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THE (m, k) - UNITARY INTERSECTION NUMBER: COMBINATORIAL ANALYSIS AND GENERATING FUNCTIONS

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Abstract. Let $\mathcal{S}$ be a set with $|\mathcal{S}| = n$, and let $2 \leq k \leq m \leq n$. An (m,k)**-unitary collection** $\boldsymbol{\zeta}$ is a collection of m subsets of $\mathcal{S}$ satisfying the following conditions:

(i) For every distinct pair $A, B \in \boldsymbol{\zeta}$, the intersection $|A \cap B| = 1$;

(ii) $\cup_{A \in \boldsymbol{\zeta}} A = S$; and

(iii) Exactly k subsets in $\boldsymbol{\zeta}$ have a common element.

The total number of distinct (m,k)-unitary collections on a set of size n is called the (m,k)-unitary intersection number of n, denoted by $\mu_{(m,k)}(n)$. This study aims to derive a formula for $\mu_{(m,k)}(n)$ in the specific cases, where $k = m$ and $k = m - 1$. Additionally, we establish recurrence relations, explicit formulas, and exponential generating functions for these two cases to better understand the structural properties of such collections.

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1. Introduction

In many fields of mathematics, such as combinatorics and optimization, understanding the intersection properties of sets is fundamental to solving real-world problems [1]. Specifically, in Enumerative Combinatorics which is primarily concerned with counting the elements contained in finite sets [2], often relies on these properties. Finite sets and intersection properties have been the subject of extensive study in mathematics over the past several decades. In 1960, a study introduced the Sunflower Lemma (also known as the Δ-System Lemma), which demonstrated that a large collection of sets must contain subfamilies where all pairwise intersections share the same common element called "core". [3]. Around the same time in 1961, a theorem by [4] established sharp bounds on the size of pairwise intersection among k-element subsets. In the later 1980s, a study by [5] explored how certain intersection sizes

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(e.g., exactly one common element) could be avoided and what limits this imposes on the collection size. This field of concept expanded in the twentieth century to include algorithmic and enumerative techniques contributing to the counting and classification of intersecting and cross-intersecting families, where two disjoint set families intersect across every pair [6], [7], [8] and [9].

This paper investigates the intersection properties of m distinct subsets from n objects such that each pair of subsets has only one common element. This study of intersection collections is significant in numerous mathematical and applied fields. For example, in information theory, unitary intersections help design error-correcting codes with specific redundancy properties to improve data transmission reliability. Similarly, in network theory, understanding these collections helps in constructing efficient and resilient networks with controlled redundancy and minimal conflicts [10], [11], [12], [13]. By establishing fundamental properties, deriving formulas or bounds, and characterizing these collections under various conditions, this research contributes to the broader understanding of combinatorial structures and their applications in real-world problems.

2. Preliminaries

This section presents the definitions and theorems in Combinatorics and Discrete Mathematics that are necessary for this paper. These concepts were taken from [14], [15], [16], [17], [18], and [19].

Theorem 2.1 [14] (Binomial Combination) Given $r, n \in \mathbb{Z}$ with $0 \leq r \leq n$. Let A be a set of n distinct objects. An r-combination of A, denoted by $\binom{n}{r}$ or C_r^n, is the number of r-element subsets of A defined by:

$$\binom{n}{r} = \begin{cases} \frac{n!}{r!(n-r)!} & \text{if } 0 \leq r \leq n \\ 0 & \text{if } r > n \text{ or } r < 0 \end{cases}$$

This will count the number of ways to select r elements from n distinct elements, where the order does not matter.

Theorem 2.2 [14] (Permutation)

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a given set of n distinct objects. For any $r, n \in \mathbb{Z}$ with $0 \leq r \leq n$, an r-permutation, denoted by $P(n, r)$, is the number of ways of arranging any r of the objects of A. This is called a permutation of A, and is defined by:

$$P(n, r) = \frac{n!}{(n-r)!}$$

By convention,$0! = 1$. Note that $P(n, 0) = 1$ and $P(n, n) = n!$.

This will count the number of ways to select r elements from n distinct elements, where the order matters.

Theorem 2.3 [15] (Recurrence Relation on Stirling Numbers)

The stirling number of the second kind $S(n, m)$ is the number of partitions of n elements into m

nonempty subsets. It is positive if $0 \leq m \leq n$, and zero for other values of m. It satisfies the recurrence relation:

$$S(n,m) = m \cdot S(n-1,m) + S(n-1,m-1)$$

This will compute Stirling number of the second kind $S(n,m)$ using the preceding number $S(n-1,m)$.

Theorem 2.4 [16] **(Exponential Generating Function of the Stirling Number)**

The exponential generating function of the Stirling numbers of the second kind is given by:

$$G_m(x) = \sum_{n=m}^{\infty} \frac{S(n,m)x^n}{n!} = \frac{(e^x-1)^m}{m!}$$

This will express and infinitely manipulate combinatorial sequences of Stirling numbers.

Theorem 2.5 [17] **(Explicit Formula on Stirling Number)**

The Stirling numbers of the second kind can also be computed using the explicit formula:

$$S(n,m) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^n$$

This will Count all functions from an n-element ordered set to a m-element as codomain that use all m values by dividing by $m!$.

Theorem 2.6 [18] **(Stirling Expansion of x^n)**

Let $0 \leq k \leq n$, then;

$$x^n = \sum_{k=0}^{\min\{x,n\}} S(n,k)P(x,k)$$

Where $S(n,k)$ is the number of ways to partition an n-element ordered set into non-empty k unordered subsets, and $P(x,k)$ is the number of ordered arrangement in assigning non-empty k unordered subset to x number of partitions.

Theorem 2.7 [19] **(Exponential Function)** The **exponential function** e^x is defined for all real or complex numbers x by the infinite series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This series converges absolutely for all $x \in \mathbb{R}$ (and $\mathbb{C}$), and defines an analytic function on the entire real (or complex) line.

3. Main Results

This section defines a new concept of counting the number of collections of m subsets from n-element set S with the condition that the pairwise intersection of m subsets has only one element and the union of all sets is equal to S. Analogously, this can be done by counting the number of ways of forming subsets from n distinct objects into m distinct boxes, provided that each box has one common element with the other boxes. This number is known as (m,k)- Unitary Intersection Number, where k is the maximum number of boxes that have exactly one common object. Additionally, k may vary depending on m. Consequently, k can be m, $m-1$ up until two only, since we cannot partition n objects into less than two box. In this paper, we focus on (m, m)- Unitary Intersection Number and $(m, m-1)$- Unitary Intersection Number.

Now, we will formally define (m, k)- Unitary Intersection Number in partitioning n distinct elements into m subsets, with k as the maximum number of subsets that have one common element with the others.

