STUDY OF UNIQUENESS FOR ALGEBROID FUNCTIONS OF FINITE ORDER WITH PSEUDO DEFICIENCY ON ANNULI

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Abstract. In this article, we study the uniqueness of algebroid functions defined on Annuli while the order, lower order, Deficiency, Reduced Deficiency and the Pseudo-deficiency are taken into consideration. Our results are extension to annuli region which extends the results of Pingyuan Zhang and Peichu Hu [11].

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1. Introduction

In the field of value distribution theory, the uniqueness theory of algebroid functions is a fascinating topic. In 1929, Valiron [1] was the first to study the uniqueness problem of algebroid functions, later Ullarich [2] extended his support to the value distribution of algebroid functions and later on, various researchers discovered several uniqueness theorems for algebroid functions in the complex plane C [3–5, 7–10, 12, 13, 16].

The Nevanlinna theory for meromorphic functions in multiple connected domains was proposed by Khrystiyanyn and Kondratyuk in 2005 [18, 19]. Cao and Yi [20] examined the uniqueness of meromorphic functions that share some values and some sets on annuli in 2009. As a result, it's worth considering the uniqueness problem of algebroid functions in multiply connected domains.

In this paper, we focus on the doubly connected domain. We presume that readers are familiar with the Nevanlinna theory of meromorphic and algebroid functions [4–6, 9, 16, 17]. Each doubly connected domain is conformally equal to the annulus $\mathbb{A}(R_1, R_2) = \{z : R_1 < |z| < R_2\}$, $0 \leq R_1 < R_2 \leq +\infty$,

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according to the doubly connected mapping theorem [28]. We just look at two scenarios:

$$(1) \quad R_1 = 0, R_2 = +\infty$$

$$(2) \quad 0 < R_1 < R_2 < +\infty.$$

In the latter situation, the homothety $z \to \frac{z}{\sqrt{R_1 R_2}}$ reduces the specified domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R_2}{R_1}}$. As a result, in two circumstances, every annulus is invariant with regard to the inversion $z \to \frac{1}{z}$.

For our convenience we let $M = \{M_i : M_i$ be ν-valued algebroid function defined on the annulus $A\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 < +\infty)$ for $i = 1, 2\}$ and a $\nu-$valued algebroid function defined by an irreducible equation given by

$$A_\nu(z)M_1^\nu + A_{\nu-1}(z)M_1^{\nu-1} + \ldots + A_1(z)M_1 + A_0(z) = 0, \tag{1}$$

with $A_\nu(z), A_{\nu-1}(z), \ldots, A_1(z), A_0(z)$ as a group of analytic functions defined on the annulus $A\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 < +\infty)$ with no common zeros.

In this article, we use the standard notations of algebroid function theory [6].

Throughout the paper, by denoting $M_1 = a \Leftrightarrow M_2 = a$, we mean to say that $M_1 - a$ and $M_2 - a$ have same zeros (IM), by $W = a \rightleftharpoons M = a$, we mean to say that $M_1 - a$ and $M_2 - a$ have same zeros (CM), by $\overline{E}_{k)}(a, M_1)$ the set of zeros of $M_1 - a$ with multiplicity $\leq k$ (IM). If $M_1 \in M$ then

$$\sigma(M_1) = \lim_{r\to+\infty} \sup \frac{\log T_0(r, M_1)}{\log r},$$

$$\mu(M_1) = \lim_{r\to+\infty} \inf \frac{\log T_0(r, M_1)}{\log r},$$

$$\delta_0(a, M_1) = 1 - \overline{\lim_{r\to R_0^-}} \frac{N_0\left(r, \frac{1}{M_1-a}\right)}{T_0(r, M_1)},$$

$$\Theta(a, M_1) = 1 - \overline{\lim_{r\to R_0^-}} \frac{\overline{N}_0\left(r, \frac{1}{M_1-a}\right)}{T_0(r, M_1)},$$

$$\delta_{k)}(a, M_1) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_{k)}\left(r, \frac{1}{M_1-a}\right)}{T_0(r, M_1)},$$

represents the order, lower order, deficiency, reduced deficiency and the Pseudo-deficiency of $M_1(z)$ respectively.

It was 1980 [15], when Ueda considered two entire functions defined on $\mathbb{C}$ with finite order and proved the uniqueness result with deficient values where as in 1985 Y.Z.He [8], obtained the uniqueness result of algebroid functions defined on $\mathbb{C}$ with their deficient values. Recently, in 2015, P. Zhang et al. [11] considered two algebroid functions defined on $\mathbb{C}$ with deficient values to showed that instead of considering the deficient values of two algebroid functions, they can be considered in one function and obtained two uniqueness results, as a glimp, in the first theorem algebroid functions with finite order has been considered whereas in the second theorem algebroid functions with finite lower order has been considered along with the deficiencies in both the theorems and they both have been proved for uniqueness.

