total distance for this phase is $1 + 2(n-1)$. We can see this in the distance graph $\mathcal{G}_n$, which is given by the path

$$\pi \xrightarrow{1} -\pi_1\pi_2\cdots\pi_n \xrightarrow{2} \pi_2 - \pi_1\pi_3\cdots\pi_n \xrightarrow{2} \pi_2\pi_3 - \pi_1\cdots\pi_n \xrightarrow{2} \cdots \xrightarrow{2} \pi_2\pi_3\cdots\pi_n - \pi_1$$

Next, we iteratively repeat the process for each subsequent element, transforming $\pi_2\pi_3\cdots\pi_n - \pi_1$ into $\pi_3\pi_4\cdots\pi_n - \pi_2 - \pi_1$, and so on until all elements are reversed and negated. At each step, the process involves:

(1) Applying η_1 to flip the sign of the next element, contributing a distance of 1.

(2) Repositioning this element to its final location through t_i operations. If the element is in position k, the number of required transpositions is $n-k$, and the corresponding distance is $2(n-k)$.

Summing the contributions from all steps, the total distance for transforming π into $(\pi^-)^{rev}$ is:

$$\begin{aligned}\overline{d_{KT}}(\pi, (\pi^-)^{rev}) &= \big[1 + 2(n-1)\big] + \big[1 + 2(n-2)\big] + \cdots + \big[1 + 2(1)\big] \\ &= n + 2\big[1 + 2 + \cdots + (n-1)\big].\end{aligned}$$

Using the formula for the sum of the first m natural numbers, the total distance becomes:

$$\begin{aligned}\overline{d_{KT}}(\pi,(\pi^-)^{rev}) &= n + 2\cdot\frac{(n-1)n}{2}\\ &= n + n^2 - n\\ &= n^2.\end{aligned}$$

Similarly, for the reversal of the negated signed permutation $(\pi^{rev})^-$, the computation follows the same steps since the operations of reversal and negation commute in this context. Thus, the generalized Kendall-τ distance satisfies:

$$\overline{d_{KT}}(\pi,(\pi^-)^{rev}) = \overline{d_{KT}}(\pi,(\pi^{rev})^-) = n^2.$$

It is worth noting that while multiple paths in $\mathcal{G}_n$ may connect π to $(\pi^-)^{rev}$, all of these paths yield the same total distance due to the intrinsic properties of the generalized Kendall-τ metric. This completes the proof. □

Theorem 5. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{B}_n$, and define $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{B}_n$ such that $\sigma_i = -\pi_i$ for some $i \in \{1,\ldots,n\}$ and $\sigma_j = \pi_j$ for all $j \neq i$. Then, the generalized Kendall-τ distance satisfies:*

$$\overline{d_{KT}}(\pi,\sigma^{rev}) = n(n-1)+1.$$

Proof. In the distance graph $\mathcal{G}_n$ of $\mathcal{B}_n$, we trace the path connecting π and $\sigma = \sigma_1\cdots\sigma_i\cdots\sigma_n$ in $\mathcal{B}_n$, where $\sigma_j = \pi_j$ for all $j \neq i$ except $\sigma_i = -\pi_i$ for $1 \le i \le n$. To establish the result, we explicitly construct a path and calculate the total distance along it.

We begin by swapping π_1 with π_2, π_1 with π_3, and so on until π_1 is placed in position n. This intermediate step transforms π into $\pi_2\pi_3\cdots\pi_n\pi_1$. The number of swaps required is $n-1$, and each swap contributes a distance of 2. Thus, the total distance for this step is $2(n-1)$. This is evident from the following path in $\mathcal{G}_n$:

$$\pi \to \pi_2\pi_1\pi_3\cdots\pi_n \to \pi_2\pi_3\pi_1\cdots\pi_n \to \cdots \pi_2\pi_3\cdots\pi_n\pi_1$$

Next, we repeat this process for π_2, swapping it with π_3, π_4, and so on, until it reaches the position $n-1$. This transforms $\pi_2\pi_3\cdots\pi_n\pi_1$ into $\pi_3\pi_4\cdots\pi_n\pi_2\pi_1$. The number of swaps required in this step is $n-2$, and the total distance contributed by these swaps is $2(n-2)$. In $\mathcal{G}_n$, this is given by the path:

$$\pi_2\pi_3\cdots\pi_n\pi_1 \to \pi_3\pi_2\pi_4\cdots\pi_n\pi_1 \to \cdots \to \pi_3\pi_4\cdots\pi_n\pi_2\pi_1$$