Definition 3.1: Let $1 \leq m \leq n$, $S = \{1, 2, \ldots, n\}$, and let $\zeta = \{A_i : A_i \subseteq S \text{ and } 1 \leq i \leq m\}$. The collection ζ is called a **Unitary Intersection** in S if $S = \bigcup_{i=1}^{m} A_i$, and $|A_i \cap A_j| = 1$ for all $1 \leq i < j \leq m$. The **index** of ζ, denoted by $\iota(\zeta)$, is defined by:

$$\iota(\zeta) = \max \left\{ |\zeta'| : \zeta' \subseteq \zeta \text{ and } \left| \bigcap_{A \in \zeta'} A \right| = 1 \right\}.$$

Let $2 \leq k \leq m \leq n$. The (m, k)**-unitary intersection number** in $S = \{1, 2, \ldots, n\}$, denoted by $\mu_{(m,k)}(S)$ or $\mu_{(m,k)}(n)$, is defined by:

$$\mu_{(m,k)}(n) = |\{\zeta : \zeta \text{ is a unitary collection in } S, |\zeta| = m, \text{ and } \iota(\zeta) = k\}|.$$

Remark 1: Suppose ζ is a unitary intersection collection in S. Then

$$2 \leq \iota(\zeta) \leq |\zeta|.$$

3.1. (m, m)-Unitary Intersection Collection.

This section establishes a formula for the (m, m)-Unitary intersection Number. Moreover, a recurrence relation, exponential generating function, and an explicit formula for this number are obtained.

Theorem 3.1.1: Let $2 \leq m \leq n$. Then,

$$\mu_{(m,m)}(n) = n \left[S(n-1, m) + S(n-1, m-1) \right],$$

where $S(n, m)$ denotes the Stirling number of the second kind.

Proof: Suppose $S = \{1, 2, \ldots, n\}$ and $\zeta = \{A_i : A_i \subseteq S \text{ and } 1 \leq i \leq m\}$ is a unitary intersection in S. Then:

$$S = \bigcup_{i=1}^{m} A_i \quad \text{and} \quad |A_i \cap A_j| = 1 \text{ for all } 1 \leq i < j \leq m.$$

Note that $|\zeta| = m$ and $\iota(\zeta) = m$. By definition:

$$\iota(\zeta) = \max \left\{ |\zeta'| : \zeta' \subseteq \zeta \text{ and } \left| \bigcap_{A \in \zeta'} A \right| = 1 \right\} = m.$$

This implies that $\zeta' = \zeta$, and therefore $|A_1 \cap A_2 \cap \cdots \cap A_m| = 1$.

Now, let;

$$B_1 = A_1 - A_2 - A_3 - \cdots - A_m,$$
$$B_2 = A_2 - A_1 - A_3 - \cdots - A_m,$$
$$\vdots$$
$$B_{m-1} = A_{m-1} - A_1 - A_2 - \cdots - A_m,$$
$$B_m = A_m - A_1 - A_2 - \cdots - A_{m-1}.$$

Now, consider the following cases:

Case 1: $B_i \neq \emptyset$ for all i, where $1 \leq i \leq m$

Note that the index of ζ is the common intersection $A_1 \cap A_2 \cap \cdots \cap A_m$ and it contains a fixed element from S. Since $|S| = n$, there are n choices for this common element. The remaining $n-1$ elements of S must be distributed among m pairwise disjoint subsets $B_1, B_2, \ldots, B_m$ of S, (see Table 1). This can be done in $S(n-1, m)$ ways (where $S(n-1, m)$ denotes the Stirling number of the second kind).

Sets	Discussion	Choices
$A_1 \cap A_2 \cap \cdots \cap A_m$	If we select and assign one fixed element from set S with n elements in this set, then there are n ways to do this.	n choices
B_1 B_2 $\vdots$ B_{m-1} B_m	The remaining $n-1$ elements of set S must be distributed to m number of sets, this case satisfies the condition that the pairwise intersection of sets in the collection has only one element. This can be done in $S(n-1, m)$ ways.	$S(n-1, m)$

Table 1. The number of choices when $B_i \neq \emptyset$

Thus, there are $n \cdot S(n-1, m)$ number of unitary intersections in this case.

Case 2: There exists i such that $B_i = \emptyset$

Suppose i is unique, then the common element is again chosen from S (still n ways), but the remaining $n-1$ elements are assigned to only $m-1$ subsets. Thus, there are $S(n-1, m-1)$ways. (see Table 2)

Sets	Discussion	Choices
$A_1 \cap A_2 \cap \cdots \cap A_m$	If we select and assign one fixed element from set S with n elements in this set, then there are n ways to do this.	n choices
B_1 B_2 $\vdots$ B_{m-1} B_m	The remaining $n-1$ elements of set S must be distributed to $m-1$ number of sets, since B_m is not assigned. This case also satisfies the condition that the pairwise intersection of sets in the collection has only one element. This can be done in $S(n-1, m-1)$ ways. (B_m is not assigned)	$S(n-1, m-1)$

TABLE 2. The number of choices when there exists i such that $B_i = \emptyset$

Hence, this case have $n \cdot S(n-1, m-1)$ unitary intersections.

However, if there are two empty collections B_i and B_j, where $i \neq j$ then $A_i = A_j$. This contradicts the requirement that subsets in a unitary intersection collection must be distinct and it has one common element.

Therefore,

$$\begin{aligned} \mu_{(m,m)}(n) &= [n \cdot S(n-1, m)] \cdot [n \cdot S(n-1, m-1)] \\ &= n\,[S(n-1, m) + S(n-1, m-1)] \end{aligned}$$

□

In the next theorem, we derive a recurrence relation for $\mu_{(m,m)}(n)$. This is necessary for constructing a sequence of (m,m)-unitary intersection numbers.

Theorem 3.1.2: Let $2 \leq m \leq n$. Then the recurrence relation for the (m,m)-unitary intersection number is given by:

$$\mu_{(m,m)}(n+1) = \frac{m(n+1)\mu_{(m,m)}(n) + n(n+1)S(n-1,m-2)}{n}$$

Proof: By Theorem 3.1.1, $\mu_{(m,m)}(n+1) = (n+1)\left[S(n,m) + S(n,m-1)\right]$, where $S(n,m)$ and $S(n,m-1)$ are stirling numbers of the second kind, then it satisfies the recurrence relationship.
Hence, by Theorem 2.3,

$$\begin{aligned}
\mu_{(m,m)}(n+1) &= (n+1)\left[m \cdot S(n-1,m) + S(n-1,m-1)\right. \\
&\quad \left. +(m-1)\cdot S(n-1,m-1) + S(n-1,m-2)\right] \\
&\quad \text{(Applying Distributive Proprty over subtraction)} \\
&= (n+1)\left[m \cdot S(n-1,m) + S(n-1,m-1) + m \cdot S(n-1,m-1)\right. \\
&\quad \left. -S(n-1,m-1) + S(n-1,m-2)\right] \quad \text{(Combining like terms)} \\
&= (n+1)\left[m \cdot S(n-1,m) + m \cdot S(n-1,m-1) + S(n-1,m-2)\right] \\
&\quad \text{(Distributing } n+1\text{)} \\
&= (n+1)m \cdot S(n-1,m) + (n+1)m \cdot S(n-1,m-1) + (n+1)S(n-1,m-2) \\
&\quad \text{(Grouping and factoring out } (n+1)\cdot m \text{ and dividing } n\text{)} \\
&= \frac{(n+1)m}{n} \cdot \mu_{(m,m)}(n) + (n+1)S(n-1,m-2) \\
&= \frac{m(n+1)\mu_{(m,m)}(n) + n(n+1)S(n-1,m-2)}{n}
\end{aligned}$$

