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THE POWER EXPONENTIAL-G FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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Abstract. A new and flexible family of continuous probability distributions, Power Exponential–G family is presented in this paper. The proposed family exploits an additional shape parameter to improve the flexibility of modeling input data structures in practice and extends the basic distribution with a power exponential transform. This formulation offers a unified approach for developing new models that can accommodate a wide range of distributional forms, such as skewness, heavy tails and different hazard rate shapes. The general mathematical properties of the Power Exponential–G family are investigated in details, such as cumulative and probability density functions, moments, quantile function and reliability measures. To demonstrate the application of the developed framework a particular family member, namely, the Power Exponential Weibull distribution is derived and studied. Maximum-likelihood estimates of the parameters and statistical performance are assessed both theoretically and by means of Monte Carlo studies for a variety of parameter settings and sample sizes. To demonstrate practical applicability, the Power Exponential Weibull distribution is fitted to several real lifetime and reliability datasets and compared with well-known competing models. The comparative results, based on information criteria and goodness-of-fit statistics, reveal that the new model provides a more accurate and adaptable representation of empirical data.

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1. Introduction

Modeling real data often calls for using more flexible probability distributions capable of accommodating various features such as asymmetry, heavy or light tails and different shapes of hazard functions. Classical distributions such as the exponential, Weibull or normal are popular choices for distributional assumptions but their intrinsic shapes seriously restricts the fitting of empirical data with

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different structure. In order to address this shortfall, generated families of probability distributions have been developed, with baseline distributions created by the introduction of more parameters. These approaches have become an active area of statistical research, particularly because of their wide applicability in survival analysis, actuarial science, hydrology, economics, and reliability theory.

An early work in this direction is the Beta-generated family proposed by Eugene et al. [1] which elevates the CDF of any base distribution by the Beta distribution. This concept received much attention in printed literature. The Kumaraswamy-generated family was introduced by Kumaraswamy [2], which provides the simpler quantile functions and numerical advantages of computational implementation over those of the Beta-generated models. A more important generalization is the Marshall–Olkin extended family [3] which added a shape parameter to determine the tilt of the hazard rate, bringing an increased level of flexibility in modeling lifetime data.

alhussaini2018, After these first families, a number of other generators have also been proposed. For example, the Exponentiated-G family [4] introduces a power parameter to model tail behavior adequate for growth mixtures models, and the Transmuted-G family [5] alhussaini2018, uses the quadratic rank transmutation mapping function to obtain more flexibility. The more general T-X family [6], bring wealthy structured language to devise new families of distributions by utilizing transformations on a baseline model. More recently, Cordeiro and Castro [7] introduced a general family that includes most of these previous proposals. Many others have been proposed recently, to name just a few [8–10].

Although these families of generations have been found to be useful, they are often highly mathematical in nature, which makes it difficult to derive statistical properties and characterized by complex mathematical structures, such as special functions. In applications this can be a drawback for calculating closed-form expressions of moments, moments generating function and simple parameter estimation. Furthermore, they tend to be computationally intensive, particularly when dealing with large data sets or in simulation studies; indeed many of these generator families are related to heavy computational algorithmic techniques. Thus, families of distributions such ones which trade-off flexibility, tractability and applicability are still sought after.

In this paper, we introduce a new family of continuous probability distributions within the class of CDF $G(x,\psi)$ and PDF $g(x,\psi)$, that includes an extra shape parameter $\theta > 0$ in any baseline distribution with known parameters ψ. The new family in terms of its cumulative distribution function.

$$F(x;\psi,\theta) = \frac{(\theta+1)^{G(x;\psi)} + G(x;\psi) - 1}{\theta+1}, \tag{1}$$

and the probability density function

$$f(x;\psi,\theta) = \frac{g(x;\psi)}{\theta+1}\left[(\theta+1)^{G(x;\psi)}\ln(\theta+1) + 1\right]. \tag{2}$$

This is a previously unknown family with several intriguing properties.(1) The CDF and PDF can be expressed as relatively simple closed forms; namely, it appears that both the analytic and numerical solutions are easier than those of many existing families. Secondly, the additional parameter θ can be used to modulate distributions by an extra weight without imposing any substantial mathematical complexity. Finally, as $\theta \to 0$, the proposed family reduces to the baseline distribution, and it is still interpretable and related with the standard models.

The main objectives of this paper are: (i) to formulate and establish characteristics of the new family, e.g., moments, quantiles, reliability measures in a rigorous manner; (ii) to provide estimation methods with some reference towards maximum likelihood estimation; and (iii) to demonstrate its potential applicability via simulation experiments and empirical studies when compared with other generator families.

The rest of the paper is organised as follows. In Section 2, we exhibit the new family and establish that it is indeed a probability distribution. Section 3 investigates some of its mathematical properties. In Section 4, the new family was used with the two-parameter Weibull distribution to generate the new Power Exponential Weibull distribution. Section 5 discusses estimation methods. The result of a Monte Carlo simulation study is presented in Section 6. Section 7 provides real data applications. Section 8 offers final comments and future research directions.

2. Definition of the PE-G Family and Some of Its Properties

In this section, we formally define the proposed family of probability distributions, which we shall refer to as the *Power Exponential – G (PE-G) family*.

2.1. **Validity as a Probability Distribution.** We verify that $F(x;\psi,\theta)$ defined in (1) is a valid CDF.

(1) Monotonicity: Since $G(x,\psi)$ is non-decreasing in x, and $(\theta+1)^{G(x,\psi)}$ is an increasing function in $G(x,\psi)$, it follows that $F(x;\psi,\theta)$ is a non-decreasing function in x.

(2) Boundary conditions:

$$\lim_{x\to-\infty} F(x;\psi,\theta) = \frac{(\theta+1)^0+0-1}{\theta+1} = 0,$$

$$\lim_{x\to\infty} F(x;\psi,\theta) = \frac{(\theta+1)^1+1-1}{\theta+1} = 1.$$

Hence, $F(x;\psi,\theta)$ satisfies the boundary conditions of a valid CDF.