We continue this process until we reposition the element π_{i-1} to obtain $\pi_i\pi_{i+1}\cdots\pi_n\pi_{i-1}\cdots\pi_2\pi_1$. Once π_i is in the first position, we apply the first negation operation η_1 to transform π_i into $-\pi_i$. This operation contributes a distance of 1. At this stage, the intermediate result is $-\pi_i\pi_{i+1}\cdots\pi_n\pi_{i-1}\cdots\pi_2\pi_1$.

Finally, to construct σ^{rev}, we reverse the order of the remaining elements. This involves applying the operation t_i for $i \in \{1, 2, \ldots, n-i\}$ to get a path connecting to $\sigma = \sigma_1 \cdots \sigma_{i-1}\sigma_i\sigma_{i+1} \cdots \sigma_n$ $\in \mathcal{B}_n$, where $\sigma_j = \pi_j$ for all $j \neq i$, and $\sigma_i = -\pi_i$ for $1 \leq i \leq n$. So, the total distance for the path is given by

$$\begin{aligned}
\overline{d_{KT}}(\pi, \sigma) &= 2(n-1) + 2(n-2) + \cdots + 2(n-(i-1)) + 1 + 2(n-i) + \cdots + 2(1) \\
&= 1 + 2[1 + 2 + \cdots + (n-1)] \\
&= 1 + 2\left[\frac{(n-1)n}{2}\right] \\
&= 1 + n(n-1).
\end{aligned}$$

Since there may exist multiple paths in $\mathcal{G}_n$ connecting π and σ, the above calculation shows that the constructed path achieves the minimum distance. Hence, the proof is completed. □

5. $\mathfrak{S}_n$-MEDIAN SET OF SIGNED PERMUTATIONS

The study of median sets in particular subgroups of the hyperoctahedral group of type $\mathcal{B}_n$ sheds light on the relationships and structural characteristics of signed permutations. In this section, we introduce the notion of an $\mathfrak{S}_n$-median set within the group of signed permutations $\mathcal{B}_n$, where $\mathfrak{S}_n$ is the symmetric group of order n, and explore some fundamental properties and results associated with this concept.

We begin by formally defining the $\mathfrak{S}_n$-median set as follows:

Definition 12 ($\mathfrak{S}_n$-median set)**.** *A set $\mathcal{A} \subseteq \mathcal{B}_n$ is said to be a $\mathfrak{S}_n$-median set if $\mathcal{M}(\mathcal{A}) = \mathfrak{S}_n$.*

To understand the behavior of distances between elements of $\mathfrak{S}_n$ and their reversals within the larger structure of $\mathcal{B}_n$, we establish a crucial identity involving the generalized Kendall-τ distance between a permutation and its reversal. The following finding is useful in describing median sets which is closed with regards to the reversal operation.

Lemma 4. *For any two permutations $\pi, \rho \in \mathfrak{S}_n$, we have the identity*

$$\overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\pi, \rho^{rev}) = n(n-1).$$

Proof. Since π and ρ are elements of the symmetric group $\mathfrak{S}_n$, any permutation ρ can b acquired from π by a sequence of transpositions t_i, where t_i swaps the elements at positions i and $i+1$, where $i \in \{1, \ldots, n-1\}$.

To establish the result, it suffices to verify the identity for two fundamental cases:

- **(i):** ρ is obtained from π by applying a single transposition operation, i.e., $\rho = t_i(\pi)$ for some $i \in \{1, \ldots, n-1\}$.

(ii): ρ is obtained from π by applying a finite number of transposition operations, i.e., $\rho = t_j(t_i(\pi))$ for some $i, j \in \{1, \ldots, n-1\}$.

