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HARDY SPACES AND TOEPLITZ OPERATORS IN THE α-CONFORMABLE FRACTIONAL DERIVATIVE SETTING

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Abstract. In this paper, we introduce and develop the theory of α-fractional Hardy spaces by using the framework of α-conformable fractional calculus. Building on the recent definitions of α-fractional analytic functions, we construct new Hardy-type spaces over the quarter unit disk namely, $D^{+}(x,y) = \{(x,y) : x^2 + y^2 < 1 \text{ and } x, y \geq 0\}$ and explore their fundamental properties. Additionally, we define and analyze α-fractional Toeplitz operators and prove several structural related theorems.

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1. Introduction

The Hardy spaces H^p, for $1 < p < \infty$, form one of the most important classes of function spaces in complex analysis, harmonic analysis and operator theory. They consist of analytic functions defined either on the unit disk $D \subset \mathbb{C}$ or the upper half-plane and are governed in norm by their boundary values. These spaces are closely related with the development of Fourier series, boundary value problems, and the theory of singular integrals. Moreover, Hardy spaces give an appropriate framework for investigating classical operators like the Toeplitz operators which are central in operator theory, systems theory, and mathematical physics.

In contrast to L^p spaces, Hardy spaces maintain analytic structure and are stable under certain integral operators such as Hilbert transform. Therefore, they are useful in applications like time frequency analysis and causality [1, 2]. From a theoretical viewpoint, Hardy spaces emerge in the examination of bounded mean oscillation (BMO) and duality theory, serving a crucial function in contemporary harmonic analysis [2].

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Definition 1.1. [1] (*Classical Hardy Space*) *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$. *The classical Hardy space* $H^p(D)$, *for* $1 < p < \infty$, *is defined as*

$$H^p(D) = \left\{ f \text{ is analytic in } D : \|f\|_{H^p} := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right)^{\frac{1}{p}} < \infty \right\}. \tag{1}$$

It should be noted that, the norm $\|f\|_{H^p}$ is derived from the boundary behavior of f on the circle ∂D, and it ensures that f has non-tangential limits almost everywhere on the boundary. The case when $p = 2$, the space H^2 has a Hilbert space structure and forms the foundation for many parts of operator theory as well as functional analysis [1].

Definition 1.2. [1] (*Toeplitz Operator*) *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$ *and* $\phi \in L^\infty(\partial D)$ *be a bounded measurable function. In classical case, the Toeplitz operator* T_ϕ *on* H^2, *is given by*

$$T_\phi f = P(\phi f), \tag{2}$$

where P *is the orthogonal projection from* $L^2(\partial D)$ *onto* H^2.

Toeplitz operators have various algebraic and spectral properties. They are widely used in control theory and signal processing. A classical example of Toeplitz operator, that has a rich structure and many applications, is the unilateral shift operator T_ϕ on H^2, where $\phi = z$. It provides a bridge between function theory and operator theory [3].

Before we discuss a nice property about T_ϕ, we introduce the following definition.

Definition 1.3. [4] (*The Fredholm Operator*) *Let* $T : X \to Y$ *be a bounded linear operator, where* X *and* Y *are two Banach spaces* (*or Hilbert spaces*). *Then* T *is called Fredholm operator if*

$i.$ $\ker(T) = \{x \in X : Tx = 0\}$ *is finite-dimensional,*

$ii.$ *The range of* T *is closed in* Y *and has finite co-dimensional.*

Definition 1.4. [4] (*The Essential Spectrum of* T_ϕ) *The set of all complex numbers* $\lambda \in \mathbb{C}$, *such that the operator* $T_\phi - \lambda I$ *is not Fredholm, is called the essential spectrum of* T_ϕ *and it is denoted by* $\sigma_{ess}(T_\phi)$.