(3) Normalization: Integrating the PDF in (2),

$$\int_{-\infty}^{\infty} f(x;\psi,\theta)\,dx = \frac{1}{\theta+1}\int_{-\infty}^{\infty}\Big[(\theta+1)^{G(x,\psi)}g(x,\psi)\ln(\theta+1)+g(x,\psi)\Big]dx.$$

With the change of variable $u=G(x,\psi)$, $du=g(x,\psi)dx$, we obtain

$$\int_{-\infty}^{\infty} f(x;\psi,\theta)\,dx = \frac{1}{\theta+1}\int_{0}^{1}\Big[(\theta+1)^{u}\ln(\theta+1)+1\Big]du.$$

Evaluating the integral,

$$\int_0^1 (\theta+1)^u \ln(\theta+1)\, du = (\theta+1)^1 - (\theta+1)^0 = \theta,$$

and

$$\int_0^1 1\, du = 1.$$

Hence,

$$\int_{-\infty}^{\infty} f(x;\psi,\theta)\, dx = \frac{\theta+1}{\theta+1} = 1.$$

Thus, $f(x;\psi,\theta)$ integrates to unity, confirming that it is a valid PDF.

Therefore, the functions (1) and (2) define CDF and PDF of the *PE-G family* respectively, a legitimate family of probability distributions for $\theta > 0$.

2.1.1. *Special Cases and Limiting Behavior.* The PE-G family includes the baseline distribution as a limiting case. As $\theta \to 0$: Using the fact that $\lim_{\theta\to 0}(\theta+1)^{G(x,\psi)} = 1$, we obtain

$$F(x;\psi,0) = G(x,\psi), \quad f(x;\psi,0) = g(x,\psi).$$

Hence, the family reduces to the baseline distribution.

3. Some Statistical Properties of the PE-G Family

Here, we present some of the useful statistical properties of PE-G family which have been derived here. These properties are quantile, moments, moment generating function, entropy survival and hazard functions.

3.1. **Quantile Function.** Let $U \sim \text{Uniform}(0,1)$. The quantile function $Q(u)$ therefore satisfies, by (1),

$$u = \frac{(\theta+1)^{G(x,\psi)} + G(x,\psi) - 1}{\theta+1}. \tag{3}$$

Let us denote $y = G(x,\psi)$ and we have:

$$(\theta+1)^y + y = (\theta+1)u + 1. \tag{4}$$

So, the quantile function of PE-G family is

$$Q(u) = G^{-1}(y), \tag{5}$$

where y is the solution to (4).

3.2. **Survival and Hazard Functions.** The (survival) reliability function is definedas follows

$$S(x;\psi,\theta) = 1 - F(x;\psi,\theta) = \frac{\theta - (\theta+1)^{G(x,\psi)} - G(x,\psi) + 2}{\theta+1}. \tag{6}$$

The hazard rate function is

$$h(x;\psi,\theta)=\frac{f(x;\psi,\theta)}{S(x;\psi,\theta)}=\frac{(\theta+1)^{G(x,\psi)}g(x,\psi)\ln(\theta+1)+g(x,\psi)}{\theta-(\theta+1)^{G(x,\psi)}-G(x,\psi)+2}. \tag{7}$$

The reverse hazard function is

$$r(x;\psi,\theta)=\frac{f(x;\psi,\theta)}{F(x;\psi,\theta)}=\frac{(\theta+1)^{G(x,\psi)}g(x,\psi)\ln(\theta+1)+g(x,\psi)}{(\theta+1)^{G(x,\psi)}+G(x,\psi)-1}. \tag{8}$$

The cumulative hazard function is

$$H(x;\psi,\theta)=-\ln S(x;\psi,\theta)=-\ln\left(\frac{\theta-(\theta+1)^{G(x,\psi)}-G(x,\psi)+2}{\theta+1}\right). \tag{9}$$

3.3. **Moments.** The r-th raw moment μ_r' of the PE-G family is

$$\begin{aligned}\mu_r'=E[x^r]&=\int_{-\infty}^{\infty}x^r\frac{(\theta+1)^{G(x,\psi)}g(x,\psi)\ln(\theta+1)+g(x,\psi)}{\theta+1}dx,\\&=\frac{E_0[x^r]}{\theta+1}+\frac{\ln(\theta+1)}{\theta+1}\int_{-\infty}^{\infty}x^r(\theta+1)^{G(x,\psi)}g(x,\psi)dx.\end{aligned} \tag{10}$$

Where $E_0[x^r]$ is the r-th raw moment of the baseline distribution. To solve the last integral, we use the expansion of the exponential function, which is a convergent series over real numbers, as follows:

$$(\theta+1)^{G(x,\psi)}=e^{G(x,\psi)\ln(\theta+1)}=\sum_{k=0}^{\infty}\frac{\ln(\theta+1)^k}{k!}(G(x,\psi))^k. \tag{11}$$

The integral in (10) can be expressed as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}x^r(\theta+1)^{G(x,\psi)}g(x,\psi)dx&=\sum_{k=0}^{\infty}\frac{\ln(\theta+1)^k}{k!}\int_0^{\infty}x^r(G(x,\psi))^k g(x,\psi)dx,\\&=\sum_{k=0}^{\infty}\frac{\ln(\theta+1)^k}{k!}E(x^r(G(x,\psi))^k)\\&=\sum_{k=0}^{\infty}\frac{\ln(\theta+1)^k}{k!}B_{(r,k)}.\end{aligned} \tag{12}$$

Where $B_{(r,k)}$ is the probability-weighted moment of order (r,k) of the baseline distribution.

So, the r-th raw moment μ_r' of the PE-G family is

$$\mu_r'=\frac{E_0[x^r]}{\theta+1}+\frac{\ln(\theta+1)}{\theta+1}\left(\sum_{k=0}^{\infty}\frac{\ln(\theta+1)^k}{k!}B_{(r,k)}\right). \tag{13}$$

3.4. **Moment Generating Function.** The moment generating function (MGF) $M_X(t)$ of the (PE-G) random variable X, is can derived as

$$M_X(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}\frac{(\theta+1)^{G(x,\psi)}g(x,\psi)\ln(\theta+1)+g(x,\psi)}{\theta+1}dx, \tag{14}$$

Taking the Taylor series for the value e^{tx}, as following

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^r x^r}{r!}. \tag{15}$$

By substituting Equation (15) into (14), then

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r. \tag{16}$$

Where μ'_r is the r-th raw moment μ'_r of the PE-G family in (13).