□

In the next result, we derive an exponential generating function for $\mu_{(m,m)}(n)$. This is necessary for creating an entire sequence of (m,m)-unitary intersection number into a more formal and analytic power series.

Theorem 3.1.3. Let $2 \leq m \leq n$. Then the exponential generating function for the (m,m)-unitary intersection number is given by:

$$G_{(m,m)}(x) = \sum_{n=m}^{\infty} \frac{\mu_{(m,m)}(n)x^n}{n!} = \frac{x\left(e^x + m - 1\right)\left(e^x - 1\right)^{m-1}}{m!}$$

Proof: By Theorem 2.4, we have,

$$G_{(m,m)}(x) = \sum_{n=m}^{\infty} \frac{\mu_{(m,m)}(n)x^n}{n!} = \sum_{n=m}^{\infty} \frac{n\left[S(n-1,m) + S(n-1,m-1)\right]x^n}{n!}$$

Expanding the function, we get

$$\sum_{n=m}^{\infty} \frac{\mu_{(m,m)}(n)x^n}{n!} = \frac{m\left[S(m-1,m)+S(m-1,m-1)\right]x^m}{m!}$$
$$+\frac{(m+1)\left[S(m,m)+S(m,m-1)\right]x^{m+1}}{(m+1)!}$$
$$+\frac{(m+2)\left[S(m+1,m)+S(m+1,m-1)\right]x^{m+2}}{(m+2)!}$$
$$+\frac{(m+3)\left[S(m+2,m)+S(m+2,m-1)\right]x^{m+3}}{(m+3)!}+\cdots$$

(simplifying terms by factoring the denominator)

$$=\frac{\left[S(m-1,m)+S(m-1,m-1)\right]x^m}{(m-1)!}$$
$$+\frac{\left[S(m,m)+S(m,m-1)\right]x^{m+1}}{m!}$$
$$+\frac{\left[S(m+1,m)+S(m+1,m-1)\right]x^{m+2}}{(m+1)!}$$
$$+\frac{\left[S(m+2,m)+S(m+2,m-1)\right]x^{m+3}}{(m+2)!}+\cdots$$

(separating terms)

$$=\frac{x^mS(m-1,m)+x^mS(m-1,m-1)}{(m-1)!}$$
$$+\frac{x^{m+1}S(m,m)+x^{m+1}S(m,m-1)}{m!}$$
$$+\frac{x^{m+2}S(m+1,m)+x^{m+2}S(m+1,m-1)}{(m+1)!}$$
$$+\frac{x^{m+3}S(m+2,m)+x^{m+3}S(m+2,m-1)}{(m+2)!}+\cdots$$

(factoring out x from each term)

$$=x\left[\frac{S(m,m)x^m}{m!}+\frac{S(m+1,m)x^{m+1}}{(m+1)!}+\cdots\right]$$
$$+x\left[\frac{S(m-1,m-1)x^m}{(m-1)!}+\frac{S(m,m-1)x^m}{m!}+\frac{S(m+1,m-1)x^{m+1}}{(m+1)!}+\cdots\right]$$

(Recognizing generating functions of stirling numbers by Theorem 2.4)

$$=x\sum_{n=m}^{\infty}\frac{S(n,m)x^n}{n!}+x\sum_{n=m-1}^{\infty}\frac{S(n,m-1)x^n}{n!}$$
$$=x\cdot\frac{(e^x-1)^m}{m!}+x\cdot\frac{(e^x-1)^{m-1}}{(m-1)!}$$

Factoring out $x \cdot \frac{(e^x-1)^{m-1}}{(m-1)!}$

$$\sum_{n=m}^{\infty} \frac{\mu_{(m,m)}(n)x^n}{n!} = x \cdot \frac{(e^x-1)^{m-1}}{(m-1)!}\left(\frac{e^x-1}{m}+1\right)$$

$$= x \cdot \frac{(e^x-1)^{m-1}}{(m-1)!} \cdot \frac{e^x+m-1}{m}$$

Therefore,

$$G_{(m,m)}(x) = \sum_{n=m}^{\infty} \frac{\mu_{(m,m)}(n)x^n}{n!} = \frac{x(e^x+m-1)(e^x-1)^{m-1}}{m!}$$

$\square$

Now, we derive an explicit formula for $\mu_{(m,m)}(n)$. This is necessary for computing directly (m,m)-unitary intersection number for large value of n.

Theorem 3.1.4: Let $2 \leq m \leq n$. Then,

$$\mu_{(m,m)}(n) = \frac{n}{(m-1)!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{1-m+j}{m}\right)j^{n-1}$$

Proof: By, Theorem 3.1, $\mu_{(m,m)}(n) = n[S(n-1,m)+S(n-1,m-1)]$. Moreover, applying the explicit stirling number formula in Theorem 2.5 to both $S(n-1,m)$ and $S(n-1,m-1)$,then;

$$\begin{aligned}
\mu_{(m,m)}(n) &= n[S(n-1,m)+S(n-1,m-1)] \\
&= n\left[\sum_{j=0}^{m}\frac{(-1)^{m-j}}{m!}\binom{m}{j}j^{n-1} + \sum_{j=0}^{m-1}\frac{(-1)^{m-1-j}}{(m-1)!}\binom{m-1}{j}j^{n-1}\right] \\
&= \sum_{j=0}^{m}\left[\frac{n(-1)^{m-j}\binom{m}{j}j^{n-1}}{m!} + \frac{n(-1)^{m-j-1}\binom{m-1}{j}j^{n-1}}{(m-1)!}\right]
\end{aligned}$$

(Combining into a single rational expression by factoring $\frac{n}{(m-1)!}$)

$$= \frac{n}{(m-1)!}\sum_{j=0}^{m}(-1)^{m-j}\left(\frac{\binom{m}{j}j^{n-1}}{m} - \binom{m-1}{j}j^{n-1}\right)$$

(By Theorem 2.1, and multipying both numerator and denominator by m)

$$= \frac{n}{(m-1)!}\sum_{j=0}^{m}(-1)^{m-j}\left(\frac{m!(1-m+j)}{j!(m-j)!m}\right)j^{n-1}$$

Thus, by Theorem 2.1;

$$\mu_{(m,m)}(n) = \frac{n}{(m-1)!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{1-m+j}{m}\right)j^{n-1}$$

$\square$

3.2. $(m, m-1)$**-Unitary Intersection Collection.**

In this section, we derive a formula for $(m, m-1)$-Unitary intersection number and obtain some properties.