Theorem 1.1. [5] *The essential spectrum of* T_ϕ *is exactly the essential range of its symbol* ϕ. *Equivalently,*

$$\sigma_{ess}(T_\phi) = ess\ range(\phi),$$

where

$$ess\ range(\phi) = \{\lambda \in \mathbb{C} : \text{ for all } \varepsilon > 0,\ \mu(\{z \in C : |\phi(z) - \lambda| < \varepsilon\}) > 0\},$$

such that $C = \{z \in \mathbb{C} : |z| = 1\}$ *is the unit circle.*

Recently, the field of fractional calculus has gained significant traction due to its ability to model phenomena involving memory, hereditary properties, and intermediate regularity. This has led to the development of fractional analogues of classical function spaces and operators.

In this paper, we define and study a new class of α-fractional Hardy spaces H^p_α, where $\alpha \in (0, 1]$ denotes the order of fractional analyticity. These spaces are built upon the recently introduced concept of fractional analytic functions using conformable fractional derivatives, as proposed by Adm and Khalil in [6]. Based on this framework, we also introduce a new definition of Toeplitz operators in the fractional setting.

2. Preliminaries

In this section, we provide the necessary definitions and background on fractional analytic functions, which form the foundation for our construction of fractional Hardy spaces and fractional Toeplitz operators.

Definition 2.1. [7] *Suppose $u : (0, \infty) \to \mathbb{R}$. Then for $x \in (0, \infty)$ the α-conformable fractional derivative of u at x denoted by $D^\alpha u(x)$ is given as*

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}, \qquad \text{where } \alpha \in (0, 1). \tag{3}$$

By the above definition, we have the following differentiation rules:

$$\begin{aligned} &(i)\ D^\alpha(c_1 u + c_2 v) = c_1 D^\alpha(u) + c_2 D^\alpha(v), \text{ for all } c_1, c_2 \in \mathbb{R}, \\ &(ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(x) = k, \\ &(iii)\ D^\alpha(uv) = uD^\alpha(v) + vD^\alpha(u), \\ &(iv)\ D^\alpha(\tfrac{u}{v}) = \tfrac{vD^\alpha(u) - uD^\alpha(v)}{v^2}, v(x) \neq 0. \end{aligned} \tag{4}$$

Also, it can be shown that

$$\begin{aligned} &(i)\ D^\alpha(x^p) = px^{p-\alpha}, \\ &(ii)\ D^\alpha(\sin(\tfrac{1}{\alpha}x^\alpha)) = \cos(\tfrac{1}{\alpha}x^\alpha), \\ &(iii)\ D^\alpha(\cos(\tfrac{1}{\alpha}x^\alpha)) = -\sin(\tfrac{1}{\alpha}x^\alpha), \\ &(iv)\ D^\alpha(e^{\frac{1}{\alpha}x^\alpha}) = e^{\frac{1}{\alpha}x^\alpha}. \end{aligned} \tag{5}$$

It should be noted that, in (5) if $\alpha = 1$, then the corresponding classical rules for ordinary derivatives are obtained.

Various differential equations, that include α-conformable fractional derivatives, have been, precisely, modeled many complex real world problems [8], [9], [10], [11].

Now, the standard notation for any complex valued function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ in complex variable z, where $z = x + iy$ for some $x; y \geq 0$, is

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part of f.

Definition 2.2. [6] (α-*Fractional Analytic*) *Let* $f : D \subseteq \{(x,y) : x,y > 0\} \to \mathbb{C}$. *Then* f *is called* α-*conformable fractional differentiable at* $z_\circ = x_\circ + iy_\circ \in D$ *and denoted by* $f^\alpha(z_\circ)$, *if*

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{f(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ + \epsilon_2 y_\circ^{1-\alpha}) - f(x,y)}{\epsilon_1 + i\epsilon_2}, \quad \text{where } \alpha \in (0,1), \tag{6}$$

exists. Moreover, if there exists $\delta > 0$ *such that* f *is* α-*differentiable for all* $z \in B(\delta, z_\circ)$ *where* $B(\delta, z_\circ)$ *is an open disc centered at* $z_\circ$, *then* f *is said to be* α-*fractional analytic at* $z_\circ$.

Theorem 2.1. [6] (*Sufficient Conditions for Complex Differentiability*) *If* $f = u + iv$ *is analytic in* $D \subseteq \{(x,y) : x,y > 0\} \subseteq \mathbb{C}$, *and* u_x^α, u_y^α,v_x^α, *and* v_y^α *exist, continuous and satisfy the* α-*Cauchy Riemann equations* (7), *then* f *is* α-*fractional analytic in* D.

