Asia Pac. J. Math. 2025 **12**:95

A SELF ADAPTIVE PARALLEL EXTRAGRADIENT METHOD AND GOLDEN RATIO FOR EQUILIBRIUM PROBLEM IN HADAMARD SPACES

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Received Aug. 9, 2025

Abstract. In this manuscript, we present a self-adaptive parallel extragradient method together with a golden ratio model for solving pseudomonotone equilibrium problem in the settings of a Hadamard space. Under certain mild conditions, we proved a $\Delta-$ convergence of the generated sequence to a solution of the equilibrium problem. Our proposed method is structured in such a way that it is independent of the Lipschitz constants of the bifunction. Lastly, we provide several consequences and a numerical example to demonstrate the performance of our method. Numerous recent findings in the literature are improved and generalized by our result.

2020 Mathematics Subject Classification. 47H06; 47H09; 47J05; 47J25.

Key words and phrases. extragradient method; Hadamard spaces; golden ratio; self-adaptive method; equilibrium problem.

1. Introduction

Let Γ be a nonempty, closed and convex subset of a metric space $\mathcal{H}$, a point $x \in \mathcal{H}$ is said to be a fixed point of a nonlinear mapping $\Psi : \mathcal{H} \to \mathcal{H}$, if $x = \Psi x$. Given a bifunction $\Upsilon : \Gamma \times \Gamma \to \mathbb{R}$ with $\Upsilon(x, x) = 0$, for all $x \in \Gamma$. We consider the equilibrium problem (EP):

$$\text{find } x \in \Gamma \text{ such that } \Upsilon(x, p) \geq 0, \ \forall\, p \in \Gamma. \tag{1}$$

Throughout this manuscript, we denote by $EP(\Upsilon)$ the solution set of EP (1). The equilibrium problem is known to include, among its specific cases, Nash equilibrium problems, convex optimization problems, variational inequality problems (monotone or otherwise), and other problems of interest in numerous applications. Numerous studies have been conducted on equilibrium problems and their generalizations involving monotone bifunctions in Hilbert, Banach, and certain topological vector spaces, (see [1, 6, 9, 28, 30, 35]). In the case when the bifunction is pseudomonotone and satisfies the Lipschitz-type

DOI: 10.28924/APJM/12-95

condition (see definition in section 2), Tran *et al.* [30] proposed the following proximal-type algorithm:

$$\begin{cases} v_t = arg\min_{h\in\Gamma}\{\mu_t\Upsilon(u_t,h)+\frac{1}{2}\|u_t-h\|^2\} := prox_{\mu_t}\Upsilon(u_t,.)(u_t), \\ u_{t+1} = arg\min_{h\in\Gamma}\{\mu_t\Upsilon(v_t,h)+\frac{1}{2}\|u_t-h\|^2\} := prox_{\mu_t}\Upsilon(v_t,.)(u_t), \end{cases} \tag{2}$$

where $\mu_t > 0$. The authors established a weak convergence result in the settings of a real Hilbert space. It is worth-mentioning that two convex minimization subproblems on Γ need to be evaluated while implementing (2) which could affect the numerical performance of (2) with some feasible set Γ.

In 2019, Vinh and Muu [33] proposed the following inertial extragradient method (combination of Tran *et al.* [30] and an inertial extrapolation technique):

Algorithm 1.1. (*An inertial extragradient algorithm for equilibrium problem*).

Step 0 (*Initialization:*) *Select* $\theta \in [0,1), 0 < \mu < \min\{\frac{1}{2g_1},\frac{1}{2g_2}\}$ *and a positive sequence* $\{\varepsilon_t\}\subset[0,\infty)$ *satisfying* $\sum_{t=0}^{\infty}\varepsilon_t < \infty$.

Choose initial iterates $u_0, u_1 \in \Gamma$ *and set* $t = 1$

Step 1: *Given the current iterates* u_{t-1} *and* u_t $(t \geq 1)$, *choose* λ_t *such that* $0 \leq \lambda_t \leq \overline{\lambda_t}$, *where*

$$\overline{\lambda_t} = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_k}{\|u_t-u_{t-1}\|}\right\}, & \text{if } u_t \neq u_{t-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3}$$

Compute

$$\begin{cases} w_t = u_t + \lambda_t(u_t-u_{t-1}), \\ y_t = arg\min_{h\in\Gamma}\{\mu\Upsilon(w_t,h)+\frac{1}{2}\|h-w_t\|^2\}. \end{cases} \tag{4}$$

If $y_t = w_t$ *then stop.* u_t *is a solution. Otherwise, go to Step 2.*

Step 2:

$$u_{t+1} = arg\min_{h\in\mathcal{H}}\{\mu\Upsilon(y_t,h)+\frac{1}{2}\|h-w_t\|^2\}.$$

Set $t := t+1$ *and return to step 1.*

Recently, Khatibzadeh and Mohebbi [22] extended the extragradient method for solving equilibrium problem to the settings of a nonlinear space (precisely, Hadamard space). It is known that extension of numerical optimization algorithms from Euclidean spaces to nonlinear spaces has significant advantages and numerous problems in applied sciences are considered in nonlinear rather than linear spaces, for example, for example, image processing and medical imaging problems in Riemannian manifolds (see [5,15,27,29]). Khatizadeh and Mohebbi [22] proposed the following extragradient method for solving equilibrium problem in Hadamard spaces:

Algorithm 1.2. *Extragradient method for equilibrium problem.*
Initialize: Choose $u_0 \in \mathcal{H},\ n := 0,\ 0 < \alpha \leq \mu_t < \min\{\frac{1}{2g_1}, \frac{1}{2g_2}\}$ *and* $t = 0, 1, 2, \cdots$