Theorem 3.2.1: Let $2 \leq m \leq n$. Then,

(i) $\mu_{(m,m-1)}(n) = n$, if $n = m$

(ii) $\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1} \sum\limits_{i=1}^{n-m} S(n-m, i) \cdot P(m, i)$, if $\lceil \frac{n}{m} \rceil \leq 2$.

(iii) $\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1} \sum\limits_{i=1}^{m} S(n-m, i) \cdot P(m, i)$, if $\lceil \frac{n}{m} \rceil > 2$.

Proof: Note that $\iota(\zeta) = \max\{|\zeta'| : \zeta' \subseteq \zeta \text{ and } |\bigcap_{A \in \zeta'} A| = 1\} = m - 1$. This implies that $\zeta' \subseteq \zeta$, i.e.,$|\,(A_1 \cap A_2 \cap \cdots \cap A_{m-1}) \setminus A_m| = 1$ and it contains a fixed element from S.

(i) Suppose $n = m$. Since $|S| = n$, then there are n choices to select a fixed element in S. Note that the disjoint collections $|A_1 \cap A_m| = |A_2 \cap A_m|...|A_{m-1} \cap A_m| = 1$. Consequently, we are distributing $n-1$ elements to $m-1$ sets, that is, $S(n-1, m-1) = 1$. (See Table 3.).

Sets	**Discussion**	**Choices**
$(A_1 \cap A_2 \cap \cdots \cap A_{m-1}) \setminus A_m$	If we assigned one fixed element from S in this set, then there are n number of ways to do this.	n
$A_1 \cap A_m$ $A_2 \cap A_m$ $\vdots$ $A_{m-1} \cap A_m$	The remaining $n-1$ elements must be assigned to $m-1$ number pairwise intersection to satisfy the condition that there must be one common element. Thus, there are $S(n-1, m-1)$ ways to do this. Since $n = m$ then there must be $S(n-1, n-1) = 1$ way.	$S(n-1, n-1)$

Table 3. The number of choices when $n = m$

Thus,

$$\mu_{(m,m-1)}(n) = n$$

(ii) Suppose $\lceil \frac{n}{m} \rceil \leq 2$, i.e $m < n \leq 2m$. Now consider the table,

Sets	**Discussion**	**Choices**
$(A_1 \cap A_2 \cap \cdots \cap A_{m-1}) \setminus A_m$	If we assigned one fixed element from S in this set, then there are n number of ways to do this.	n
$A_1 \cap A_m$ $A_2 \cap A_m$ $\vdots$ $A_{m-1} \cap A_m$	The remaining $n-1$ elements must be assigned to $m-1$ number of pairewise intersection to satisfy the condition that there must be one common element. Thus there are $\binom{n-1}{m-1}$ to do this.	$\binom{n-1}{m-1}$
$A_1 \setminus (A_2 \cup A_3 \cup \cdots \cup A_m)$ $A_2 \setminus (A_1 \cup A_3 \cup \cdots \cup A_m)$ $\vdots$ $A_{m-1} \setminus (A_1 \cup A_2 \cup \cdots \cup A_m)$ $A_m \setminus (A_1 \cup A_2 \cup \cdots \cup A_{m-1})$	Since there are already a total m number of elements assigned in the above sets, then there are still remaining $n-m$ elements to be assigned into m sets.	(See next Table)

Table 4. Distribution of $n-m$ elements into m subsets when $\lceil \frac{n}{m} \rceil \leq 2$

Now, in distributing $n-m$ elements to m number of sets, consider the following table,

Number of sets to be assigned	**Discussion**	**Choices**
1	If $n-m$ elements are assigned to only one set, then this can be done in $S(n-m,1)$ ways. Meanwhile, since there are m pairwise disjoint sets, the $n-m$ elements can also be assigned to one of them in $P(m,1)$ ways. Hence, this case has $S(n-m,1)\cdot P(m,1)$ choices.	$S(n-m,1)\cdot P(m,1)$
2	If $n-m$ elements are split and assigned to two sets, then this can be done in $S(n-m,2)$ ways. Since there are m disjoint sets, we can choose 2 from them in $P(m,2)$ ways. So this case has $S(n-m,2)\cdot P(m,2)$ choices.	$S(n-m,2)\cdot P(m,2)$
$\vdots$	$\vdots$	$\vdots$
$n-m$	If $n-m$ elements are split and assigned to $n-m$ sets, then this can be done in $S(n-m,n-m)$ ways. The sets can be chosen from m disjoint sets in $P(m,n-m)$ ways. Hence, this case has $S(n-m,n-m)\cdot P(m,n-m)$ choices.	$S(n-m,n-m)\cdot P(m,n-m)$

TABLE 5. Distribution of $n-m$ elements into $n-m$ subsets

Column three shows the different cases when selecting the number of sets to be assigned with $n-m$ remaining elements. This suggests that there are $\sum_{i=1}^{n-m} S(n-m,i)\cdot P(m,i)$ number of ways.

By summaring all the choices in the two tables, thus;

$$mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}\sum_{i=1}^{n-m} S(n-m,i)\cdot P(m,i), \text{if } \left\lceil \frac{n}{m} \right\rceil \leq 2$$

(iii) Suppose $\left\lceil \frac{n}{m} \right\rceil > 2$, i.e $n > 2m$. Now considering back Table 4, we are still left with $n-m$ elements, but in this case, it will be distributed to m number of sets (see Table 6).