With regard to the multiple values in Annuli, in 2000, Zongsheng Gao [21] proved the existence of the sequence of filling disks and Borel directions dealing with its multiple values, where as in 2017, Ashok Rathod [22] investigated about the multiple values and deficiencies and proved some uniqueness theorems. Further concerning deficiency in Annuli, in 1993, Lianzhong Yang [23] studied on the deficiencies of an algebroid function with finite lower order and provided some results. While in 2019, Ashok Rathod [24] studied and obtained the relationship between the deficiency of an algebroid function on annuli and of their derivatives.

Further, in 2015, Yang Tan et al [25] proposed an extension of Nevanlinna value distribution theory for algebroid functions on annuli. He obtained Analogs of the Cartan theorem, the first fundamental theorem, the second fundamental theorem, deficient values, and the uniqueness of algebroid functions on annuli. Again In 2016, Yang Tan [26] discussed about the uniqueness problem of algebroid functions on annuli, and gave several uniqueness theorems of algebroid functions on annuli, which extended the Nevanlinna value distribution theory for algebroid functions on annuli. In 2017, Ashok Rathod [22], obtained Xiong inequality of algebroid function on annuli and using this result they proved uniqueness theorem of algebroid functions on annuli concerning to their multiple values and derivative.

Motivated by all these studies, it was natural to ask about the nature or the behavior of algebroid functions defined on annuli when the order, lower order, deficiency, reduced deficiency and Pseudo-deficiency are taken into account. As an affirmative answer, we have obtained three results which are stated and proved in the section 3.

2. Lemmas

We highlight some of the Lemmas required as follows. We consdier , $M_1(z)$, $M_2(z)$ as two ν-valued algebroid functions.

(1) If the set of $M_1(0)$ and $M_2(0)$ have no poles, then we have

$$T_0(r, M_1/M_2) \leq T_0(r, M_1) + T_0(r, M_2) + O(1), \tag{2}$$

[11].

(2) If $\sigma(M_1)$ as the order of $M_1(z)$ and $\mu(M_2)$ as the lower order of $M_1(z)$ with $\sigma(M_1) < \mu(M_2) < \infty$, then

$$T_0(r, M_1) = o(T_0(r, M_2)) \quad (r \to \infty), \tag{3}$$

[11].

(3) For Algebroid functions Second Fundamental Theorem for a_j $(j = 1, 2, \ldots, q)$ we have

$$(q-2\nu)T_0(r, M_1) < \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{M_1 - a_j}\right) + S(r, M_1), \tag{4}$$

[14].

3. Main Results

Theorem: Let M_1, $M_2 \in M$ with finite lower order $\mu(M_1)$, then write $\frac{M_1}{M_2} = G$ and let $\mu(M_1) \neq \mu(G) = \sigma(G)$. Presume that $M_1 = 0 \Leftrightarrow M_2 = 0$. If there exists a_j $(j = 1, 2, \ldots, 2\nu + 1)$ which are distinct and non-zero such that

$$\overline{E}_{1)}(a_j, M_1) = \overline{E}_{1)}(a_j, M_2), \tag{5}$$

and

$$\sum_{j=1}^{2\nu+1} max\{\Theta(0, M_1), \Theta(a, M_1), \delta(a_j, M_1)\} > 1, \tag{6}$$

then $M_1 \equiv M_2$.

Proof. For the value of $j = 1, 2, \ldots, 2\nu + 1$, following we have

$$\overline{N}_0\left(r, \frac{1}{M_1 - a_j}\right) \leq \frac{1}{2}\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + \frac{1}{2}N_0\left(r, \frac{1}{M_1 - a_j}\right). \tag{7}$$

Thus,

$$\overline{N}_0\left(r, \frac{1}{M_1 - a_j}\right) \leq \frac{1}{2}\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + \frac{1}{2}T_0(r, M_1) + O(1).$$