Theorem 2.2. [6] (α-*Cauchy Riemann Equations*) *If* $f = u + iv$ *is analytic in* $D \subseteq \{(x,y) : x,y > 0\} \subseteq \mathbb{C}$ *then,* u *and* v *satisfy the* α-*Cauchy Riemann equations*

$$u_x^\alpha = v_y^\alpha \text{ and } u_y^\alpha = -v_x^\alpha \text{ in } D. \tag{7}$$

Theorem 2.3. [6] (α-*Conformable Fractional Cauchy Integral Formula*) *Let* f *be* α-*fractional analytic everywhere inside and on a simple closed* α-*contour* C *taken in the positive sense such that*

$$C := \{z^\alpha = x^\alpha + iy^\alpha : |(x^\alpha + iy^\alpha) - (\xi_\circ^\alpha + i\xi_\circ^\alpha)| = r\}.$$

Then for all $z_\circ \in C^\circ$, *the interior of* C, *we have*

$$f(z_\circ^\alpha) = \frac{\alpha}{2\pi i} \oint_C \frac{f(z^\alpha)}{z^\alpha - z_\circ^\alpha} dz^\alpha. \tag{8}$$

3. Main Results

Let

$$D^+(x,y) = \left\{(x,y) : x^2 + y^2 < 1 \text{ and } x,y \geq 0\right\}, \tag{9}$$

and

$$T^+ = T_1 \cup T_2 \cup T_3, \tag{10}$$

where

$$T_1 = \left\{(x,y) : (x^\alpha)^2 + (y^\alpha)^2 = 1 \text{ and } x,y \geq 0\right\},$$
$$T_2 = \{x : 0 \leq x^\alpha \leq 1\},$$
and
$$T_3 = \{y : 0 \leq y^\alpha \leq 1\}. \tag{11}$$

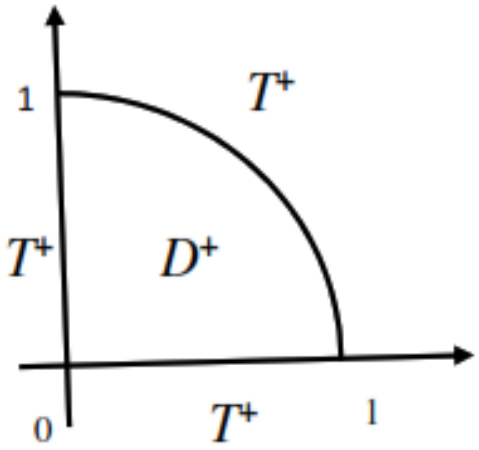

FIGURE 1. Quarter unit disk.

Consequently, we present the following main results.

3.1. α-Fractional Hardy Spaces.

Definition 3.1. (α-*fractional Hardy spaces* H_α^2) *Let* $f : D^+(0,1) \to \mathbb{C}$ *be an* α*-fractional analytic. Then we say that* f *is in the* α*-fractional Hardy space, that is denoted by* $H_\alpha^2(D^+(0,1))$*, if*

$$\begin{aligned}
&(a) \quad \lim_{r\to 1} f\left(re^{i\theta}\right) = f^*\left(e^{i\theta}\right) \text{ exists,}\\
&(b) \quad \lim_{\theta\to 0} f\left(re^{i\theta}\right) = f^*(r) \text{ exists on } [0,1] \text{ on the } x\text{-axis}\\
&(c) \quad \lim_{\theta\to \frac{\pi}{2}} f\left(re^{i\theta}\right) = f^*(i\theta) \text{ exists on } [0,1] \text{ on the } y\text{-axis}\\
&\quad \text{and}\\
&(d) \quad \int_\gamma |f^*(\omega)|^2 |d\omega| < \infty,
\end{aligned} \tag{12}$$

where $\gamma = T^+ = T_1 \cup T_2 \cup T_3$.