Number of sets to be assigned	Discussion	Choices
1	If $n-m$ elements are assigned to only one set, then this can be done in $S(n-m,1)$ ways. Meanwhile, since there are m pairwise disjoint sets, the $n-m$ elements can also be assigned to one of them in $P(m,1)$ ways. Hence, this case has $S(n-m,1)\cdot P(m,1)$ choices.	$S(n-m,1)\cdot P(m,1)$
2	If $n-m$ elements are split and assigned to two sets, then this can be done in $S(n-m,2)$ ways. Since there are m disjoint sets, we can choose the two sets in $P(m,2)$ ways. So this case has $S(n-m,2)\cdot P(m,2)$ choices.	$S(n-m,2)\cdot P(m,2)$
$\vdots$	$\vdots$	$\vdots$
m	If $n-m$ elements are split and assigned to m sets, then this can be done in $S(n-m,m)$ ways. The sets to be assigned can be chosen from m disjoint sets in $P(m,m)$ ways. Hence, this case has $S(n-m,m)\cdot P(m,m)$ choices.	$S(n-m,n-m)\cdot P(m,n-m)$

TABLE 6. Distribution of $n-m$ elements into m subsets when $\left\lceil \frac{n}{m} \right\rceil > 2$

Column three shows the different cases when selecting the number of sets to be assigned with $n-m$ remaining elements. This suggests that the number of ways is $\sum_{i=1}^{m} S(n-m,i)\cdot P(m,i)$. Therefore, considering back Table 4,

$$\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}\sum_{i=1}^{m} S(n-m,i)\cdot P(m,i), \text{ if } \left\lceil \frac{n}{m} \right\rceil > 2$$

$\square$

Now, we proceed by deriving an explicit formula for $\mu_{(m,m-1)}(n)$.This is necessary for computing directly $(m,m-1)$-unitary intersection number for large value of n.

Theorem 3.2.2: Let $n \geq m$. Then for all $2 \leq m \leq n$,

$$\mu_{(m,m-1)}(n) = \frac{n!\, m^{n-m}}{(n-m)!(m-1)!}$$

Proof: Suppose $n = m$. Then, by Theorem 3.2.1 (i),

$$\mu_{(m,m-1)}(n) = \frac{n!\, m^{n-m}}{(n-m)!(m-1)!}$$

$$
\begin{aligned}
&= \frac{m!\, m^{m-m}}{(m-m)!(m-1)!} \\
&= \frac{m!}{(m-1)!} \\
&= m
\end{aligned}
$$

Note that $2 \leq m \leq n$, which means that m must be at least 2 and $n \geq 2$. This implies that either $\lceil n/m \rceil \leq 2$ or $\lceil n/m \rceil > 2$.

Suppose that $\lceil n/m \rceil \leq 2$, by Theorem 3.2.1 (ii),

$$\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}\sum_{i=1}^{n-m} S(n-m,i)P(m,i)$$

since $\lceil n/m \rceil \leq 2$, then $n \leq 2m$. Consequenlty, $n-m \leq m$. This further implies that $\min\{n-m, m\} = n-m$.

By Theorem 2.6,

$$\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}m^{n-m}.$$

Suppose that $\lceil n/m \rceil > 2$. by Theorem 3.2.1 (iii),

$$\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}\sum_{i=1}^{m} S(n-m,i)P(m,i)$$

Since $\lceil n/m \rceil > 2$, then $n > 2m$. Consequenlty, $n-m > m$, so $\min\{n-m, m\} = m$. Hence, without loss of generality, by Theorem 2.6,

$$\mu_{(m,m-1)}(n) = n\binom{n-1}{m-1}m^{n-m}.$$

Now, simplifying:

$$
\begin{aligned}
\mu_{(m,m-1)}(n) &= n \cdot \frac{(n-1)!}{(n-m)!(m-1)!}\, m^{n-m} \\
&= \frac{n!\, m^{n-m}}{(n-m)!(m-1)!}.
\end{aligned}
$$

□

Here, we derive a recurrence relation for $\mu_{(m,m-1)}(n)$. This is necessary for constructing a sequence of $(m, m-1)$-unitary intersection number.

Theorem 3.2.3: Let $2 \leq m \leq n$. Then,

$$\mu_{(m,m-1)}(n+1) = \frac{m(n+1)}{n-m+1}\, \mu_{(m,m-1)}(n).$$

Proof: By Theorem 3.2.2, we have:

$$\mu_{(m,m-1)}(n) = \frac{n!\, m^{n-m}}{(n-m)!(m-1)!}$$

Thus,

$$\begin{aligned}
\mu_{(m,m-1)}(n+1) &= \frac{(n+1)!\, m^{n-m+1}}{(n-m+1)!(m-1)!} \\
&= \frac{m\, m^{n-m}(n+1)n!}{(n-m+1)(n-m)!(m-1)!} \\
&= \frac{m(n+1)\, m^{n-m}n!}{(n-m+1)(n-m)!(m-1)!} \\
&= \frac{m(n+1)}{n-m+1} \times \frac{m^{n-m}n!}{(n-m)!(m-1)!} \\
&= \frac{m(n+1)}{n-m+1}\, \mu_{(m,m-1)}(n).
\end{aligned}$$

□

Lastly, we derive an exponential generating function for $\mu_{(m,m-1)}(n)$. This is necessary for creating an entire sequence of $(m, m-1)$-unitary intersection numbers into a more formal and analytic power series.

Theorem 3.2.4: Let $2 \leq m \leq n$. Then,

$$G_{(m,m-1)}(x) = \sum_{n=m}^{\infty} \frac{\mu_{m,m-1}(n)x^n}{n!} = \frac{x^m}{(m-1)!}e^{mx}$$

Proof: By Theorem 2.4, we have

$$\begin{aligned}
\sum_{n=m}^{\infty} \frac{\mu_{m,m-1}(n)x^n}{n!} &= \sum_{n=m}^{\infty} \frac{m^{n-m}x}{(n-m)!(m-1)!} \\
&= \frac{1}{(m-1)!} \sum_{n=m}^{\infty} \frac{m^{n-m}x}{(n-m)!}
\end{aligned}$$

Now, let $k = n - m$. So, we have:

$$\begin{aligned}
\sum_{n=m}^{\infty} \frac{\mu_{m,m-1}(n)x^n}{n!} &= \frac{1}{(m-1)!} \sum_{k=0}^{\infty} \frac{m^k x^{m+k}}{k!} \\
&= \frac{1}{(m-1)!} \sum_{k=0}^{\infty} \frac{m^k x^m x^k}{k!} \\
&= \frac{x^m}{(m-1)!} \sum_{k=0}^{\infty} \frac{m^k x^k}{k!} \\
&= \frac{x^m}{(m-1)!} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!}
\end{aligned}$$

(By Theorem 2.7)

$$= \frac{x^m}{(m-1)!}e^{mx}.$$

Therefore,

$$G_{m,m-1}(x) = \sum_{n=m}^{\infty} \frac{\mu_{m,m-1}(n)x^n}{n!} = \frac{x^m}{(m-1)!}e^{mx}$$

□

3.3. $(m, 2)$-Unitary Intersection Collection.

In this section, we derive a formula for $(m, 2)$-Unitary intersection number and obtain some properties.

Theorem 3.3.1: Let $2 \leq m \leq n$. Then,

(i) $\mu_{(m,2)}(n) = 0$, if $n < \binom{m}{2}$

(ii) $\mu_{(m,2)}(n) = \frac{n!}{m!}$, if $n = \binom{m}{2}$.