3.5. **Shannon Entropy.** The Shannon entropy is defined as

$$H(x) = -E[\ln f(x;\theta)]. \tag{17}$$

Substituting the PDF, we obtain

$$H(x) = -E[\ln g(x,\psi)] + \ln(\theta+1) - E\left[\ln\left(1 + (\theta+1)^{G(x,\psi)}\ln(\theta+1)\right)\right]. \tag{18}$$

3.6. **Remarks.**

- The PE-G family provides flexible hazard shapes (increasing, decreasing, bathtub, or upside-down bathtub), depending on the parameter θ and the baseline distribution.
- Moments and MGFs can be expressed in terms of baseline moments and quantile functions, which simplifies their derivation in practice.
- The entropy expression shows that the new parameter θ perturbs the baseline entropy by an additive correction term.
- Simulation from the PE-G family can be carried out via inversion using the quantile function (5).

4. The Power Exponential Weibull (PEW) Distribution

In this section, we demonstrate the flexibility of the Power Exponential-G (PE-G) family by applying it to two-parameter baseline Weibull distribution [11] to obtain a two-parameter new distribution. The newly formulated distribution is designated as the Power Exponential Weibull (PEW) Distribution. We derive the cumulative distribution function (CDF), the probability density function (PDF), the survival function, and the hazard function, and provide simplified expressions for moments and the moment generation function (MGF).

Consider the Weibull distribution with rate parameter $\lambda > 0, \beta > 0$. The baseline CDF and PDF are

$$G(x;\lambda,\beta) = 1 - e^{-(x/\lambda)^\beta}. \tag{19}$$

$$g(x;\lambda,\beta) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-(x/\lambda)^\beta}, \quad x \geq 0. \tag{20}$$

Therefore, through the combination of the CDF for the Weibull distribution and the CDF of the

Power Exponential-G (PE-G) Family in Eq (1), we obtain the CDF for the Power Exponential Weibull (PEW) Distribution as follows.

$$F_{\text{PEW}}(x;\lambda,\beta,\theta)=\frac{(\theta+1)^{1-e^{-(x/\lambda)^{\beta}}}-e^{-(x/\lambda)^{\beta}}}{\theta+1},\quad x\geq 0. \tag{21}$$

The corresponding PDF is derived by differentiating Eq (21) with respect to x, resulting in:

$$f_{\text{PEW}}(x;\lambda,\beta,\theta)=\frac{\beta}{\lambda(\theta+1)}\left(\frac{x}{\lambda}\right)^{\beta-1}e^{-(x/\lambda)^{\beta}}\left[(\theta+1)^{1-e^{-(x/\lambda)^{\beta}}}\ln(\theta+1)+1\right],\quad x\geq 0. \tag{22}$$

The Survival and Hazard Functions for the Power Exponential Weibull (PEW) Distribution are as follows. By definition $S_{\text{PEW}}(x)=1-F_{\text{PEW}}(x)$, hence

$$S_{\text{PEW}}(x;\lambda,\beta,\theta)=\frac{\theta-(\theta+1)^{1-e^{-(x/\lambda)^{\beta}}}+e^{-(x/\lambda)^{\beta}}+1}{\theta+1}. \tag{23}$$

Thus, the hazard function is

$$h_{\text{PEW}}(x;\lambda,\beta,\theta)=\frac{\frac{\beta}{\lambda(\theta+1)}\left(\frac{x}{\lambda}\right)^{\beta-1}e^{-(x/\lambda)^{\beta}}\left[(\theta+1)^{1-e^{-(x/\lambda)^{\beta}}}\ln(\theta+1)+1\right]}{\theta-(\theta+1)^{1-e^{-(x/\lambda)^{\beta}}}+e^{-(x/\lambda)^{\beta}}+1}. \tag{24}$$

The subsequent figures illustrate the graphical representations of various parameter values selected for the Probability Density Function (PDF) and the Cumulative Distribution Function (CDF) of the new PEW distribution.

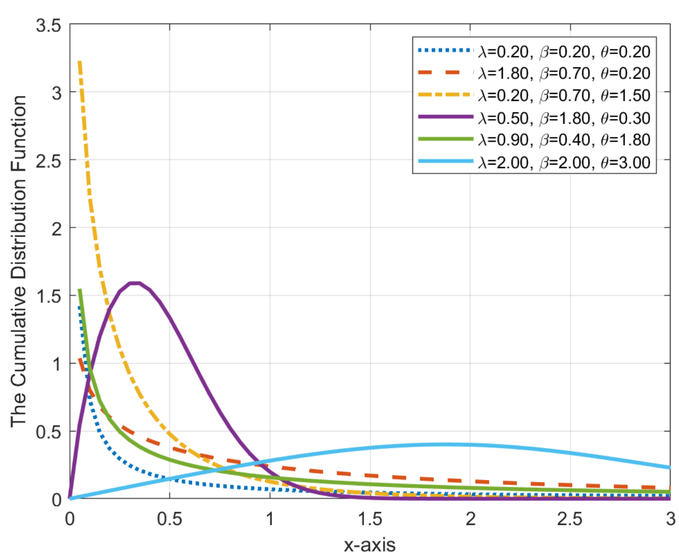

FIGURE 1. Plots of the PDF of the PEW distribution for different values of λ, β and θ.