It is crucial to highlight that

$$\begin{aligned}
&\text{on } T_1\text{: } \omega = e^{i\theta}, |d\omega| = d\theta,\\
&\text{on } T_2\text{: } \omega = (x,0), |d\omega| = dx,\\
&\qquad \text{and}\\
&\text{on } T_3\text{: } \omega = (0,y), |d\omega| = dy.
\end{aligned} \tag{13}$$

Similar definitions can be given for H_α^1 and H_α^∞ on $D^+(0,1)$.

Theorem 3.1. *Let* $\phi \in H_\alpha^\infty$ *and* $f \in H_\alpha^2$*. Then* $\phi f \in H_\alpha^2$.

Proof. Let $f = u_1 + iv_1$ and $\phi = u_2 + iv_2$. Then

$$\begin{aligned}\phi f &= u_1u_2 - v_1v_2 + i\left(v_1u_2 + u_1v_2\right)\\ &= P + iQ,\end{aligned} \tag{14}$$

where $P = u_1u_2 - v_1v_2$ and $Q = v_1u_2 + u_1v_2$.

Now,

$$\begin{aligned}P_x^\alpha &= u_{1x}^\alpha u_2 + u_1u_{2x}^\alpha - v_{1x}^\alpha v_2 - v_1v_{2x}^\alpha,\\ Q_x^\alpha &= u_{1x}^\alpha v_2 + u_1v_{2x}^\alpha + v_{1x}^\alpha u_2 + v_1u_{2x}^\alpha,\\ P_y^\alpha &= u_{1y}^\alpha u_2 + u_1u_{2y}^\alpha - v_{1y}^\alpha v_2 - v_1v_{2y}^\alpha,\end{aligned}$$

and

$$Q_y^\alpha = u_{1y}^\alpha v_2 + u_1v_{2y}^\alpha + v_{1y}^\alpha u_2 + v_1u_{2y}^\alpha. \tag{15}$$

But, since

$$\begin{aligned}u_{1x}^\alpha &= v_{1y}^\alpha, \quad u_{1y}^\alpha = -v_{1x}^\alpha,\\ u_{2x}^\alpha &= v_{2y}^\alpha, \quad u_{2y}^\alpha = -v_{2x}^\alpha,\end{aligned} \tag{16}$$

then one can easily deduce that

$$P_x^\alpha = Q_y^\alpha \text{ and } P_y^\alpha = -Q_x^\alpha. \tag{17}$$

But because u_{1x}^α, u_{2x}^α, v_{1x}^α, v_{2x}^α, u_{1y}^α, u_{2y}^α, v_{1y}^α, and v_{2y}^α are continuous as both ϕ and f are α-fractional analytic, then by Theorem 2.2, it follows that ϕf is analytic. Moreover, since $\phi \in H_\alpha^\infty$, then this implies that $\phi f \in H_\alpha^2$. Thus, the proof is complete.

Theorem 3.2. *Let $H_\alpha^2\left(D^+\left(0,1\right)\right)$ be α-fractional Hardy space. Then $H_\alpha^2\left(D^+\left(0,1\right)\right)$ is complete with the $\|\cdot\|_2$-norm on T^+.*

Proof. Let $f \in H_\alpha^2\left(D^+\left(0,1\right)\right)$. Then f is α-fractional analytic over $D^+\left(0,1\right)$. Thus, by Theorem 2.3, we have

$$f(z^\alpha) = \frac{\alpha}{2\pi i}\oint_\gamma \frac{f(w^\alpha)}{w^\alpha - z^\alpha}dw^\alpha, \tag{18}$$

where $z^\alpha = x^\alpha + iy^\alpha \in \gamma = T^+ = T_1 \cup T_2 \cup T_3$, $w^\alpha = a^\alpha + ib^\alpha$, and $dw^\alpha = a^{\alpha-1}da + ib^{\alpha-1}db$.

Now, by using the same technique as the usual Cauchy integral formula, we get

$$f(z^\alpha) = \sum_{n=0}^{\infty} a_n\left(z^\alpha\right)^n. \tag{19}$$

This is because the curve $\gamma = T^+ = T_1 \cup T_2 \cup T_3$ is $\left(x^\alpha\right)^2 + \left(y^\alpha\right)^2 = 1$ where $x, y \geq 0$ such that $0 \leq x^\alpha \leq 1$ and $0 \leq y^\alpha \leq 1$. Hence, the map $\psi : [0,\infty) \to [0,\infty)$ where $\psi\left(x\right) = x^\alpha$ is injective and surjective. Similarly, one can deduce a function say, π, on the non-negative y-axis to be injective and surjective.