(iii) $\mu_{(m,2)}(n) = \frac{\mathcal{P}(n,\binom{m}{2})}{m!}[\sum_{k=1}^{n-\binom{m}{2}} S(n-\binom{m}{2}, k) \cdot P(m,k)]$, if $n - \binom{m}{2} < m$.

(iv) $\mu_{(m,2)}(n) = \frac{\mathcal{P}(n,\binom{m}{2})}{m!}[\sum_{k=1}^{m} S(n-\binom{m}{2}, k) \cdot P(m,k)]$, if $n - \binom{m}{2} > m$.

Proof: Note that $\iota(\zeta) = \max\{|\zeta'| : \zeta' \subseteq \zeta$ and $|\bigcap_{A\in\zeta'} A| = 1\} = 2$. This implies that $\zeta' \subseteq \zeta$, i.e.,$|A_1 \cap A_m| = |A_2 \cap A_m| = ... = |A_{m-1} \cap A_m| = 1$. Hence there are $\binom{m}{2}$ number of distinct pairs, $A_i \cap A_j$ that is,

$$B_1 = (A_1 \cap A_2) \setminus (A_3 \cup A_4 \cup \cdots \cup A_m)$$

$$B_2 = (A_1 \cap A_3) \setminus (A_2 \cup A_4 \cup \cdots \cup A_m)$$

$$\vdots$$

$$B_{m-1} = (A_1 \cap A_m) \setminus (A_2 \cup A_3 \cup \cdots \cup A_{m-1})$$

$$B_{m+1} = (A_2 \cap A_3) \setminus (A_1 \cup A_4 \cup \cdots \cup A_m)$$

$$B_{m+2} = (A_2 \cap A_4) \setminus (A_1 \cup A_5 \cup \cdots \cup A_m)$$

$$\vdots$$

$$B_{2m-2} = (A_2 \cap A_m) \setminus (A_1 \cup A_3 \cup \cdots \cup A_{m-1})$$

$$\vdots$$

$$B_{\binom{m}{2}-2} = (A_{m-2} \cap A_{m-1}) \setminus (A_1 \cup A_2 \cup \cdots \cup A_m)$$

$$B_{\binom{m}{2}-1} = (A_{m-2} \cap A_m) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{m-1})$$

$$B_{\binom{m}{2}} = (A_{m-1} \cap A_m) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{m-2})$$

Each of these sets must be assigned with one element from S to form a unitary collection in S.

(i) Suppose $n < \binom{m}{2}$ and there exists unitary collection $\zeta = \{A_1, A_2, \cdots A_m\}$. Note that there are $\binom{m}{2}$ number of distinct pairs $A_i \cap A_j$. Now, $n < \binom{m}{2}$, hence, there exists i and j

such that $A_i \cap A_j = \emptyset$. This contradicts to the condition that for all i and j, $|A_i \cap A_j| = 1$. Thus, in the case, there is no unitary collection that can be formed. Therefore, $\mu_{(m,2)}(n) = 0$. $\square$

(ii) Suppose $n = \binom{m}{2}$. Now, consider the table below,

Sets	**Discussion**	**Choices**
B_1	Now we assign one fixed element from S in this set, and there are n number of ways to do this.	n
B_2 B_3 $\vdots$ $B_{\binom{m}{2}}$	The remaining $n-1$ elements must be assigned to $\binom{m}{2} - 1$ a number of disjoint pairwise intersections. Since, $n = \binom{m}{2}$ then each of the remaining distinct pairs has exactly one element.	$(n-1)$ $(n-2)$ $\vdots$ 2 1

Table 7. The number of choices when $n = \binom{m}{2}$

Since ζ is an unordered collection of subsets in S and $|\zeta| = m$, then there are $m!$ collections treated as one. Thus, there are $\frac{n!}{m!}$ number of unitary collections in this case.

Therefore,

$$\mu_{(m,2)}(n) = \frac{n!}{m!}$$

$\square$

(iii) Suppose $n > \binom{m}{2}$. This implies that either $n - \binom{m}{2} \leq m$ or $n - \binom{m}{2} > m$. Now, suppose we have $n - \binom{m}{2} \leq m$. Consider the next table.

Sets	**Discussion**	**Choices**
B_1	When we assign one fixed element from S in this set, then there are n number of ways to do this.	n
B_2 B_3 $\vdots$ $B_{\binom{m}{2}}$	The remaining $n-1$ elements must be assigned to $\binom{m}{2} - 1$ a number of disjoint pairwise intersections. Since, $n > \binom{m}{2}$ then, there are $P(n-1, \binom{m}{2} - 1)$ ways to do this. Note that there are $m!$ number of identical collections. Hence,$P(n-1, \binom{m}{2} - 1)$ must be divided by $m!$.	$\frac{P(n-1,\binom{m}{2}-1)}{m!}$
$A_1 \setminus (A_2 \cup A_3 \cup \cdots \cup A_m)$ $A_2 \setminus (A_1 \cup A_3 \cup \cdots \cup A_m)$ $\vdots$ $A_{m-1} \setminus (A_1 \cup A_2 \cup \cdots \cup A_m)$ $A_m \setminus (A_1 \cup A_2 \cup \cdots \cup A_{m-1})$	Since there are already $\binom{m}{2}$ number of elements assigned in the above sets, then there are still remaining $n - \binom{m}{2}$ elements to be assigned into $n - \binom{m}{2}$ sets.	(See Table 9 and 10)

Table 8. The number of choices when $n - \binom{m}{2} \leq m$

Number of sets to be assigned	**Discussion**	**Choices**
1	If $n-\binom{m}{2}$ elements are assigned to only one set, then this can be done in $S(n-\binom{m}{2},1)$ ways. Meanwhile, since there are m pairwise disjoint sets, the $n-\binom{m}{2}$ elements can also be assigned to one of them in $P(m,1)$ ways. Hence, this case has $S(n-\binom{m}{2},1)\cdot P(m,1)$ choices.	$S(n-\binom{m}{2},1)\cdot P(m,1)$
2	If $n-m$ elements are split and assigned to two sets, then this can be done in $S(n-\binom{m}{2},2)$ ways. Since there are m disjoint sets, we can choose the two sets in $P(m,2)$ ways. So this case has $S(n-\binom{m}{2},2)\cdot P(m,2)$ choices.	$S(n-\binom{m}{2},2)\cdot P(m,2)$
$\vdots$	$\vdots$	$\vdots$
$n-\binom{m}{2}$	If $n-\binom{m}{2}$ elements are split and assigned to $n-\binom{m}{2}$ sets, then this can be done in $S(n-\binom{m}{2},n-\binom{m}{2})$ ways. The sets to be assigned can be chosen from m disjoint sets in $P(m,n-\binom{m}{2})$ ways. Hence, this case has $S(n-\binom{m}{2},n-\binom{m}{2})\cdot P(m,n-\binom{m}{2})$ choices.	$S(n-\binom{m}{2},m)\cdot P(m,n-\binom{m}{2})$

TABLE 9. Distribution of $n-\binom{m}{2}$ elements into $n-\binom{m}{2}$ subsets when $n-\binom{m}{2}\le m$

Column three shows the different cases when selecting the number of sets to be assigned with $n-\binom{m}{2}$ remaining elements. This suggests that the number of ways is $\sum\limits_{k=1}^{n-\binom{m}{2}} S(n-\binom{m}{2},k)\cdot P(m,k)$.