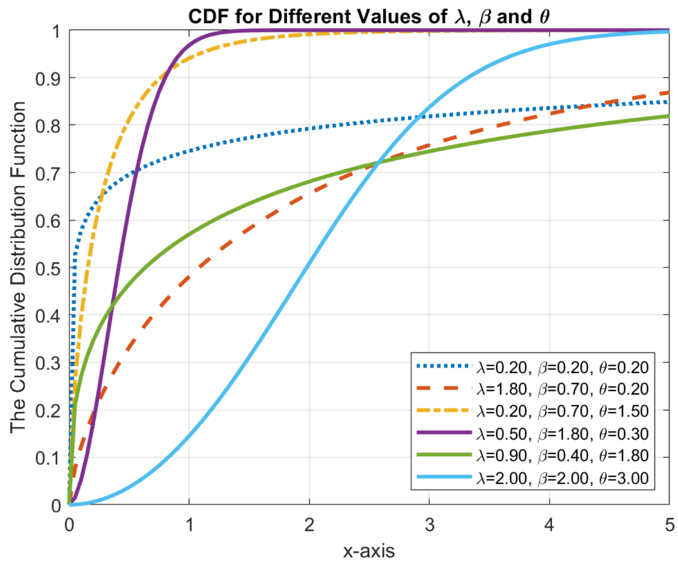

FIGURE 2. Plots of the CDF of the PEW distribution for different values of λ, β and θ.

4.1. **Moments of the PEW Distribution.** Here, we will discuss the moment of the order r about the origin for PEW distribution, by using the general formula 13.

Let X be a random variable has a PEW distribution, and let r be a nonnegative intege, and let k be a nonnegative integer. The probability weighted moment of order (r, k) for Weibull distribution,

$$\begin{aligned}\beta_{r,k} &= \mathbb{E}\left[x^r \left(G(x;\lambda,\beta)\right)^k\right] = \int_0^\infty x^r \left(G(x;\lambda,\beta)\right)^k g(x;\lambda,\beta)\,dx, \\ &= \int_0^\infty x^r \left(1-e^{-\left(\frac{x}{\lambda}\right)^\beta}\right)^k \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^\beta}\,dx.\end{aligned} \tag{25}$$

Using the substitution $t = (x/\lambda)^\beta$ (so $x = \lambda t^{1/\beta}$, $dx = \lambda \frac{1}{\beta} t^{1/\beta - 1} dt$) one obtains the convenient reduction

$$\beta_{r,k} = \lambda^r \int_0^\infty t^{\frac{r}{\beta}} e^{-t}(1-e^{-t})^k dt. \tag{26}$$

Here k is a nonnegative integer ($k \in \{0, 1, 2, \ldots\}$) then expand $(1-e^{-t})^k$ by the binomial theorem:

$$(1-e^{-t})^k = \sum_{j=0}^{s} (-1)^j \binom{k}{j} e^{-jt}.$$

which is a convergent series. Hence,

$$\begin{aligned}\int_0^\infty t^{\frac{r}{\beta}} e^{-t}(1-e^{-t})^k dt &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \int_0^\infty t^{\frac{r}{\beta}} e^{-(j+1)t} dt, \\ &= \Gamma\left(\frac{r}{\beta}+1\right) \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^{-\left(\frac{r}{\beta}+1\right)}.\end{aligned} \tag{27}$$

So for integer k,

$$\beta_{r,k} = \lambda^r \Gamma\left(1+\frac{r}{\beta}\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+1)^{-(1+\frac{r}{\beta})}. \tag{28}$$

And the moment of the order r about the origin for Weibull distribution is $E_0[x^r] = \lambda^r\Gamma\left(1+\frac{r}{\beta}\right)$, then by Eq. (13), the r-th moment of PEW distribution is

$$\mu_r' = \frac{\lambda^r\Gamma\left(1+\frac{r}{\beta}\right)}{\theta+1} + \frac{\ln(\theta+1)}{\theta+1}\left(\sum_{k=0}^{\infty}\left(\frac{\ln(\theta+1)^k}{k!}\lambda^r\Gamma\left(1+\frac{r}{\beta}\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+1)^{-(1+\frac{r}{\beta})}\right)\right). \tag{29}$$

4.2. **Moment Generating Function of the PEW Distribution.** Let $X \sim \text{PEW}(\lambda,\beta,\theta)$ with PDF $f_{\text{PEW}}(x)$ as defined in Eq. (22). The moment generating function is

$$M_X(t) = \mathbb{E}\big[e^{tx}\big] = \int_0^\infty e^{tx}\, f_{\text{PEW}}(x)\,dx.$$

Introduce the change of variable

$$u = \exp\Big(-\big(\tfrac{x}{\lambda}\big)^\beta\Big) \in (0,1), \quad x = \lambda(-\ln u)^{1/\beta}, \quad dx = -\frac{\lambda}{\beta}(-\ln u)^{\frac{1}{\beta}-1}\frac{du}{u}.$$

With this transform, the PEW density simplifies so that $f_{\text{PEW}}(x)\,dx$ reduces to

$$f_{\text{PEW}}(x)\,dx = -\frac{1}{\theta+1}\Big((\theta+1)^{1-u}\ln(\theta+1)+1\Big)\,du. \tag{30}$$

Hence

$$M_X(t) = \frac{1}{\theta+1}\int_0^1 \exp\{t\lambda(-\ln u)^{1/\beta}\}\Big((\theta+1)^{1-u}\ln(\theta+1)+1\Big)\,du. \tag{31}$$

Equivalently, with $u=e^{-z}$ (so $z\in(0,\infty)$ and $du=-e^{-z}dz$), we obtain the more convenient form

$$M_X(t) = \int_0^\infty e^{-z}\,e^{t\lambda z^{1/\beta}}\left[\ln(\theta+1)\,e^{-e^{-z}\ln(\theta+1)} + \frac{1}{\theta+1}\right]dz. \tag{32}$$

Series representation. Expanding $e^{t\lambda z^{1/\beta}} = \sum_{r=0}^\infty \frac{(t\lambda)^r}{r!}\,z^{r/\beta}$ and $e^{-e^{-z}\ln(\theta+1)} = \sum_{m=0}^\infty \frac{(-\ln(\theta+1))^m}{m!}\,e^{-mz}$, the integrals in (32) yield

$$\int_0^\infty e^{-(m+1)z}\,z^{r/\beta}\,dz = \frac{\Gamma(1+\frac{r}{\beta})}{(m+1)^{1+r/\beta}}.$$

Therefore,

$$M_X(t) = \sum_{r=0}^\infty \frac{\Gamma(1+\frac{r}{\beta})}{r!}(t\lambda)^r\left[\frac{1}{\theta+1} + \ln(\theta+1)\sum_{m=0}^\infty\frac{(-\ln(\theta+1))^m}{m!\,(m+1)^{1+r/\beta}}\right]. \tag{33}$$

The double series in (33) converges absolutely for all t in the domain where $M_X(t)$ exists (see the discussion below). Equation (33) is convenient for numerical evaluation: truncating at moderate r, m provides accurate approximations.