For case (i), let ρ be given by $\rho = \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+2} \cdots \pi_n$, which means that t_i swaps π_i and π_{i+1}. It is evident by looking at the distance graph $\mathcal{G}_n$'s structure that the distance

$$\overline{d_{KT}}(\pi, \rho) = 2. \tag{5}$$

Now, we analyze $\overline{d_{KT}}(\pi, \rho^{rev})$. The reversed signed permutation ρ^{rev} is given by:

$$\rho^{rev} = \pi_n\pi_{n-1} \cdots \pi_{i+2}\pi_i\pi_{i+1}\pi_{i-1} \cdots \pi_2\pi_1.$$

To compute $\overline{d_{KT}}(\pi, \rho^{rev})$, we trace a path in the distance graph $\mathcal{G}_n$ connecting π to ρ^{rev}.

The transformation proceeds as follows:

- Swap π_1 iteratively with $\pi_2, \pi_3, \ldots, \pi_n$, placing π_1 in position n. This requires $n-1$ swaps, contributing a total distance of $2(n-1)$. This intermediate step transforms π into $\pi_2\pi_3 \cdots \pi_n\pi_1$. This is evident from the following path in $\mathcal{G}_n$:

$$\pi \to \pi_2\pi_1\pi_3 \cdots \pi_n \to \pi_2\pi_3\pi_1 \cdots \pi_n \to \cdots \pi_2\pi_3 \cdots \pi_n\pi_1$$

- Repeat this process for π_2, moving it to position $n-1$ through $n-2$ swaps, contributing $2(n-2)$. In $\mathcal{G}_n$, this is given by the path:

$$\pi_2\pi_3 \cdots \pi_n\pi_1 \to \pi_3\pi_2\pi_4 \cdots \pi_n\pi_1 \to \cdots \to \pi_3\pi_4 \cdots \pi_n\pi_2\pi_1$$

- Continue this process for all π_k until we reach π_{i-1}, which is moved to its required position with $n-(i-1)$ swaps, to obtain

$$\pi_i\pi_{i+1} \cdots \pi_n\pi_{i-1} \cdots \pi_2\pi_1.$$

- Once π_i is in the first position, we begin to swap π_{i+1} to the right until π_{i+1} reaches a position immediately right to π_n which is acquired in $n-(i-1)$ steps to get

$$\pi_i\pi_{i+2} \cdots \pi_n\pi_{i+1}\pi_{i-1} \cdots \pi_2\pi_1.$$

- Now again we start swapping π_i with π_{i+2}, so on and place π_i in between π_n and π_{i+1}. At this stage, the intermediate result is

$$\pi_{i+2} \cdots \pi_n\pi_i\pi_{i+1}\pi_{i-1} \cdots \pi_2\pi_1.$$

  Then we again continue the process of swapping elements to finally arrive at

$$\rho^{rev} = \pi_n\pi_{n-1} \cdots \pi_{i+2}\pi_i\pi_{i+1}\pi_{i-1} \cdots \pi_2\pi_1.$$

The total distance accumulated through these swaps is given by:

$$\begin{aligned}
\overline{d_{KT}}(\pi, \rho^{rev}) &= 2(n-1) + 2(n-2) + \cdots + 2(n-(i-1)) + 2(n-(i-1)) \\
&\quad +2(n-(i+1)) + \cdots + 2(2) + 2(1) \\
&= 2\left\{[(n-1) + (n-2) + \cdots + (n-(i-1)) + (n-i) + (n-(i+1))\right. \\
&\quad \left. + \cdots + 2 + 1] - [(n-i) + n - (i-1)]\right\} \\
&= 2\{[1 + 2 + \cdots + (n-2) + (n-1)] - 1\} \\
&= 2\left[\sum_{k=1}^{n-1} k - 1\right] \\
&= \left[2\frac{(n-1)n}{2} - 2\right] \\
&= n(n-1) - 2. \qquad (6)
\end{aligned}$$

Thus, we obtain from (5) and (6),

$$\overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\pi, \rho^{\text{rev}}) = n(n-1).$$

For case (ii), when $\rho = t_j(t_i(\pi))$ with $j = i + 1$, the second transposition t_j reverses the effect of the first t_i, restoring π. Hence, we have

$$\overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\pi, \rho^{rev}) = \overline{d_{KT}}(\pi, \pi) + \overline{d_{KT}}(\pi, \pi^{rev}) = n(n-1).$$