Step 1: *Solve the following minimization problem and let* y_t *be the solution i.e.*

$$y_t \in Arg \min_{h \in \mathcal{H}} \{\Upsilon(u_t, h) + \frac{1}{2\mu_t} d^2(u_t, h)\}.$$

Step 2: *Solve the following minimization problem and let* u_{t+1} *be the solution, i.e.*

$$u_{t+1} \in Arg \min_{h \in \mathcal{H}} \{\Upsilon(y_t, h) + \frac{1}{2\mu_t} d^2(y_t, h)\}.$$

Step 3: $t := t + 1$ *and go back to step 1.*

They proved a $\triangle-$ convergence of the sequence generated by Algorithm 1.2. We observed that the bifunction in (2), Algorithm 1.1 and Algorithm 1.2 depends on Lipschitz constant. In view of this, Aremu *et al.* [3] proposed a self-adaptive extragardient method for solving a common solution of a pseudomonotone equilibrium problem and fixed point problem for multi-valued nonexpansive mapping in Hadamard spaces. Readers should consult ([2,17,24] and the references therein) for more results on extragradient method.

In this manuscript, we propose a self-adaptive extragradient method together with a golden ratio model for solving an equilibrium problem with the following contributions:

(1) Our proposed algorithm include a golden ratio technique with a self-adaptive stepsize. Our golden ratio value is not restricted to $\frac{1+\sqrt{5}}{2}$. The algorithms in [3,22] are special cases of our algorithm when $i = 1$.
(2) Our result extends the result of [30,33] from linear spaces to nonlinear spaces.
(3) Numerical experiments are presented to demonstrate the performance of our method.

The manuscript is organized as follows: In section 2, we present some definitions and preliminary results for further use. In section 3, we present our iterative algorithm and discuss its convergence results. We also provide some consequences of our results in section 3. In section 4, we report some numerical examples to demonstrate the performance of our iterative method and compare with the un accelerated algorithm.

2. Preliminaries

Let $\mathcal{H}$ be a metric space and $x, y \in \mathcal{H}$. A geodesic from x to y is a map (or a curve) c from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to $\mathcal{H}$ such that $c(0) = x,\ c(d(x, y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of c is called a geodesic segment joining from x to y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space $(\mathcal{H}, d)$ is said to be a geodesic space if every two points of $\mathcal{H}$ are joined by a geodesic, and $\mathcal{H}$ is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in \mathcal{H}$. A subset Γ of a geodesic space $\mathcal{H}$ is said to be convex, if for any two points $x, y \in \Gamma$, the geodesic joining x and y is contained in Γ, that is, if $c : [0, d(x,y)] \to \mathcal{H}$ is a geodesic such that $x = c(0)$ and $y = c(d(x,y))$, then $c(t) \in \Gamma \;\forall\, t \in [0, d(x,y)]$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(\mathcal{H}, d)$ consists of three vertices (points in $\mathcal{H}$) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$, such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic space $\mathcal{H}$ is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in $\mathcal{H}$ respectively, then Δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{5}$$

Let x, y, z be points in $\mathcal{H}$ and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{6}$$

Berg and Nikolaev [4] introduced the notion of quasi-linearization in a CAT(0) space as follows: Let a pair $(a, b) \in \mathcal{H} \times \mathcal{H}$ denoted by $\overrightarrow{ab}$, be called a vector. Then, the quasilinearization map $\langle ., . \rangle : (\mathcal{H} \times \mathcal{H}) \times (\mathcal{H} \times \mathcal{H}) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \text{ for all } a, b, c, d \in \mathcal{H}. \tag{7}$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in \mathcal{H}$. Furthermore, a geodesic space X is said to satisfy the Cauchy-Schwartz inequality, if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),$$

for all $a, b, c, d \in \mathcal{H}$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality [13]. Also, it is known that complete CAT(0) spaces are called Hadamard spaces.

In 2010, Kakavandi and Amini [19] introduced the dual space of a Hadamard space $\mathcal{H}$ as follows: Consider the map $\Theta : \mathbb{R} \times \mathcal{H} \times \mathcal{H} \to C(\mathcal{H}, \mathbb{R})$ define by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \; (t \in \mathbb{R}, \; a, b, x \in \mathcal{H}),$$

where $C(\mathcal{H}, \mathbb{R})$ is the space of all continuous real-valued functions on $\mathcal{H}$. Then the Cauchy-Schwartz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$ $(t \in \mathbb{R}, a, b \in \mathcal{H})$, where $L(\phi) = \sup\{(\phi(x) - \phi(y))/d(x, y) : x, y \in \mathcal{H}, \; x \neq y\}$ is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric Γ on $\mathbb{R} \times \mathcal{H} \times \mathcal{H}$ is defined by

$$\mathcal{D}((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \; (t, s \in \mathbb{R}, a, b, c, d \in \mathcal{H}).$$

In an Hadamard space $\mathcal{H}$, the psuedometric space $(\mathbb{R} \times \mathcal{H} \times \mathcal{H}, \mathcal{D})$ can be considered as a subset of the pseudometric space of all real-valued Lipschitz functions $(Lip(\mathcal{H}, \mathbb{R}), L)$. It is well known from [19] that $\mathcal{D}((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in \mathcal{H}$. Thus, $\mathcal{D}$ induces an equivalence relation on $\mathbb{R} \times \mathcal{H} \times \mathcal{H}$, where the equivalence class of (t,a,b) is defined as

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : \mathcal{D}((t,a,b),(s,c,d)) = 0\}.$$