Existence domain. Because the PEW tail is Weibull-type (i.e., behaves like $\exp\{-(x/\lambda)^\beta\}$ up to a bounded factor), the existence of $M_X(t)$ for $t>0$ follows the usual Weibull conditions:

$$\begin{cases} \beta > 1: & M_X(t) \text{ exists for all } t \in \mathbb{R}, \\ \beta = 1: & M_X(t) \text{ exists for } t < 1/\lambda, \\ 0 < \beta < 1: & M_X(t) \text{ exists for } t \leq 0 \text{ (Laplace transform region)}. \end{cases}$$

In all cases, $M_X(0) = 1$.

5. Maximum Likelihood Estimations of PEW Distribution Parameters

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Power Exponential Weibull (PEW) distribution with cumulative distribution function (CDF) and probability density function (PDF) given by

$$f_{\text{PEW}}(x;\lambda,\beta,\theta) = \frac{\beta}{\lambda(\theta+1)}\left(\frac{x}{\lambda}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\left[(\theta+1)^{1-\exp[-(x/\lambda)^{\beta}]}\ln(\theta+1)+1\right], \qquad x \geq 0, \tag{34}$$

where $\lambda > 0$, $\beta > 0$, and $\theta > 0$ are the scale and shape parameters, respectively.

Log-likelihood Function. Given a random sample $\{x_1, x_2, \ldots, x_n\}$, the likelihood function of the PEW distribution is

$$L(\lambda,\beta,\theta) = \prod_{i=1}^{n} f_{\text{PEW}}(x_i;\lambda,\beta,\theta), \tag{35}$$

and the corresponding log-likelihood function is expressed as

$$\begin{aligned}\ell(\lambda,\beta,\theta) = n(\ln\beta - \ln\lambda - \ln(\theta+1)) + (\beta-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\beta} \\ + \sum_{i=1}^{n}\ln\left[(\theta+1)^{1-\exp[-(x_i/\lambda)^{\beta}]}\ln(\theta+1)+1\right].\end{aligned} \tag{36}$$

Maximum Likelihood Estimation. The maximum likelihood estimates (MLEs) $(\hat{\lambda}, \hat{\beta}, \hat{\theta})$ are obtained by solving the system of nonlinear equations

$$\frac{\partial \ell}{\partial \lambda} = 0, \quad \frac{\partial \ell}{\partial \beta} = 0, \quad \frac{\partial \ell}{\partial \theta} = 0,$$

where each score function is given by differentiating $\ell(\lambda,\beta,\theta)$ with respect to the corresponding parameter.

Since these equations do not admit closed-form analytical solutions, numerical optimization procedures such as the Newton–Raphson or BFGS algorithms are employed to obtain the MLEs. The observed Fisher information matrix $\mathcal{I}(\hat{\lambda}, \hat{\beta}, \hat{\theta})$ is computed as the negative of the Hessian matrix of ℓ evaluated at the MLEs. The asymptotic variance–covariance matrix of the estimators is then given by $\mathcal{I}^{-1}(\hat{\lambda}, \hat{\beta}, \hat{\theta})$, and the corresponding standard errors are obtained from its diagonal entries.

Statistical Properties. Under standard regularity conditions, the MLEs of the parameters of the PEW distribution are consistent, asymptotically efficient, and asymptotically normally distributed. That is,

$$\sqrt{n}\left(\begin{bmatrix}\hat{\lambda}\\ \hat{\beta}\\ \hat{\theta}\end{bmatrix}-\begin{bmatrix}\lambda\\ \beta\\ \theta\end{bmatrix}\right)\xrightarrow{d} N_3\big(0,\mathcal{I}^{-1}(\lambda,\beta,\theta)\big)\,,$$

where $\mathcal{I}(\lambda,\beta,\theta)$ is the Fisher information matrix per observation. All numerical estimations and optimizations reported in this paper were implemented in MATLAB R2022b.

6. Simulation Study

In this section, we implement a Monte Carlo simulation to evaluate the performance of the maximum likelihood estimators (MLEs) for parameters of PEW distribution. In the simulation we can study the bias, mean squared error (MSE) and efficiency of estimators when changing parameter settings and sample sizes.

6.1. **Simulation Design.** The simulation experiment is performed as follows:

(1) Specify the replication size N of samples and sample size n.
(2) Setting the values of parameters (λ,β,θ) for all Scenario j.
(3) Generate N samples of size n from the PEW distribution using the inversion method through (5).
(4) Calculate the MLEs $(\hat{\lambda},\hat{\beta},\hat{\theta})$ with a numerical optimizer for every sample.
(5) Add the estimates by all replications.

6.2. **Performance Measures.** The following performance measures are computed over N replications:

Mean Estimate.

$$\overline{\lambda}_j=\frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_j^{(i)},\quad \overline{\beta}_j=\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_j^{(i)},\quad \overline{\theta}_j=\frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_j^{(i)}.$$

Bias.

$$\text{Bias}(\hat{\lambda}_j)=\overline{\lambda}_j-\lambda_j,\quad \text{Bias}(\hat{\beta}_j)=\overline{\beta}_j-\beta_j,\quad \text{Bias}(\hat{\theta}_j)=\overline{\theta}_j-\theta_j.$$

Mean Squared Error (MSE).

$$\text{MSE}(\hat{\lambda}_j)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\lambda}_j^{(i)}-\lambda_j\right)^2,\quad \text{MSE}(\hat{\beta}_j)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\beta}_j^{(i)}-\beta_j\right)^2,\quad \text{MSE}(\hat{\theta}_j)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_j^{(i)}-\theta_j\right)^2.$$

6.3. **Selecting parameter values and simulation results.** Consider the four scenarios, Scenario 1: $(\lambda,\beta,\theta)=(2,3,1.5)$, Scenario 2: $(\lambda,\beta,\theta)=(0.8,2,0.9)$, Scenario 3:$(\lambda,\beta,\theta)=(3,1.2,2.2)$ and Scenario

4: $(\lambda, \beta, \theta) = (1.5, 4, 1.1)$. For $n = 30, 50, 80, 100, 150, 200, 300, 500, 800, 1000$, and $N = 1000$ replications, we compute the mean estimates, biases, and MSEs for $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\theta}$.