When $j \neq i + 1$, without loss of generality, assume $i < j$. Then ρ differs from π by a finite number of swaps (say k), contributing $\overline{d_{KT}}(\pi, \rho) = 2k$ by the distance graph $\mathcal{G}_n$. Following an argument analogous to case (i), we conclude that the identity holds for this case as well. □

Remark 3. *It is clear from the distance diagram $\mathcal{G}_n$ of $\mathcal{B}_n$ that the converse also holds. i.e., for $\rho \in \mathfrak{S}_n$, if we have $\overline{d_{KT}}(\pi, \rho) + \overline{d_{KT}}(\pi, \rho^{rev}) = n(n-1)$, then necessarily $\pi \in \mathfrak{S}_n$.*

Building on these distance properties, we now present a characterization result for subsets of $\mathfrak{S}_n$ that are closed under reversal. The following theorem shows that such a subset necessarily forms an $\mathfrak{S}_n$-median set, and moreover, provides an explicit distance formula from any $\mathfrak{S}_n$-element to the set.

Theorem 6. *If $\mathcal{A}$ is a subset of the symmetric group $\mathfrak{S}_n \subset \mathcal{B}_n$ of cardinality m and $\mathcal{A}$ is closed under the unary operation 'rev' (i.e., whenever $\pi \in \mathcal{A}$, then $\pi^{rev} \in \mathcal{A}$), then $\mathcal{A}$ is a $\mathfrak{S}_n$-median set. That is,*

$$\mathcal{M}(\mathcal{A}) = \mathfrak{S}_n.$$

and also for any $\sigma \in \mathfrak{S}_n$,

$$\overline{d_{KT}}(\sigma, \mathcal{A}) = \frac{m}{2} n(n-1).$$

Proof. Let $\mathcal{A} = \{\pi^1, \pi^2, \ldots, \pi^m\}$ be a set of m permutations in $\mathfrak{S}_n$. Since $\mathcal{A}$ is closed under the reversal operation rev, it follows that if $\pi \in \mathcal{A}$, then $\pi^{rev} \in \mathcal{A}$. Observe that for every $i \in \{1, 2, \ldots, m\}$, we can find $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ such that $\pi^j = (\pi^i)^{rev}$. We arrange these elements in pairs such that $(\pi^i)^{rev}$ is followed by π^i, and rename this to get

$$\mathcal{A} = \{\sigma^1, (\sigma^1)^{rev}, \sigma^2, (\sigma^2)^{rev}, \ldots, \sigma^{m/2}, (\sigma^{m/2})^{rev}\},$$

We want to prove that $\mathcal{M}(\mathcal{A}) = \mathfrak{S}_n$. To do so, we need to show the following.

(i): $\overline{d_{KT}}(\rho, \mathcal{A}) = \frac{m}{2} n(n-1)$, for every $\rho \in \mathfrak{S}_n$,

(ii): $\overline{d_{KT}}(\rho, \mathcal{A}) < \overline{d_{KT}}(\alpha, \mathcal{A})$ for every $\rho \in \mathfrak{S}_n, \alpha \in \mathcal{B}_n \setminus \mathfrak{S}_n$.

The median set $\mathcal{M}(\mathcal{A})$ consists of signed permutations ρ that minimize $\overline{d_{KT}}(\rho, \mathcal{A})$. Part (i) shows that all elements in $\mathfrak{S}_n$ are equidistant from $\mathcal{A}$ since the distance for any $\rho \in \mathfrak{S}_n$ is exactly $\frac{m}{2} n(n-1)$, which will imply $\mathcal{M}(\mathcal{A}) \supseteq \mathfrak{S}_n$. Then this with part (ii) completes the proof.