Now, by 4, we have

$$\begin{aligned}(2\nu + 1 - 2\nu)T_0(r, M_1) &\leq \sum_{j=1}^{2\nu+1} \overline{N}_0\left(r, \frac{1}{M_1 - a_j}\right) + S(r, M_1),\\
&\leq \overline{N}_0\left(r, \frac{1}{M_1}\right) + \sum_{j=1}^{2\nu+1} \frac{1}{2}\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + \sum_{j=1}^{2\nu+1} \frac{1}{2}N_0\left(r, \frac{1}{M_1 - a_j}\right) + S(r, M_1),\\
&\leq \overline{N}_0\left(r, \frac{1}{M_1}\right) + \sum_{j=1}^{2\nu+1} \frac{1}{2}\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + \left(\frac{2\nu+1}{2}\right)T_0\left(r, M_1\right) + S(r, M_1).\end{aligned} \tag{8}$$

By 8 and assumptions, we get

$$\begin{aligned}T_0(r, M_1) &\leq \left(\frac{2}{1-2\nu}\right)\overline{N}_0\left(r, \frac{1}{M_1}\right) + \left(\frac{2}{1-2\nu}\right)\sum_{j=1}^{2\nu+1} \frac{1}{2}\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + S(r, M_1),\\
T_0(r, M_1) &\leq \left(\frac{3+2\nu}{1-2\nu}\right)T_0\left(r, M_2\right) + S(r, M_1).\end{aligned} \tag{9}$$

Similarly, we can get

$$T_0(r, M_2) \leq \left(\frac{3+2\nu}{1-2\nu}\right)T_0\left(r, M_1\right) + S(r, M_2). \tag{10}$$

This yields

$$\mu(M_2) \leq \left(\frac{3+2\nu}{1-2\nu}\right)\mu(M_1) + S(r, M_2), \tag{11}$$

and

$$\mu(M_1) \leq \left(\frac{3+2\nu}{1-2\nu}\right)\mu(M_2) + S(r, M_1). \tag{12}$$

This implies,

$$\mu(M_1) = \mu(M_2). \tag{13}$$

By 10, 2 and assumptions, we get

$$\begin{aligned} T_0(r,G) &\leq T_0(r,M_1) + T_0(r,M_2) + O(1), \\ T_0(r,G) &\leq T_0(r,M_1) + \left(\frac{3+2\nu}{1-2\nu}\right)T_0\left(r,M_1\right) + S(r,M_1), \\ T_0(r,G) &\leq \left(\frac{4}{1-2\nu}\right)T_0\left(r,M_1\right) + S(r,M_1). \end{aligned} \tag{14}$$

Again, on calculation, we see that $\mu(G) \leq \mu(M_1)$. However $\sigma(G) = \mu(G) \neq \mu(M_1)$. Thus we get

$$\sigma(G) < \mu(M_1). \tag{15}$$

On the contrary, if suppose, $M_1 \not\equiv M_2$. Letting $\{z_n\}$ be all simple zeros of $M_1 - a_1$. By hypothesis $\overline{E}_{1)}(a_1, M_1) = \overline{E}_{1)}(a_1, M_2)$ which infers that $\{z_n\}$ are simple zeros of $M_2 - a_1$. Also by hypothesis noting that $\frac{M_1}{M_2} = G$, which infers that $G \not\equiv 1$. But $G(z_n)$ is 1, therefore we obtain

$$\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_1}\right) \leq N\left(r, \frac{1}{G-1}\right) \leq T_0(r,G) + O(1). \tag{16}$$

By 15 and 16

$$\lim_{r\to+\infty}\sup\frac{\log \overline{N}_0^{1)}\left(r, \frac{1}{M_1-a_1}\right)}{\log r} \leq \sigma(G) < \mu(M_1).$$

By 7 and 4, we obtain

$$\begin{aligned} (2\nu+1-2\nu)T_0(r,M_1) &\leq \overline{N}_0\left(r,\frac{1}{M_1}\right) + \sum_{j=1}^{2\nu+1}\overline{N}_0\left(r,\frac{1}{M_1-a_j}\right) + S_0(r,M_1), \\ T_0(r,M_1) &\leq \overline{N}_0\left(r,\frac{1}{M_1}\right) + \frac{1}{2}\sum_{j=1}^{2\nu+1}\overline{N}_0^{1)}\left(r,\frac{1}{M_1-a_j}\right) + \frac{1}{2}\sum_{j=1}^{2\nu+1}N_0\left(r,\frac{1}{M_1-a_j}\right). \end{aligned} \tag{17}$$

which gives

$$\Theta(0,M_1) + \frac{1}{2}\Theta(a,M_1) + \frac{1}{2}\sum_{j=0}^{2\nu+1}\delta(a_j,M_1) \leq 1.$$

This is contradiction to 6. This infers that $M_1 \equiv M_2$. □

Theorem 3.1. *Let $M_1, M_2 \in M$ with finite lower order $\mu(M_1)$, then write $\frac{M_1}{M_2} = G$ and let $\sigma(M_1) \neq \sigma(G)$. Presume that $M_1 = 0 \Leftrightarrow M_2 = 0$. If there exists a_j $(j = 1, 2, \ldots, 2\nu + 1)$ which are distinct and non-zero such that*

$$\overline{E}_{1)}(a_j, M_1) = \overline{E}_{1)}(a_j, M_2),$$

and

$$\sum_{j=1}^{2\nu+1} max\{\Theta(0, M_1), \delta(a_j, M_1)\} > \frac{1}{2},$$

then $M_1 \equiv M_2$.