The set $\mathcal{H}^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times \mathcal{H} \times \mathcal{H}\}$ is a metric space with the metric $\mathcal{D}([t\overrightarrow{ab}], [s\overrightarrow{cd}) := \mathcal{D}((t,a,b),(s,c,d))$. The pair $(\mathcal{H}^*, d)$ is called the dual space of the metric space $(\mathcal{H}, d)$. It is shown in [19] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is H and $t(b-a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R},\ a, b \in H$. We also note that $\mathcal{H}^*$ acts on $\mathcal{H} \times \mathcal{H}$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle,\ (x^* = [t\overrightarrow{ab}] \in \mathcal{H}^*,\ x, y \in \mathcal{H}).$$

Let $\{x_t\}$ be a bounded sequence in $\mathcal{H}$ and $r(., \{x_t\}) : \mathcal{H} \to [0, \infty)$ be a continuous mapping defined by

$$r(x, \{x_t\}) = \limsup_{t \to \infty} d(x, x_t).$$

The asymptotic radius of $\{x_t\}$ is given by

$$r(\{x_t\}) : \inf\{r(x, x_t) : x \in \mathcal{H}\},$$

while the asymptotic center of $\{x_t\}$ is the set

$$A(\{x_t\}) = x \in \mathcal{H} : r(x, \{x_t\}) = r(\{x_t\}).$$

It is well known from [12, 23] that in a complete CAT(0) space $\mathcal{H}$, $A(\{x_t\})$ consists of exactly one point. A sequence $\{x_t\}$ in $\mathcal{H}$ is said to be Δ-convergent to a point $x \in \mathcal{H}$ if $A(\{x_{t_k}\}) = \{x\}$ for every subsequence $\{x_{t_k}\}$ of $\{x_t\}$. In this case, we write $\Delta - \lim\limits_{t \to \infty} x_t = x$.
Let $\mathcal{H}$ be an Hadamard space and $\Gamma \subset \mathcal{H}$ and $\Upsilon : \Gamma \times \Gamma \to \mathbb{R}$ be a bifunction. Υ is said to be monotone, if

$$\Upsilon(x,y) + \Upsilon(y,x) \leq 0,\ \forall\, x, y \in \mathcal{H},$$

and pseudo monotone, if

$$\Upsilon(x,y) \geq 0 \implies \Upsilon(y,x) \leq 0,\ \forall\, x, y \in \mathcal{H}.$$

Definition 2.1. Let $\mathcal{H}$ be a Hadamard space. A function $\Upsilon : \mathcal{H} \to (-\infty, \infty]$ is said to be

(i) convex, if

$$\Upsilon(\rho u \oplus (1-\rho)v) \leq \rho\Upsilon(u) + (1-\rho)\Upsilon(v),\ \forall\, u, v \in \mathcal{H},\ \rho \in (0,1),$$

(ii) lower semicontinuous (or upper semicontinuous) at a point $u \in \Gamma$, if

$$\Upsilon(u) \leq \liminf_{t\to\infty} \Upsilon(u_t),$$

for each sequence $\{u_t\}$ in Γ such that $\lim_{t\to\infty} u_t = u$.

The convex programming associated with the proper, convex and lower semicontinuous function Υ for all $\mu > 0$ is given by:

$$arg\min_{h\in\mathcal{H}} \big(\Upsilon(u) + \frac{1}{2\mu} d^2(u,v)\big), \tag{8}$$

for all $v \in \mathcal{H}$.

Remark 2.2. [18] The subproblem (8) is well-defined for all $\mu > 0$.

To study the EP, it is required that the the bifuntion Υ satisfy the following conditions:

(B1) $\Upsilon(x, .)$ is convex and lower semicontinuous for all $x \in \mathcal{H}$,

(B2) $\Upsilon(., y)$ is Δ-upper semicontinuous for all $y \in \mathcal{H}$,

(B3) Υ is Lipschitz-type continuous, i.e. there exist two positive constants g_1 and g_2 such that

$$\Upsilon(x,y) + \Upsilon(y,z) \geq \Upsilon(x,z) - g_1 d^2(x,y) - g_2 d^2(y,z),\ \forall\, x,y,z \in \mathcal{H}. \tag{9}$$

(B4) Υ is pseudo-monotone.

Below is an example of equilibrium problem in a Hadamard space.

Example 2.3. Let $\{w_j\}_{j=1}^p \subset \mathcal{H}$ and $\{\alpha_j\}_{j=1}^p$ be positive weights satisfying $\sum\limits_{j=1}^p \alpha_j = 1$. We define the geometric median for $\{w_j\}_{j=1}^p$ as

$$Arg\min_{h\in\mathcal{H}} \sum_{j=1}^p \alpha_j d(u, w_j),$$

and the Fréchet mean as

$$Arg\min_{h\in\mathcal{H}} \sum_{j=1}^p \alpha_j d(u, w_j).$$

Now, we define $\theta_i : \mathcal{H} \to \mathbb{R}$ as $\theta_i(u) = \sum\limits_{j=1}^p \alpha_j d^i(u, w_j)$ and we consider $\Upsilon_j : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ as $\Upsilon_i(u,v) = \theta_i(v) - \theta_i(u)$, for $i = 1, 2$. It is clear that Υ_i satisfies $(B1) - (B4)$ and $\Omega \neq \emptyset$, and any equilibrium point of $EP(\Upsilon_i, \mathcal{H})$ is the minimum point of $\theta_i,\ i = 1, 2$.

Lemma 2.4. [10, 13] *Let $\mathcal{H}$ be a Hadamard space. Then for all $w, x, y, z \in \mathcal{H}$ and all $t \in [0, 1]$, we have*

(1) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$,

(2) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$,

(3) $d^2(z, tx \oplus (1-t)y) \leq t^2 d^2(z,x) + (1-t)^2 d^2(z,y) + 2t(1-t)\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle$.