For part(i), we have for any $\rho \in \mathfrak{S}_n$, the distance to the set $\mathcal{A}$ is given by

$$\overline{d_{KT}}(\rho, \mathcal{A}) = \overline{d_{KT}}(\rho, \sigma^1) + \overline{d_{KT}}(\rho, (\sigma^1)^{rev}) + \cdots + \overline{d_{KT}}(\rho, \sigma^{m/2}) + \overline{d_{KT}}(\rho, (\sigma^{m/2})^{rev}).$$

Using Lemma 4, we have,

$$\begin{aligned} \overline{d_{KT}}(\rho, \mathcal{A}) &= n(n-1) + \cdots + n(n-1) \quad \left(\text{summed for } \frac{m}{2} \text{ pairs}\right). \\ \text{Thus, } \overline{d_{KT}}(\rho, \mathcal{A}) &= \frac{m}{2} n(n-1). \end{aligned}$$

For part(ii), using the triangle inequality and Remark 3, for any $\alpha \in \mathcal{B}_n \setminus \mathfrak{S}_n$, we have,

$$\begin{aligned} \overline{d_{KT}}(\alpha, \sigma^i) + \overline{d_{KT}}(\alpha, \sigma^{i^{rev}}) &> \overline{d_{KT}}(\sigma^i, \sigma^{i^{rev}}) = n(n-1) \\ \Rightarrow \overline{d_{KT}}(\alpha, \sigma^i) + \overline{d_{KT}}(\alpha, \sigma^{i^{rev}}) &> \overline{d_{KT}}(\rho, \sigma^i) + \overline{d_{KT}}(\rho, \sigma^{i^{rev}}), \text{ for } \rho \in \mathfrak{S}_n \end{aligned}$$

Summing over all pairs in $\mathcal{A}$, we obtain

$$\overline{d_{KT}}(\alpha, \mathcal{A}) > \overline{d_{KT}}(\rho, \mathcal{A}) = \frac{m}{2} n(n-1).$$

This confirms that for any $\alpha \notin \mathfrak{S}_n$,

$$\overline{d_{KT}}(\rho, \mathcal{A}) < \overline{d_{KT}}(\alpha, \mathcal{A}).$$

Thus, $\mathcal{M}(\mathcal{A})$ cannot contain any element from $\mathcal{B}_n \setminus \mathfrak{S}_n$, implying

$$\mathcal{M}(\mathcal{A}) = \mathfrak{S}_n.$$

□

6. Conclusion and future works

In this paper, we have extended the classical median problem for permutations to the setting of signed permutations by developing a generalized Kendall-τ distance that accounts for both order disagreements and sign differences. This advancement enables a rigorous study of median sets within the hyperoctahedral group of type $\mathcal{B}_n$, significantly broadening the scope of rank aggregation beyond the symmetric group $\mathfrak{S}_n$. We have introduced refined notions of order disagreements (Type I and Type II), formalized the new distance metric, and provided theoretical characterizations for special classes of median sets, namely, universal-median sets and $\mathfrak{S}_n$-median sets, which are sets closed under negation and invariant under reversal, respectively. Furthermore, the construction of a weighted distance graph $\mathcal{G}_n$ over $\mathcal{B}_n$ allows for efficient computation and visualization of transformations between signed permutations.

Our approach is particularly motivated by applications in computational biology, such as aggregating signed gene rankings from gene regulatory networks across experimental conditions. By capturing both the relative order and the regulatory direction (upregulation or downregulation) of genes, our method provides a principled framework for consensus analysis in biologically relevant settings.

There are several promising directions for future research. First and foremost, the computational complexity of the generalized median problem in $\mathcal{B}_n$ remains an open question for larger sets; a formal classification as polynomial-time solvable or NP-hard would deepen our understanding of the algorithmic landscape. While we have observed experimentally that for small values of m (e.g., $m = 2$), where m is the cardinality of a subset $\mathcal{A}$ of $\mathcal{B}_n$, the median set can be computed efficiently and seems to be polynomial-time solvable, the problem appears to scale unfavorably as m increases, because the number of candidate medians grows rapidly with n, since $|\mathcal{B}_n| = 2^n n!$. Therefore, while an exact complexity classification remains open, it is reasonable to conjecture that the median problem in $\mathcal{B}_n$ under the Generalized Kendall-τ distance is NP-hard for general m. Second, these insights suggest a need for heuristic methods, exploring approximation algorithms, or structural restrictions to make median computation feasible in practice. Our future work will aim to find special sets A for which it is easy to find medians in polynomial time. Automedian sets of signed permutations are such a case. Lastly, it would be fruitful to investigate analogous median problems under other combinatorial structures, such as Coxeter groups, signed posets, or graphs, to generalize the theory further.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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