The results can be summarized in the following Tables 1, 2, 3 and 4, also in the Figures 3 and 4.

Table 1. Monte Carlo (Scenario 1): true $(\lambda, \beta, \theta) = (2, 3, 1.5)$.

n	Mean λ	Bias λ	MSE λ	Mean β	Bias β	MSE β	Mean θ	Bias θ	MSE θ
30	1.9556	-0.0444	0.0529	3.0399	0.0399	0.3532	5.6318	4.1318	205.5931
50	1.9541	-0.0459	0.0474	2.989	-0.011	0.2386	5.2159	3.7159	180.6107
80	1.9391	-0.0609	0.0505	2.9552	-0.0448	0.2225	6.1441	4.6441	224.193
100	1.9548	-0.0452	0.0433	2.9635	-0.0365	0.1798	5.2312	3.7312	184.3133
150	1.9454	-0.0546	0.0396	2.925	-0.075	0.1458	4.4371	2.9371	89.4549
200	1.9483	-0.0517	0.0364	2.9376	-0.0624	0.1375	4.3296	2.8296	78.0594
300	1.9663	-0.0337	0.0293	2.9392	-0.0608	0.1093	3.4614	1.9614	48.3654
500	1.9636	-0.0364	0.0259	2.9301	-0.0699	0.0924	3.1777	1.6777	27.907
800	1.9733	-0.0267	0.0176	2.95	-0.05	0.0631	2.5408	1.0408	14.6998
1000	1.9917	-0.0083	0.0138	2.9855	-0.0145	0.0476	2.0691	0.5691	9.1757

Table 2. Monte Carlo (Scenario 2): true $(\lambda, \beta, \theta) = (0.8, 2, 0.9)$.

n	Mean λ	Bias λ	MSE λ	Mean β	Bias β	MSE β	Mean θ	Bias θ	MSE θ
30	0.7646	-0.0354	0.016	2.0209	0.0209	0.1326	3.6444	2.7444	71.2195
50	0.7557	-0.0443	0.0159	1.9476	-0.0524	0.0972	5.1931	4.2931	185.9215
80	0.7492	-0.0508	0.0156	1.9145	-0.0855	0.0797	4.6827	3.7827	124.3122
100	0.7619	-0.0381	0.0123	1.9495	-0.0505	0.061	3.6863	2.7863	84.2532
150	0.7571	-0.0429	0.0128	1.9303	-0.0697	0.0659	3.7847	2.8847	81.3677
200	0.763	-0.037	0.0115	1.935	-0.065	0.0578	3.5059	2.6059	71.2822
300	0.7713	-0.0287	0.0087	1.9479	-0.0521	0.0406	2.5668	1.6668	29.388
500	0.7672	-0.0328	0.0092	1.941	-0.059	0.0401	2.9513	2.0513	46.9663
800	0.776	-0.024	0.0057	1.9526	-0.0474	0.0251	2.0023	1.1023	17.7805
1000	0.7878	-0.0122	0.0043	1.975	-0.025	0.0182	1.5201	0.6201	9.7176

Table 3. Monte Carlo (Scenario 3): true $(\lambda, \beta, \theta) = (3, 1.2, 2.2)$.

n	Mean λ	Bias λ	MSE λ	Mean β	Bias β	MSE β	Mean θ	Bias θ	MSE θ
30	3.0952	0.0952	0.6797	1.2626	0.0626	0.0652	5.9499	3.7499	250.8881
50	2.9349	-0.0651	0.6078	1.209	0.009	0.0455	7.479	5.279	315.606
80	2.9505	-0.0495	0.5708	1.191	-0.009	0.0326	6.8743	4.6743	247.2792
100	2.9549	-0.0451	0.5868	1.1915	-0.0085	0.0341	7.1246	4.9246	285.655
150	2.9186	-0.0814	0.5262	1.1827	-0.0173	0.0288	6.2801	4.0801	169.206
200	2.9806	-0.0194	0.436	1.1906	-0.0094	0.0229	5.0988	2.8988	104.1694
300	2.9311	-0.0689	0.3837	1.1808	-0.0192	0.0195	4.6795	2.4795	65.8516
500	2.9546	-0.0454	0.2925	1.1833	-0.0167	0.0142	3.7562	1.5562	28.4988
800	3.0067	0.0067	0.2433	1.1944	-0.0056	0.011	3.1221	0.9221	17.2785
1000	3.0102	0.0102	0.2118	1.1964	-0.0036	0.0098	2.9609	0.7609	14.1479

Table 4. Monte Carlo (Scenario 4): true $(\lambda, \beta, \theta) = (1.5, 4, 1.1)$.

n	Mean λ	Bias λ	MSE λ	Mean β	Bias β	MSE β	Mean θ	Bias θ	MSE θ
30	1.4663	-0.0337	0.018	4.0828	0.0828	0.6739	4.2043	3.1043	111.445
50	1.4589	-0.0411	0.0166	3.928	-0.072	0.3956	4.7547	3.6547	135.5318
80	1.4571	-0.0429	0.0165	3.9043	-0.0957	0.3218	5.1677	4.0677	162.7011
100	1.4647	-0.0353	0.0146	3.9104	-0.0896	0.3059	4.3249	3.2249	120.8009
150	1.4623	-0.0377	0.0136	3.8738	-0.1262	0.2493	4.4439	3.3439	125.8317
200	1.4728	-0.0272	0.0106	3.9136	-0.0864	0.1985	3.2032	2.1032	55.4906
300	1.4635	-0.0365	0.0114	3.868	-0.132	0.2108	3.6812	2.5812	61.7893
500	1.4701	-0.0299	0.009	3.886	-0.114	0.1581	2.933	1.833	33.9569
800	1.4835	-0.0165	0.0053	3.9359	-0.0641	0.0983	2.0744	0.9744	18.3269
1000	1.4856	-0.0144	0.0045	3.9425	-0.0575	0.0815	1.8175	0.7175	10.0112