Definition 2.5. [14] Let Γ be a nonempty, closed and subset of $\mathcal{H}$ and $\{x_t\}$ be a sequence in $\mathcal{H}$. Then $\{x_t\}$ is said to be Fejèr convergent with respect to Γ if for all $p \in \Gamma$ and $t \in \mathbb{N}$,

$$d(x_{t+1}, p) \leq d(x_t, p).$$

Lemma 2.6. [14] *Let Γ be a nonempty, closed and closed subset of $\mathcal{H}$ and $\{x_t\} \subset \mathcal{H}$ be a sequence such that $\{x_t\}$ be a Fejér convergent with respect to Γ. Then the following hold:*

(i) *For every $p \in \Gamma,\ d(x_t, p)$ converges,*

(ii) *$\{x_t\}$ is bounded,*

(iii) *Assume that every cluster point of $\{x_t\}$ belongs to Γ, then $\{x_t\}$ converges to a point in Γ.*

Lemma 2.7. [13] *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergence subsequence.*

Lemma 2.8. [7] *Let $\{a_t\}$ and $\{b_t\}$ be nonnegative sequences verifying*

$$a_{t+1} \leq a_t - b_t,\ \forall\, t > N,$$

where N is some nonnegative integer. Then, $\lim_{t\to\infty} b_t = 0$ and $\lim_{t\to\infty} a_t$ exists.

3. Main results

In this section, we propose a self adaptive parallel algorithm with a golden ratio technique for approximating solution of pseudomonotone equilibrium problem in the context of Hadamard spaces. Now, let Γ be a nonempty, closed and convex subset of a Hadamard space $\mathcal{H},\ \Upsilon_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, i = 1, 2, \cdots, m$ satisfying $(B1) - (B4)$ and suppose that $\Omega \neq \emptyset$.
Now, we present our algorithm as follows:

Algorithm 3.1. *Parallel Extragradient method for EP with golden ratio (PEMGR).*

Initialization: *Choose $\mu_1 > 0,\ \alpha \in (0,1), \varphi \in (1, +\infty)$ and $u_0, w_1 \in \mathcal{H}$. Let $\{\sigma_t\}$ be a nonnegative real number satisfying $\sum_{t=1}^{\infty} \sigma_t < \infty$. Given the current iterate w_t, compute $\{w_{t+1}\}$ via the following procedure:*

Step 1: *Compute*

$$u_t = \frac{\varphi - 1}{\varphi} w_t \oplus \frac{1}{\varphi} u_{t-1}.$$

Step 2: *Evaluate*

$$v_t^i = arg \min_{h \in \mathcal{H}} \{\Upsilon_i(u_t, h) + \frac{1}{2\mu_t} d^2(u_t, h)\},\ i = 1, 2, \cdots, m. \tag{10}$$

If $u_t = v_t^i$, then stop. Otherwise go to step 3.

Step 3: *Calculate*

$$w_{t+1}^i = arg \min_{h \in \mathcal{H}} \{\Upsilon_i(v_t^i, h) + \frac{1}{2\mu_t} d^2(u_t, h)\},\ i = 1, 2, \cdots .m. \tag{11}$$

Step 4: *Find among $w^i_{t+1}, i = 1, 2, \cdots, m$, the farthest element from u_t, that is*

$$i_t = arg\max\{d(w^i_{t+1}, u_t) : i = 1, 2, \cdots, m\},\ w_{t+1} := w^{i_t}_{t+1}.$$

and

$$\mu_{t+1} = \begin{cases} \min\limits_{1\le i\le m}\left\{\dfrac{\alpha\big[d^2(u_t, v^i_t) + d^2(w^i_{t+1}, v^i_t)\big]}{2\big[\Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(v^i_t, u_t) - \Upsilon_i(v^i_t, w^i_{t+1})\big]},\ \mu_t + \sigma_t\right\} & \text{if } \Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(v^i_t, u_t) \\ & -\Upsilon_i(v^i_t, w^i_{t+1}) > 0, \quad i = 1, 2, \ldots, m, \\ \mu_t + \sigma_t, \qquad \text{otherwise.} & \end{cases} \tag{12}$$

Lemma 3.2. *For any $q \in EP(\Upsilon_i)$, the sequences generated by Algorithm 3.1 satisfy*

$$d^2(w^i_{t+1}, q) \le d^2(u_t, q) - (1 - \alpha\frac{\mu_t}{\mu_{t+1}})d^2(u_t, v^i_t) - (1 - \alpha\frac{\mu_t}{\mu_{t+1}})d^2(w^i_{t+1}, v^i_t).$$

Proof. Let $q \in EP(\Upsilon_i),\ i = 1, 2, \cdots, m$, since w^i_{t+1} is a solution of the minimization problem in (34), let $h = \lambda q \oplus (1-\lambda)w^i_{t+1}$, where $\lambda \in [0, 1)$. By applying Lemma 2.4 (1), we have

$$\begin{aligned} \Upsilon_i(u_t, w^i_{t+1}) + \frac{1}{2\mu_t}d^2(w^i_{t+1}, u_t) &\le \Upsilon_i(u_t, h) + \frac{1}{2\mu_t}d^2(h, u_t) \\ &\le \Upsilon_i(u_t, \lambda q \oplus (1-\lambda)w^i_{t+1}) + \frac{1}{2\mu_t}d^2(\lambda q \oplus (1-\lambda)w^i_{t+1}, u_t) \\ &\le \lambda\Upsilon_i(u_t, q) + (1-\lambda)\Upsilon_i(u_t, w^i_{t+1}) + \frac{1}{2\mu_t}\Big[\lambda d^2(q, u_t) + (1-\lambda)d^2(w^i_{t+1}, u_t) \\ &- \lambda(1-\lambda)d^2(q, w^i_{t+1})\Big]. \end{aligned} \tag{13}$$