Therefore,

$$\mu_{(m,2)(n)} = n\cdot\frac{P(n-1,\binom{m}{2}-1)}{m!}\cdot\sum_{k=1}^{n-\binom{m}{2}} S(n-\binom{m}{2},k)\cdot P(m,k)$$
$$= \frac{P(n,\binom{m}{2})}{m!}\cdot\sum_{k=1}^{n-\binom{m}{2}} S(n-\binom{m}{2},k)\cdot P(m,k)$$

□

(iv) Suppose $n - \binom{m}{2} > m$. Now considering back Table 8, we are still left with $n - \binom{m}{2}$ elements, but in this case, it will be distributed to m number of sets.(see Table 10).

Number of sets to be assigned	**Discussion**	**Choices**
1	If $n - \binom{m}{2}$ elements are assigned to only one set, then this can be done in $S(n - \binom{m}{2}, 1)$ ways. Meanwhile, since there are m pairwise disjoint sets, the $n - \binom{m}{2}$ elements can also be assigned to one of them in $P(m, 1)$ ways. Hence, this case has $S(n - \binom{m}{2}, 1) \cdot P(m, 1)$ choices.	$S(n - \binom{m}{2}, 1) \cdot P(m, 1)$
2	If $n - m$ elements are split and assigned to two sets, then this can be done in $S(n - \binom{m}{2}, 2)$ ways. Since there are m disjoint sets, we can choose the two sets in $P(m, 2)$ ways. So this case has $S(n - \binom{m}{2}, 2) \cdot P(m, 2)$ choices.	$S(n - \binom{m}{2}, 2) \cdot P(m, 2)$
$\vdots$	$\vdots$	$\vdots$
m	If $n - \binom{m}{2}$ elements are split and assigned to m sets, then this can be done in $S(n - \binom{m}{2}, m)$ ways. The sets to be assigned can be chosen from m disjoint sets in $P(m, m)$ ways. Hence, this case has $S(n - \binom{m}{2}, m) \cdot P(m, m)$ choices.	$S(n - \binom{m}{2}, m) \cdot P(m, m)$

TABLE 10. Distribution of $n - \binom{m}{2}$ elements into m subsets when $n - \binom{m}{2} > m$

Column three shows the different cases when selecting the number of sets to be assigned with $n - \binom{m}{2}$ remaining elements. This suggests that the number of ways is $\sum\limits_{k=1}^{m} S(n - \binom{m}{2}, k) \cdot P(m, k)$. Therefore,

$$\begin{aligned} \mu_{(m,2)(n)} &= n \cdot \frac{P(n-1, \binom{m}{2} - 1)}{m!} \cdot \sum_{k=1}^{m} S(n - \binom{m}{2}, k) \cdot P(m, k) \\ &= \frac{P(n, \binom{m}{2})}{m!} \cdot \sum_{k=1}^{m} S(n - \binom{m}{2}, k) \cdot P(m, k) \end{aligned}$$

$\square$

Now, we proceed by deriving an explicit formula for $\mu_{(m,2)}(n)$.This is necessary for computing directly $(m,2)$-unitary intersection number for large value of n.

Theorem 3.3.2: Let $n \geq m$. Then for all $2 \leq m \leq n$,

$$\mu_{(m,2)}(n) = \frac{n!(m^{n-\frac{m(m-1)}{2}})}{m!(n-\frac{m(m-1)}{2})!}$$

Proof: Suppose $n = \binom{m}{2}$. Then, by Theorem 3.3.2 (ii),

$$\begin{aligned}\mu_{(m,2)}(n) &= \frac{n!(m^{n-\frac{m(m-1)}{2}})}{m!(n-\frac{m(m-1)}{2})!} \\ &= \frac{n!m^{n-\binom{m}{2}}}{m!(n-\binom{m}{2})!} \\ &= \frac{n!m^{\binom{m}{2}-\binom{m}{2}}}{m!(\binom{m}{2}-\binom{m}{2})!} \\ &= \frac{n!}{m!}\end{aligned}$$

Suppose that $n > \binom{m}{2}$. This implies that either $n - \binom{m}{2} \leq m$ or $n - \binom{m}{2} > m$. Now, let $n - \binom{m}{2} \leq m$, by Theorem 3.3.1 (ii),

$$\mu_{(m,2)(n)} = \frac{P(n, \binom{m}{2})}{m!} \cdot \sum_{k=1}^{n-\binom{m}{2}} S(n - \binom{m}{2}, k) \cdot P(m,k)$$

since $n - \binom{m}{2} \leq m$, this implies that $\min\{n - \binom{m}{2}, m\} = n - \binom{m}{2}$.

By Theorem 2.6,

$$\mu_{(m,2)(n)} = \frac{P(n, \binom{m}{2})}{m!} m^{n-\binom{m}{2})}.$$

Suppose that $n - \binom{m}{2} > m$. by Theorem 3.3.1 (iv),

$$\mu_{(m,2)(n)} = \frac{P(n, \binom{m}{2})}{m!} \cdot \sum_{k=1}^{m} S(n - \binom{m}{2}, k) \cdot P(m,k)$$

Since $n - \binom{m}{2} > m$, then $\min\{n - \binom{m}{2}, m\} = m$. Hence, without loss of generality, by Theorem 2.6,

$$\mu_{(m,2)}(n) = \frac{P(n, \binom{m}{2})}{m!} m^{n-\binom{m}{2})}.$$

Now, simplifying:

$$\begin{aligned}\mu_{(m,2)}(n) &= \frac{n!}{m!(n-\binom{m}{2})!} m^{n-\binom{m}{2})} \\ &= \frac{n!m^{n-\binom{m}{2}}}{m!(n-\binom{m}{2})!}. \\ &= \frac{n!(m^{n-\frac{m(m-1)}{2}})}{m!(n-\frac{m(m-1)}{2})!}\end{aligned}$$

$\square$

Here, we derive a recurrence relation for $\mu_{(m,2)}(n)$. This is necessary for constructing a sequence of $(m,2)$-unitary intersection number.