Now, the quarter xy-plane, $x, y \geq 0$, is the quarter $x^\alpha y^\alpha$-plane, $x^\alpha, y^\alpha \geq 0$. Hence, in the $x^\alpha y^\alpha$-plane, $x^\alpha, y^\alpha \geq 0$, the equation $(x^\alpha)^2 + (y^\alpha)^2 = 1$ represents say, the α-arc of the circle in such plane. Thus,

$$z^\alpha = re^{i\theta} = (x^\alpha, y^\alpha),$$
$$(x^\alpha)^2 + (y^\alpha)^2 = r^2, \text{ and } \theta = \tan^{-1}\frac{y^\alpha}{x^\alpha}. \tag{20}$$

Consequently,

$$f(z^\alpha) = \sum_{n=0}^{\infty} a_n (z^\alpha)^n = \sum_{n=0}^{\infty} a_n \left(r^n e^{in\theta}\right). \tag{21}$$

Therefore, on $T_1 = \left\{(x^\alpha, y^\alpha) : (x^\alpha)^2 + (y^\alpha)^2 = 1 \text{ and } x^\alpha, y^\alpha \geq 0\right\}$, we have $f(z^\alpha) = \sum\limits_{n=0}^{\infty} a_n \left(r^n e^{in\theta}\right)$ with $\|f\|_2 = \langle f, f\rangle = \sum\limits_{n=0}^{\infty} a_n^2 < \infty$. So, H_α^2 is closed in $L^2(T_1)$.

Further, on $T_2 = \{z^\alpha = x^\alpha : 0 \leq x^\alpha \leq 1\}$, we have

$$f(z^\alpha) = \sum_{n=0}^{\infty} a_n (x^\alpha)^n \text{ where } \theta = 0. \tag{22}$$

So, $f(z^\alpha) = \sum\limits_{n=0}^{\infty} a_n r^n$, $0 < r < 1$with $\|f\|_2 = \langle f, f\rangle = \sum\limits_{n=0}^{\infty} a_n^2 r^{2n} < \infty$. Similarly, for $T_3 = \{z^\alpha = iy^\alpha : 0 \leq y^\alpha \leq 1 \text{ and } y \geq 0\}$.

Thus, one can easily see that $H_\alpha^2(T^+)$ is complete.

3.2. α-Fractional Toeplitz Operators.

Definition 3.2. (α-*fractional Toeplitz Operator*) *Let $\phi \in L^\infty(T^+)$. Then the α-fractional Toeplitz operator, $T_{\phi,\alpha}$, on $H_\alpha^2(T^+)$ is given by*

$$T_{\phi,\alpha} : H_\alpha^2\left(T^+\right) \to H_\alpha^2\left(T^+\right) \text{ such that}$$
$$T_{\phi,\alpha}(f) = P(\phi f), \; f \in H_\alpha^2\left(T^+\right),$$

where P is the orthogonal projection from $L^2(T^+)$ onto $H_\alpha^2(T^+)$.

Now, before continuing with $T_{\phi,\alpha}$, let us present the following nice result on $H_\alpha^2(T^+)$.

Theorem 3.3. *Let $\phi \in L^\infty(T^+)$. Then $\phi H_\alpha^2(T^+) \subseteq H_\alpha^2(T^+)$ if and only if $\phi \in H_\alpha^\infty(T^+)$.*

Proof. Let $\phi H_\alpha^2(T^+) \subseteq H_\alpha^2(T^+)$. Then $\phi \cdot 1 \in H_\alpha^2(T^+)$, and thus, $\phi \in H_\alpha^2(T^+)$. Moreover, since by definition $\phi \in L^\infty(T^+)$, ϕ is bounded. So, ϕ is a bounded function in $H_\alpha^2(T^+)$. This implies $\phi \in H_\alpha^\infty(T^+)$.