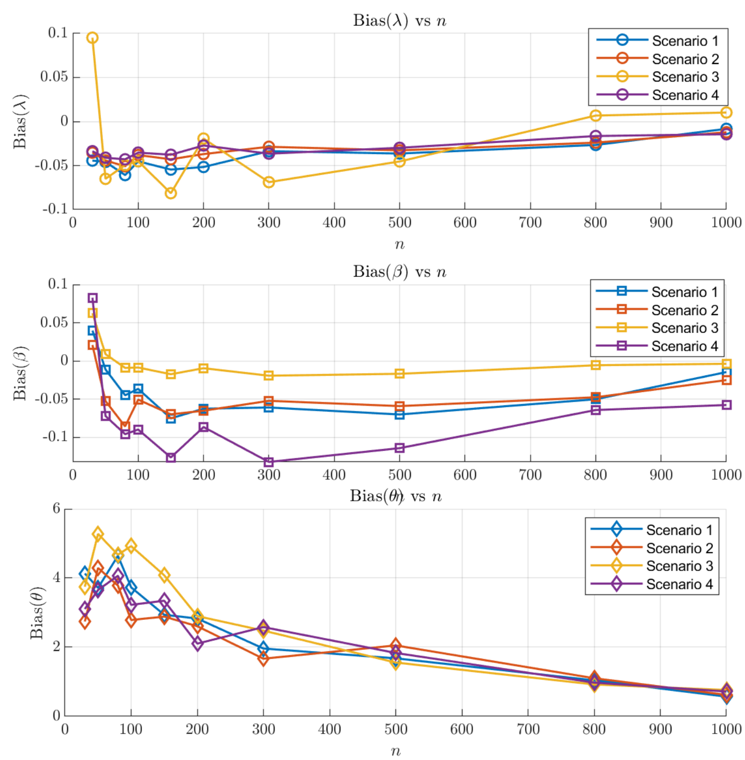

Figure 3. Bias of the estimators as a function of sample size.

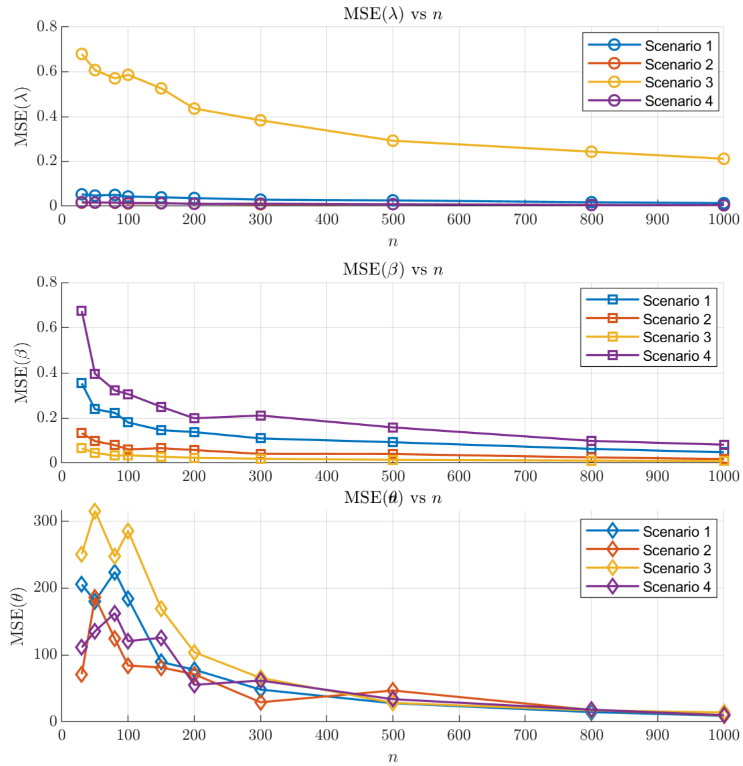

FIGURE 4. MSE of the estimators as a function of sample size.

6.4. **Discussion of Simulation Results.** The Monte Carlo results give a summary of the finite-sample performance of the MLEs of PEW distribution parameters. The overall behavior of the estimators follows the predicted asymptotic properties, but with apparent improvement as sample size grows.

Throughout all scenarios, the biases of $\hat{\lambda}$,$\hat{\beta}$, and $\hat{\theta}$ decrease monotonically for larger n until they become insignificant for moderate to large sample sizes. Although the estimator of θ exhibits more bias in smaller samples, it appears to quickly dissipate as n becomes larger, an observation that is consistent with Figure 3.

The MSEs of estimators also decrease as the sample size grows (Figure 4), which indicates an improvement in efficiency. These are used below for Monte Carlo simulations (Tables 1,2), which show that the estimators of λ and β stabilize rather rapidly, while the MSE of $\hat{\theta}$ declines at a slower pace but reaches vanishing values as $n \to \infty$.

Differences among the four situations appear mostly in small sample sizes, specially when true parameters are larger that increases variability, as expected. Nevertheless, such dissimilarities decrease as n increases which indicate that the MLEs are fairly robust under different parameter settings.

To conclude, the simulation results confirm that the MLEs for PEW distribution parameters are consistent and asymptotically unbiased, and also more efficient as sample sizes increase, while being well-suited for statistical applications.