Since $\Upsilon_i(q, w^i_{t+1}) \ge 0, i = 1, 2, \cdots, m$, the pseudomonotonicity of Υ_i implies that $\Upsilon_i(w^i_{t+1}, q) \le 0$. Thus (13) can be reduced to

$$\begin{aligned} &\frac{1}{2\mu_t}\Big[d^2(w^i_{t+1}, u_t) - d^2(q, u_t) - (1-\lambda)d^2(q, w^i_{t+1})\Big] \\ &\le \Upsilon_i(u_t, q) - \Upsilon_i(u_t, w^i_{t+1}). \end{aligned} \tag{14}$$

If $\lambda \to 1^{-1}$ in (14), we get

$$\begin{aligned} &\frac{1}{2\mu_t}\Big[d^2(w^i_{t+1}, u_t) - d^2(q, u_t) - d^2(q, w^i_{t+1})\Big] \\ &\le \Upsilon_i(u_t, q) - \Upsilon_i(u_t, w^i_{t+1}), \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\Big[d^2(w^i_{t+1}, u_t) - d^2(q, u_t) - d^2(q, w^i_{t+1})\Big] \\ &\le \mu_t\Big[\Upsilon_i(u_t, q) - \Upsilon_i(u_t, w^i_{t+1})\Big]. \end{aligned} \tag{15}$$

From (12), we have

$$\Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(v^i_t, u_t) - \Upsilon_i(v^i_t, w^i_{t+1}) \leq \frac{\alpha}{2\mu_{t+1}}\Big[d^2(u_t, v^i_t) + d^2(w^i_{t+1}, v^i_t)\Big]. \tag{16}$$

Since $\mu_t > 0$, we obtain from (16) that

$$\begin{aligned}\mu_t \Upsilon_i(u_t, w^i_{t+1}) \leq \mu_t \Big[\Upsilon_i(v^i_t, u_t) + \Upsilon_i(v^i_t, w^i_{t+1})\Big]\\ + \frac{\alpha}{2\mu_{t+1}}\Big[d^2(u_t, v^i_t) + d^2(w^i_{t+1}, v^i_t)\Big].\end{aligned} \tag{17}$$

Using the quasilinearization properties and (14), we obtain that

$$\langle \overrightarrow{w^i_{t+1}q}, \overrightarrow{u_t w^i_{t+1}}\rangle \geq \mu_t\Big[\Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(u_t, q)\Big]. \tag{18}$$

Thus, from (16) and (17), we have

$$\begin{aligned}\langle \overrightarrow{w^i_{t+1}q}, \overrightarrow{u_t w^i_{t+1}}\rangle \geq \mu_t\Big[\Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(u_t, q)\Big] - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(u_t, v^i_t)\\ - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(w^i_{t+1}, v^i_t).\end{aligned} \tag{19}$$

From (34) and Remark 8, we get that

$$\mu_t\big[\Upsilon_i(u_t, w^i_{t+1}) - \Upsilon_i(u_t, v^i_t)\big] \geq \langle \overrightarrow{v^i_t u_t}, \overrightarrow{v^i_t w^i_{t+1}}\rangle. \tag{20}$$

Thus, using (19) and (20), we have

$$\begin{aligned}\langle \overrightarrow{w^i_{t+1}q}, \overrightarrow{u_t w^i_{t+1}}\rangle \geq \langle \overrightarrow{v^i_t u_t}, \overrightarrow{v^i_t w^i_{t+1}}\rangle - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(u_t, v^i_t)\\ - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(w^i_{t+1}, v^i_t).\end{aligned} \tag{21}$$

Using quasilinearization property, we have the following

$$\begin{cases} 2\langle \overrightarrow{v^i_t u_t}, \overrightarrow{v^i_t w^i_{t+1}}\rangle = \{d^2(v^i_t, w^i_{t+1}) + d^2(u_t, v^i_t) - d^2(u_t, w^i_{t+1})\} \\ 2\langle \overrightarrow{u_t w^i_{t+1}}, \overrightarrow{w^i_{t+1}q}\rangle = \{d^2(u_t, q) - d^2(u_t, w^i_{t+1}) - d^2(w^i_{t+1}, q)\}. \end{cases} \tag{22}$$

Thus, from (21) and (22), we obtain

$$\begin{aligned}d^2(w^i_{t+1}, q) \leq d^2(u_t, q) - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(u_t, v^i_t)\\ - (1 - \alpha\frac{\mu_t}{2\mu_{t+1}})d^2(w^i_{t+1}, v^i_t),\end{aligned}$$

as desired. □

Theorem 3.3. *Suppose that he bifunction $\Upsilon_i, i = 1, 2, \cdots, m$ satisfies $(B1) - (B4)$. Then the sequence $\{w_t\}$ generated by Algorithm 3.1 Δ-converges to a solution of Ω.*

Proof. Let $q \in \Omega$, then it follows from the definition of u_t in Algorithm 3.1 that

$$w_t = \frac{\varphi}{\varphi-1}u_t - \frac{1}{\varphi-1}u_{t-1},$$

Thus, using Lemma 2.4 (2), we get

$$\begin{aligned} d^2(w_t,q) &\le \frac{\varphi}{\varphi-1}d^2(u_t,q) - \frac{1}{\varphi-1}d^2(u_{t-1},q) + \frac{\varphi}{(\varphi-1)^2}d^2(u_t,u_{t-1}) \\ &= \frac{\varphi}{\varphi-1}d^2(u_t,q) - \frac{1}{\varphi-1}d^2(u_{t-1},q) + \frac{1}{\varphi}d^2(w_t,u_{t-1}). \end{aligned} \tag{23}$$