Proof. If suppose $M_1 \not\equiv M_2$. As proceeded in the above theorem, we can obtain the following

$$T_0(r, G) \leq \left(\frac{4}{1 - 2\nu}\right) T_0\left(r, M_1\right) + S(r, M_1),$$

and since $\sigma(G) \neq \sigma(M_1)$

$$\sigma(G) < \sigma(M_1), \tag{18}$$

similarly, from 16 we have

$$\overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_1}\right) \leq N\left(r, \frac{1}{G - 1}\right) \leq T_0(r, G) + O(1). \tag{19}$$

By 17 and 19

$$\begin{aligned} T_0(r, W) &\leq \overline{N}_0\left(r, \frac{1}{M_1}\right) + \frac{1}{2}\sum_{j=1}^{2\nu+1} T_0(r, G) + \frac{1}{2}\sum_{j=1}^{2\nu+1} N_0\left(r, \frac{1}{M_1 - a_j}\right) + S(r, M_1), \\ &\leq (1 - \Theta(0, M_1))T_0(r, M_1) + \frac{1}{2}\sum_{j=1}^{2\nu+1} T_0(r, G) + \frac{1}{2}\sum_{j=1}^{2\nu+1}(1 - \delta(a_j, w))T_0(r, M_1) + S(r, M_1), \end{aligned} \tag{20}$$

$$T_0(r, M_1)\left(\Theta(0, M_1) + \frac{1}{2}\sum_{j=1}^{2\nu+1} \delta(a_j, w) + o(1)\right) < \frac{2\nu + 1}{2} T_0(r, G). \tag{21}$$

3.1 and 21 yield $\sigma(M_1) \leq \sigma(G)$, a contradiction to 18, hence we have $M_1 \equiv M_2$. □

Theorem 3.2. *Let $M_1, M_2 \in M$ with finite lower order $\mu(M_1)$, then write $\frac{M_1}{M_2} = G$ and let $\mu(M_1) \neq \mu(G) = \sigma(G)$. Presume that $M_1 = 0 \Leftrightarrow M_2 = 0$. If there exists a_j $(j = 1, 2, \ldots, 2\nu + 1)$ which are distinct and non-zero such that*

$$\overline{E}_{1)}(a_j, M_1) = \overline{E}_{1)}(a_j, M_2), \tag{22}$$

and

$$\sum_{j=1}^{2\nu+1} max\{\Theta(0, M_1), \delta_{1)}(a_j, M_1), \delta(a_j, M_1)\} > 1, \tag{23}$$

then $M_1 \equiv M_2$.

Proof. Proof of this theorem is similar to the proof of the theorem **??**. By 8 we have

$$T_0(r, M_1) \leq \overline{N}_0\left(r, \frac{1}{M_1}\right) + \frac{1}{2}\sum_{j=1}^{2\nu+1} \overline{N}_0^{1)}\left(r, \frac{1}{M_1 - a_j}\right) + \frac{1}{2}\sum_{j=1}^{2\nu+1} N_0\left(r, \frac{1}{M_1 - a_j}\right). \tag{24}$$

Then we can write by using the definition of Pseudo-deficiency

$$\begin{aligned} T_0(r, M_1) \leq (1 - \Theta(0, M_1))T_0(r, M_1) + \frac{1}{2}\sum_{j=1}^{2\nu+1}(1 - \delta_{1)}(a_j, M_1))T_0(r, M_1) \\ + \frac{1}{2}\sum_{j=1}^{2\nu+1}(1 - \delta(a_j, M_1))T_0(r, M_1), \end{aligned}$$

$$\Theta(0, M_1) + \frac{1}{2}\sum_{j=1}^{2\nu+1}\delta_{1)}(a_j, M_1) + \frac{1}{2}\sum_{j=1}^{2\nu+1}\delta(a_j, M_1) \leq 1.$$

This contradicts (23). Hence, $M_1 \equiv M_2$ □

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