Theorem 3.3.3: Let $2 \le m \le n$. Then,

$$\mu_{(m,2)}(n+1) = \frac{2m(n+1)}{2n - m(m-1) + 2}\,\mu_{(m,2)}(n).$$

Proof: By Theorem 3.2.2, we have:

$$\mu_{(m,2)}(n) = \frac{n!(m^{n-\frac{m(m-1)}{2}})}{m!(n - \frac{m(m-1)}{2})!}$$

Thus,

$$\begin{aligned}
\mu_{(m,2)}(n+1) &= \frac{(n+1)!(m^{n-\frac{m(m-1)}{2}+1})}{m!(n-\frac{m(m-1)}{2}+1)!} \\
&= \frac{(n+1)n!m(m^{n-\frac{m(m-1)}{2}})}{m!(n-\frac{m(m-1)}{2}+1)(n-\frac{m(m-1)}{2})!} \\
&= \frac{m(n+1)\,m^{n-\frac{m(m-1)}{2}}n!}{(n-\frac{m(m-1)}{2}+1)(n-\frac{m(m-1)}{2})!m!} \\
&= \frac{m(n+1)}{n-\frac{m(m-1)}{2}+1} \times \frac{m^{n-\frac{m(m-1)}{2}}n!}{(n-\frac{m(m-1)}{2})!m!} \\
&= \frac{2m(n+1)}{2n-m(m-1)+2}\,\mu_{(m,2)}(n)
\end{aligned}$$

□

Lastly, we derive an exponential generating function for $\mu_{(m,2)}(n)$. This is necessary for creating an entire sequence of $(m,2)$-unitary intersection numbers into a more formal and analytic power series.

Theorem 3.3.4: Let $2 \le m \le n$. Then,

$$G_{(m,2)}(x) = \sum_{n=\binom{m}{2}}^{\infty} \frac{\mu_{(m,2)}(n)x^n}{n!} = \frac{x^{\frac{m(m-1)}{2}}}{m!}e^{mx}$$

Proof: By Theorem 2.4, we have

$$\begin{aligned}
\sum_{n=\binom{m}{2}}^{\infty} \frac{\mu_{(m,2)}(n)x^n}{n!} &= \sum_{n=\binom{m}{2}}^{\infty} \frac{m^{n-\binom{m}{2}}x^n}{m!(n-\binom{m}{2})!} \\
&= \frac{1}{m!}\sum_{n=\binom{m}{2}}^{\infty} \frac{m^{n-\binom{m}{2}}x^n}{(n-\binom{m}{2})!}
\end{aligned}$$

Now, let $k = n - \binom{m}{2}$. So, we have:

$$\sum_{n=\binom{m}{2}}^{\infty} \frac{\mu_{(m,2)}(n)x^n}{n!} = \frac{1}{m!}\sum_{k=0}^{\infty} \frac{m^k x^{\binom{m}{2}+k}}{k!}$$

$$= \frac{1}{m!}\sum_{k=0}^{\infty}\frac{m^k x^{\binom{m}{2}} x^k}{k!}$$

$$= \frac{x^{\binom{m}{2}}}{m!}\sum_{k=0}^{\infty}\frac{m^k x^k}{k!}$$

$$= \frac{x^{\frac{m(m-1)}{2}}}{m!}\sum_{k=0}^{\infty}\frac{(mx)^k}{k!}$$

(By Theorem 2.7)

$$= \frac{x^{\frac{m(m-1)}{2}}}{m!}e^{mx}.$$

Therefore,

$$G_{(m,2)}(x) = \sum_{n=\binom{m}{2}}^{\infty}\frac{\mu_{(m,2)}(n)x^n}{n!} = \frac{x^{\frac{m(m-1)}{2}}}{m!}e^{mx}$$

□

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References

[1] I. Anderson, Combinatorics of Finite Sets, Courier Corporation, (2002).

[2] R.P. Stanley, Enumerative Combinatorics, Volume 1, Cambridge Studies in Advanced Mathematics, Cambridge University Press, (2011).

[3] P. Erdös, R. Rado, Intersection Theorems for Systems of Sets, J. Lond. Math. Soc. s1-35 (1960), 85–90. https://doi.org/10.1112/jlms/s1-35.1.85.

[4] P. Erdós, C. Ko, R. Rado, Intersection Theorems for Systems of Finite Sets, Q. J. Math. 12 (1961), 313–320. https://doi.org/10.1093/qmath/12.1.313.

[5] P. Frankl, Z. Füredi, Forbidding Just One Intersection, J. Comb. Theory Ser. A 39 (1985), 160–176. https://doi.org/10.1016/0097-3165(85)90035-4.

[6] R. Anstee, P. Keevash, Pairwise Intersections and Forbidden Configurations, Eur. J. Comb. 27 (2006), 1235–1248. https://doi.org/10.1016/j.ejc.2006.05.002.

[7] P. Damaschke, Finding and Enumerating Large Intersections, Theor. Comput. Sci. 580 (2015), 75–82. https://doi.org/10.1016/j.tcs.2015.02.034.

[8] P. Frankl, A. Kupavskii, Counting Intersecting and Pairs of Cross-Intersecting Families, Comb. Probab. Comput. 27 (2017), 60–68. https://doi.org/10.1017/s0963548317000529.

[9] P. Keevash, D. Mubayi, R.M. Wilson, Set Systems with No Singleton Intersection, SIAM J. Discret. Math. 20 (2006), 1031–1041. https://doi.org/10.1137/050647372.

[10] W.W. Peterson, E. J. Weldon Jr., Error-Correcting Codes, The MIT Press, (1972).

[11] K. Phelps, C. Rodger, Designs and Error-Correcting Codes, in: Encyclopedia of Physical Science and Technology, Elsevier, 2003: pp. 335–342. https://doi.org/10.1016/b0-12-227410-5/00167-8.

[12] M. Toloo, S. Talatahari, I. Rahimi, Multi-Objective Combinatorial Optimization Problems and Solution Methods, Academic Press, 2022.

[13] W. Lu, K. Gu, Z. Lu, H. Peng, Resource Allocation for Secure Communication in Cooperative Cr Networks, in: Lecture Notes in Electrical Engineering, Springer, Singapore, 2018, pp. 1199-1206. https://doi.org/10.1007/978-981-10-6571-2_143.

[14] C.C. Chen, K.M. Koh, K. Khee-Meng, Principles and Techniques in Combinatorics, World Scientific, 2004.

[15] K.H. Rosen, Discrete Mathematics and Its Applications, McGraw-Hill, (2012).

[16] C.A. Charalambides, Enumerative Combinatorics, Chapman and Hall/CRC, 2018. https://doi.org/10.1201/9781315273112.

[17] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, Springer, 1974. https://doi.org/10.1007/978-94-010-2196-8.

[18] R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison-Wesley, (1994).

[19] L. Euler. Introductio in Analysin Infinitorum, Lausannae, (1748).