On substituting (23) into Lemma 3.2, and applying step 4 of Algorithm 3.1, we get

$$\begin{aligned} d^2(w_{t+1},q) - d^2(w_t,q) \le \frac{-1}{\varphi-1}d^2(u_t,q) + \frac{1}{\varphi-1}d^2(u_{t-1},q) - \frac{1}{\varphi}d^2(w_t,u_{t-1}) \\ - (1-\alpha\frac{\mu_t}{\mu_{t+1}})\Big[d^2(u_t,v_t^i) + d^2(w_{t+1}^i,v_t^i)\Big], \end{aligned} \tag{24}$$

which implies that

$$\begin{aligned} d^2(w_{t+1},q) + \frac{1}{\varphi-1}d^2(u_t,q) \le d^2(w_t,q) + \frac{1}{\varphi-1}d^2(u_{t-1},q) - \frac{1}{\varphi}d^2(w_t,u_{t-1}) \\ - (1-\alpha\frac{\mu_t}{\mu_{t+1}})\Big[d^2(u_t,v_t^i) + d^2(w_{t+1}^i,v_t^i)\Big]. \end{aligned} \tag{25}$$

By settings $r_t = d^2(w_t,q)+\frac{1}{\varphi-1}d^2(u_{t-1},q)$ and $b_t = \frac{1}{\varphi}d^2(w_t,u_{t-1})+(1-\alpha\frac{\mu_t}{\mu_{t+1}})\Big[d^2(u_t,v_t^i)+d^2(w_{t+1}^i,v_t^i)\Big]$, we infer that

$$r_{t+1} \le r_t - b_t.$$

Also, we obtain that $r_t \ge 0$ and $b_t \ge 0$. Using Lemma 2.8, we can see that $\lim_{t\to\infty} b_t = 0$ and that $\lim_{t\to\infty} r_t$ exists. Thus, we obtain that

$$\lim_{t\to\infty} d(u_t,v_t^i) = 0 = \lim_{t\to\infty} d(w_{t+1}^i,v_t^i), \tag{26}$$

and

$$\lim_{t\to\infty} d(u_t,w_{t+1}^i) = 0. \tag{27}$$

From the definition of u_t, it is easy to see that

$$\begin{aligned} r_{t+1} &= d^2(w_{t+1},q) + \frac{1}{\varphi-1}d^2(u_t,q) \\ &= \frac{\varphi}{\varphi-1}d^2(u_{t+1},q) + \frac{\varphi}{(\varphi-1)^2}d^2(u_{t+1},u_t) - \frac{1}{\varphi-1}d^2(u_t,q) + \frac{1}{\varphi-1}d^2(u_t,q) \\ &= \frac{\varphi}{\varphi-1}d^2(u_{t+1},q) + \frac{1}{\varphi}d^2(w_{t+1},u_t). \end{aligned}$$

It is obvious that the limit of $\{d^2(u_{t+1},q)\}$ exists. We also conclude that the limit of $\{d^2(w_{t+1},q)\}$ exists and the sequences $\{u_t\}$ and $\{w_t\}$ are bounded. Consequently, $\{v_t^i\}, i = 1,2,\cdots,m$ are also bounded. We now establish that $\{w_t\}$ Δ-converges to a point $\overline{q} \in \Omega$. Since $\{w_t\}$ is bounded, using Lemma 2.7,

there exists a subsequence $\{w_{t_l}\}$ of $\{w_t\}$ such that $\Delta - \lim_{t\to\infty} w_{t_l} = \overline{q}$, for some $\overline{q} \in \mathcal{H}$.
From Algorithm 3.1, $w^i_{t+1}, i = 1, 2, \cdots, m$ solves (34). Let $r = \lambda w^i_{t+1} \oplus (1-\lambda)h$ such that $\lambda \in [0, 1)$ and $h \in \mathcal{H}$, we get

$$\begin{aligned}\Upsilon_i(v^i_t, w^i_{t+1}) + \frac{1}{2\mu_t} d^2(u_t, w^i_{t+1}) &\le \Upsilon_i(v^i_t, r) + \frac{1}{2\mu_t} d^2(u_t, r)\\ &\le \Upsilon_i(v^i_t, \lambda w^i_{t+1} \oplus (1-\lambda)h) + \frac{1}{2\mu_t} d^2(u_t, \lambda w^i_{t+1} \oplus (1-\lambda)h)\\ &\le \lambda\Upsilon_i(v^i_t, w^i_{t+1}) + (1-\lambda)\Upsilon_i(v^i_t, h) + \frac{1}{2\mu_t}\{\lambda d^2(u_t, w^i_{t+1})\}\\ &+ (1-\lambda)d^2(u_t, h) - \lambda(1-\lambda)d^2(w^i_{t+1}, h)\}. \end{aligned} \tag{28}$$

Following a similar approach as in (16)-(18), we obtain from

$$\Upsilon_i(v^i_t, w^i_{t+1}) - \Upsilon_i(v^i_t, h) \le \frac{1}{2\mu_t}\big[d^2(u_t, h) - d^2(u_t, w^i_{t+1}) - \lambda d^2(w^i_{t+1}, h)\big]. \tag{29}$$