In contrast, let $\phi \in H_\alpha^\infty(T^+)$. Then for any $0 < r < 1$, we have

$$\oint_{\gamma_r} \left|\phi\left(re^{i\theta}\right) f\left(re^{i\theta}\right)\right|^2 d\theta$$
$$\leq \|\phi\|_\infty^2 \|f\|_2^2,$$

where $\gamma = \left\{(x,y) : x^2 + y^2 = r^2\right\} \cup \{x : 0 \leq x \leq r\} \cup \{y : 0 \leq y \leq r\}$.

So, $\phi f \in H^2_\alpha(T^+)$. Thus, $\phi H^2_\alpha(T^+) \subseteq H^2_\alpha(T^+)$. Noting that ϕf is α-fractional analytic if both ϕ and f are α-fractional analytic.

Theorem 3.4. *If $\phi \in L^\infty(T^+)$ and $\varphi \in H^\infty_\alpha(T^+)$, then $T_{\phi,\alpha}T_{\varphi,\alpha} = T_{\phi\varphi,\alpha}$.*

Proof. Let $f \in H^2_\alpha(T^+)$. Then by the previous theorem $\varphi f \in H^2_\alpha(T^+)$. So,

$$\begin{aligned} T_{\phi,\alpha}T_{\varphi,\alpha}(f) &= T_{\phi,\alpha}(P(\varphi f)) \\ &= T_{\phi,\alpha}(\varphi f) \\ &= P(\phi\varphi f) \\ &= T_{\phi\varphi,\alpha}(f). \end{aligned}$$

This completes the proof.

Theorem 3.5. *If $\phi \in L^\infty(T^+)$ such that $T^{-1}_{\phi,\alpha}$ exists, then $\phi^{-1} \in L^\infty(T^+)$, where $T_{\phi,\alpha} : H^2_\alpha(T^+) \to H^2_\alpha(T^+)$.*

Proof. Suppose $T^{-1}_{\phi,\alpha}$ exists. Equivalently, there exists $\varepsilon > 0$ such that

$$\|T_{\varphi,\alpha}(f)\|_2 \geq \varepsilon \|f\|_2 \quad \text{for all } f \in H^2_\alpha\left(T^+\right).$$

Now,

$$\|\phi \cdot f\|_2 \geq \|T_{\varphi,\alpha}(f)\|_2 = \|P(\phi f)\|_2 \geq \varepsilon \|f\|_2.$$

Hence, $M_\phi(f) = \phi \cdot f$ is invertible, where M_ϕ is the multiplication operator on $L^2(T^+)$. Thus, $\phi^{-1} \in L^\infty(T^+)$.

A consequence of the previous theorem is the following corollary.

Corollary 1. *The essential spectrum of $T_{\phi,\alpha}$ is exactly the essential range of its symbol ϕ. Equivalently,*

$$\sigma_{ess}(T_{\phi,\alpha}) = ess\ range(\phi).$$

Proof. Noting that $(T_{\phi,\alpha} - \lambda I) f = T_{\phi,\alpha}(f) - \lambda f = P(\phi f) - \lambda f$. In addition, $(T_{\phi,\alpha} - \lambda I)^{-1}$ exists and implies that $M_{(\phi-\lambda)}$ is invertible. But,

$$\left(M_{(\phi-\lambda)}\right)^{-1} \text{ exists if and only if } (\phi - \lambda)^{-1} \text{ exists in } L^\infty\left(T^+\right).$$

So, $(\phi - \lambda)^{-1}$ does not exist if $\phi(t) - \lambda = 0$ which is in the essential range of $\phi - \lambda$. Thus, $\sigma_{ess}(T_{\phi,\alpha}) = ess\ range(\phi)$ as required.

4. Conclusions

In this work, we introduced a new class of function spaces called α-fractional Hardy spaces based on the α-conformable fractional derivative. These spaces generalize the classical Hardy spaces and preserve key properties such as completeness and boundary behavior. We also extended the concept of Toeplitz operators to this fractional context and showed that many classical results such as the multiplicative algebra property and the essential spectrum formula still hold.

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