Let $\lambda \to 1^-$, thus we get

$$\Upsilon_i(v^i_t, w^i_{t+1}) - \Upsilon_i(v^i_t, h) \le \frac{1}{2\mu_t}\big[d^2(u_t, h) - d^2(u_t, w^i_{t+1}) - \lambda d^2(w^i_{t+1}, h)\big]. \tag{30}$$

This implies from (30) that

$$\Upsilon_i(v^i_t, h) \ge \frac{1}{2\mu_t}\left\{d^2(u_t, w^i_{t+1}) + d^2(w^i_{t+1}, h) - d^2(u_t, h)\right\} + \Upsilon_i(v^i_t, w^i_{t+1}), \tag{31}$$

which from the quasilinearization property implies that

$$\Upsilon_i(v^i_t, h) \ge \frac{1}{2\mu_t}\langle \overrightarrow{w^i_{t+1}h}, \overrightarrow{u_t w^i_{t+1}}\rangle + \Upsilon_i(v^i_t, w^i_{t+1}). \tag{32}$$

Hence, using (26) and (27) and $\mu_t > 0$, we obtain that $\Upsilon_i(v^i_t, h) \ge 0, i = 1, 2, \cdots, m$. Since $\{u_t\}$ is Δ-convergent to $\overline{q}$, by the fact that $\Upsilon_i(v^i_t, h) \ge 0$ and Assumption (B2), we conclude that $\Upsilon_i(\overline{q}, h) \ge 0,\ i = 1, 2, \cdots, m$. Thus $\overline{q} \in \Omega$. Also, since q is taken arbitrarily from Ω, then using Lemma 2.6, we conclude that $\{w_t\}$ Δ-converges to $q \in \Omega$ as asserted. □

We present a consequence of our result as follows:

Corollary 3.4.

Algorithm 3.5. *Extragradient method for EP with golden ratio (EMGR).*
Initilization: *Choose $\mu_1 > 0, \alpha \in (0, 1), \varphi \in (1, +\infty)$ and $u_0, w_1 \in \mathcal{H}$. Let $\{\sigma_t\}$ be a nonnegative real number satisfying $\sum_{t=1}^{\infty} \sigma_t < \infty$. Given the current iterate w_t, compute $\{w_{t+1}\}$ via the following procedure:*
Step 1: *Compute*

$$u_t = \frac{\varphi - 1}{\varphi} w_t \oplus \frac{1}{\varphi} u_{t-1}.$$

Step 2: *Evaluate*

$$v_t = arg\min_{h\in\mathcal{H}}\{\Upsilon(u_t,h) + \frac{1}{2\mu_t}d^2(u_t,h)\}. \tag{33}$$

If $u_t = v_t$, then stop. Otherwise go to step 3.

Step 3: *Calculate*

$$w_{t+1} = arg\min_{h\in\mathcal{H}}\{\Upsilon(v_t,h) + \frac{1}{2\mu_t}d^2(u_t,h)\}. \tag{34}$$

Suppose that he bifunction Υ, satisfies $(B1)-(B4)$. Then the sequence $\{w_t\}$ generated by Algorithm 3.1 Δ-converges to a solution of Ω.

4. Numerical Examples

In this section, we present some numerical experiments to demonstrate the performance of our main result.

Example 4.1. Let $P(n,\mathbb{R})$ be the space of $(n\times n)$ positive symmetric definite matrices endowed with the Riemannian metric

$$\langle G,H\rangle_E := Tr(E^{-1}GE^{-1}H),$$

for all $G,H\in T_E(P(n,\mathbb{R}))$ and every $E\in P(n,\mathbb{R})$. The pair $(P(n,\mathbb{R}),\langle G,H\rangle_E)$ is a Hadamard space, (see [22]). Let $\mathbb{R}^+$ be the set of positive real numbers. Now, consider the space $P(n,\mathbb{R})$ such that n=1 with an inner product $\langle a,b\rangle_\lambda = \frac{1}{\lambda^2}ab$ for $\lambda>0$ and $a,b\in T_\lambda\mathbb{R}^+=\mathbb{R}$. Let $(\mathcal{H},d)$ be a metric space with $\mathcal{H}=\mathbb{R}^+$ and $d:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be defined by

$$d(a,b) = |\ln a - \ln b|,$$

with the geodesic between $a,b\in\mathcal{H}$ defined as $\gamma(\kappa) = a(\frac{a}{b})^\kappa$. Therefore, the pair $(\mathcal{H},d)$ is a CAT(0) space with the geodesic between a and b given as

$$ln\gamma(\kappa) = \ln a(\frac{b}{a})^\kappa = \ln a + \kappa(\ln b - lna) = (1-\kappa)lna + \kappa lnb. \tag{35}$$

Now let $\Upsilon:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be bifunctions by $\Upsilon(u,v) - \ln x(\ln\frac{y}{x})$. From (35), we have that

$$\begin{aligned}\Upsilon(u,\gamma(\kappa)) &= \ln u\big(\ln\frac{\gamma(\kappa)}{u}\big) = (1-\kappa)\ln u(\ln\frac{a}{u}) + \kappa\ln u(\ln\frac{b}{u})\\ &= (1-\kappa)\Upsilon(u,a) + \kappa\Upsilon(u,b).\end{aligned} \tag{36}$$

It is obvious that Υ satisfies (B1) and (B2). It has been established in [3] that Υ satisfies (B3) AND (B4). Hence Υ is monotone (and thus pseudomonotone).

For the sake of numerical computation, we choose $\mu_1 = 0.9,\ \alpha = 0.6,\ \varphi = 1.93$ and $\mu_t = \frac{1}{2g_1}$, where $g_1 = g_2 = \frac{1}{2}$. We terminate the execution of the process at $E_n = d(x_{n+1},x_n) = 10^{-6}$ and make a comparison of Algorithm 3.1 with an unaccelerated version of it. The result of this experiment is shown in Figure 1.

Case 1: $x_0 = 1.8$ and $x_1 = 0.6$;

Case 2: $x_0 = 1.5$ and $x_1 = 0.5$;

Case 3: $x_0 = 2.5$ and $x_1 = 0.5$;

Case 4: $x_0 = 0.7$ and $x_1 = 0.6$.

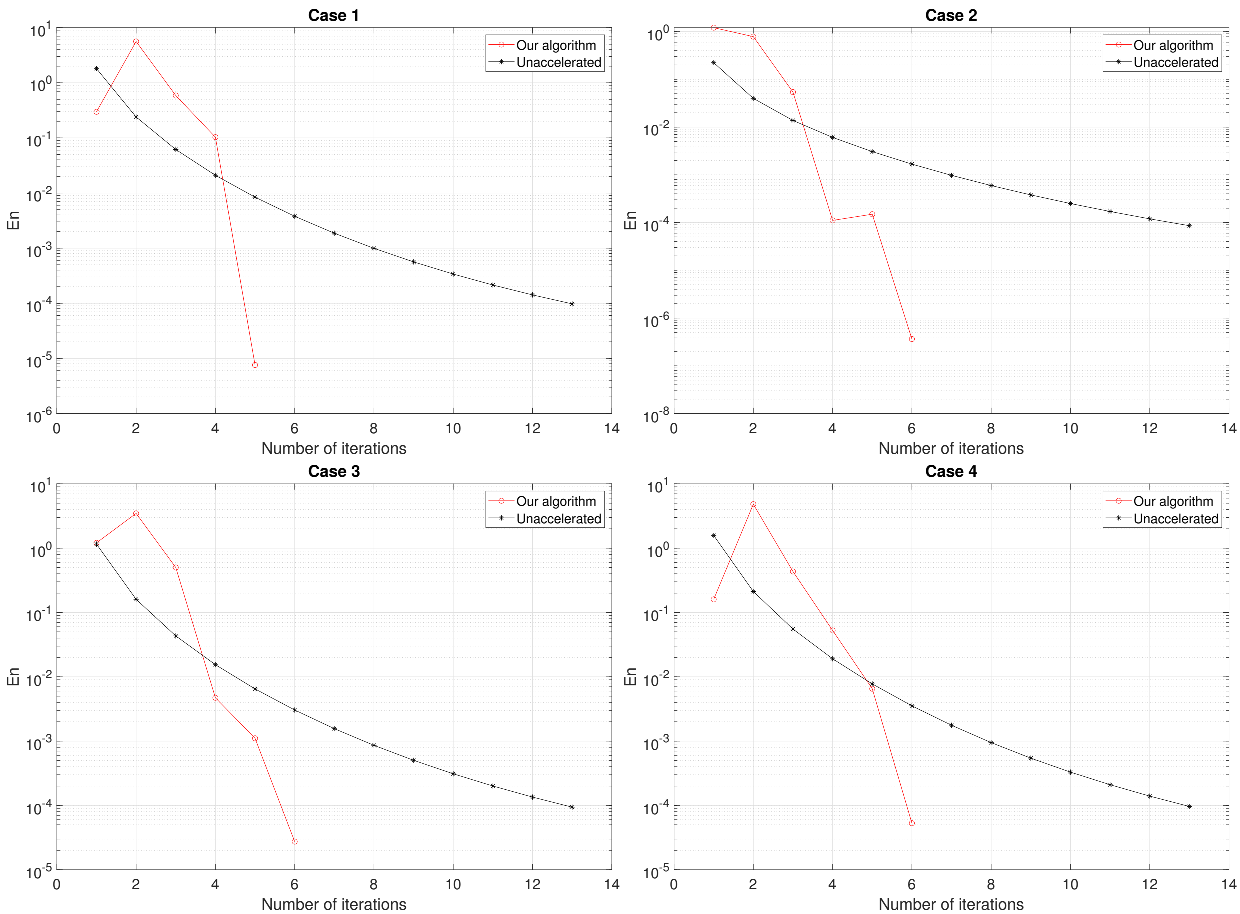

FIGURE 1. Example 4.1. Top Left: Case 1, Top Right: Case 2, Bottom right: Case 3, Bottom left: Case 4

Example 4.2. We consider the Nash-Cournot Oligopolistic equilibrium model in [26] with bifunctions $\Upsilon_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Upsilon_i(u, v) = (P_i u + Q_i v + h_i)^T (u - v),\ i = 1, 2, \cdots, m,$$

where for each $i = 1, 2, \cdots, m$, h_i is a vector in $\mathbb{R}^m$, P_i and Q_i are $m \times m$ matrices such that Q_i is positive symmetric semi-definite and $Q_i - P_i$ is negative semi-definite. In this case, the bifunction Υ_i satisfy $(B1) - (B3)$. The vector $h_{i's}$ are generated randomly and uniformly with the entries being in $[-2, 2]$

and the matrices $P_{i's}$ and $Q_{i's}$ are generated randomly such that the properties are satisfied. Let all the parameter remain as stated in Example 4.1.

5. Conclusion

We introduced a self-adaptive parallel extragradient method together with a golden ratio technique for solving pseudomonotone equilibrium problem in the settings of a Hadamard space. Under some suitable conditions, we established a $\Delta-$ convergence of the generated sequence to a solution of the equilibrium problem. The proposed method is designed in such a way that it is independent of the Lipschitz constant of the bifunction. Lastly, we presented some numerical examples to demonstrate the performance of our method.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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