7. Applications to Real Datasets

In this section, we empirically show the flexibility of our PEW distribution by analyzing three real-world datasets. Namely, we want to contrast the PEW distribution with the competing models below and their PDFs (definitions) and CDFs. The model selection is based on a full set of statistical tests including information theoretic criterions as well as goodness-of-fit tests. The difference is formalised in terms of the so-called Akaike Information Criterion (AIC) [12], $\text{AIC} = 2k - 2\ln(\mathcal{L})$, which measures the Kullback-Leibler divergence between a true model and its approximations. The Bayesian Information Criterion (BIC) [13], $\text{BIC} = k\ln(n) - 2\ln(\mathcal{L})$, yields consistent model selection with a greater degree of complexity penalty. Additional refinements include the Consistent AIC (CAIC) [14], $\text{CAIC} = -2\ln(\mathcal{L}) + k[\ln(n)+1]$, and the Hannan-Quinn Criterion (HQC) [15], $\text{HQC} = -2\ln(\mathcal{L}) + 2k\ln(\ln(n))$, which bridge AIC and BIC properties. For distributional adequacy assessment, the Kolmogorov-Smirnov statistiC [16], $D_n = \sup_x |F_n(x) - F(x)|$, measures maximum CDF discrepancy, while the Anderson-Darling test [17], $A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\ln F(X_i) + \ln(1 - F(X_{n-i+1}))]$, provides enhanced tail sensitivity. The Cramér-von Mises statistic [18], $W^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(X_i) - \frac{2i-1}{2n}\right]^2$, offers balanced fit assessment across the distribution support. Parameter uncertainty is quantified via standard errors derived from the observed Fisher information matrix [19] $\text{SE}(\hat{\theta}i) = \sqrt{[\mathcal{I}^{-1}(\hat{\theta})]ii}$, ensuring comprehensive model evaluation from both information-theoretic and goodness-of-fit perspectives. The numerical results in this section were obtained using MATLAB R2022b. The models for comparison are as follows:

- Weibull Distribution (W): The probability density function (PDF) and cumulative distribution function (CDF) of the Weibull distribution are given by [11]:

$$f(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right], \quad x > 0. \tag{37}$$

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right], \tag{38}$$

  where $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter.

- Exponentiated Weibull Distribution (EW): The exponentiated Weibull distribution has the following PDF and CDF [4]:

$$f(x) = \theta\,\frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\right\}^{\theta-1}, \quad x > 0. \tag{39}$$

$$F(x) = \left\{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\right\}^{\theta}, \tag{40}$$

  where $\lambda > 0$ is the scale parameter, $\beta > 0$ is the shape parameter, and $\theta > 0$ is the exponentiation parameter.

- Kumaraswamy Weibull Distribution (KW): The Kumaraswamy-Weibull distribution is defined by the following PDF and CDF [20]:

$$f(x) = a\,\theta\,\frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\right\}^{\theta-1}$$
$$\times\left[1-\left\{1-\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\right\}^{\theta}\right]^{a-1}, \quad x>0. \tag{41}$$

$$F(x) = 1-\left[1-\left\{1-\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\right\}^{\theta}\right]^{a}, \tag{42}$$

where $\lambda > 0$, $\beta > 0$, $\theta > 0$, and $a > 0$ are parameters.

- Transmuted Weibull Distribution (TW): The transmuted Weibull distribution has the following PDF and CDF [21]:

$$f(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]\left[(1+\theta)-2\theta F_{\text{Weibull}}(x)\right], \quad x>0. \tag{43}$$

$$F(x) = (1+\theta)F_{\text{Weibull}}(x) - \theta[F_{\text{Weibull}}(x)]^2, \tag{44}$$

where $F_{\text{Weibull}}(x) = 1-\exp\left[-\left(\frac{x}{\lambda}\right)^{\beta}\right]$ and $\lambda > 0$, $\beta > 0$, $-1 \le \theta \le 1$.

- Modified Weibull Distribution (MW): The modified Weibull distribution is characterized by the following PDF and CDF [22]:

$$f(x) = (\theta\beta x^{\beta-1}+\lambda)\exp\left(-\theta x^{\beta}-\lambda x\right), \quad x>0. \tag{45}$$

$$F(x) = 1-\exp\left(-\theta x^{\beta}-\lambda x\right), \tag{46}$$

where $\lambda > 0$, $\beta > 0$, and $\theta > 0$ are parameters.

7.1. **First Dataset.** The first dataset details the fatigue fracture lifespan of Kevlar 373/epoxy under a constant pressure at a 90% stress level until complete failure occurred. This data was sourced from Abdul-Moniem and Seham (2015) [23]. The first dataset is; 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.565, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.912, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.257, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.488, 1.5728, 1.5733, 1.7083, 1.7263, 1.746, 1.763, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558,2.0048, 2.0408, 2.0903, 2.1093, 2.133, 2.21, 2.246, 2.2878, 2.3203, 2.347, 2.3513, 2.4951, 2.526, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.096.

Table 5 presents the maximum likelihood estimates of the parameters, along with their standard errors in parentheses, and Table 6 provides the criteria for comparison. Figure 5 represent the empirical density and the cumulative density of the data considered. Figures 6 and 7 represent the P-P plot and the Q-Q plot, respectively, for all distributions used for comparison at the first dataset.

TABLE 5. Parameter estimates with standard errors (The first dataset).

Model	λ	β	θ	a
W	2.1328 (0.1945)	1.3256 (0.1138)		
EW	1.6409 (0.5871)	1.1013 (0.2629)	1.4426 (0.6436)	
KW	5.27 (15.4796)	0.7919 (0.9041)	2.0021 (2.4976)	6.4133 (29.8024)
TW	2.9417 (0.5143)	1.4311 (0.1277)	0.7114 (0.3017)	
MW	0.0000 (0.3207)	1.3256 (0.2635)	0.3664 (0.3305)	
PEW	0.6952 (0.2699)	0.7818 (0.1390)	111.6686 (182.96)	

TABLE 6. Model comparison (The first dataset).

Model	logLik	AIC	BIC	CAIC	HQC	KS	p_{KS}	AD	p_{AD}	W^2
W	-122.5247	249.0494	253.7108	255.7108	250.9123	0.1099	0.2953	0.7889	0.0408	0.1354
EW	-122.1636	250.3272	257.3194	260.3194	253.1216	0.0988	0.4217	0.6563	0.0867	0.1093
KW	-122.0635	252.1270	261.4499	265.4499	255.8528	0.0973	0.4409	0.6360	0.0974	0.1055
TW	-121.7353	249.4706	256.4628	259.4628	252.2650	0.0958	0.4600	0.6028	0.1176	0.1000
MW	-122.5247	251.0494	258.0416	261.0416	253.8438	0.1099	0.2953	0.7889	0.0408	0.1354
PEW	-120.6149	247.2298	254.2220	257.2220	250.0242	0.0879	0.5698	0.4003	0.3619	0.0672

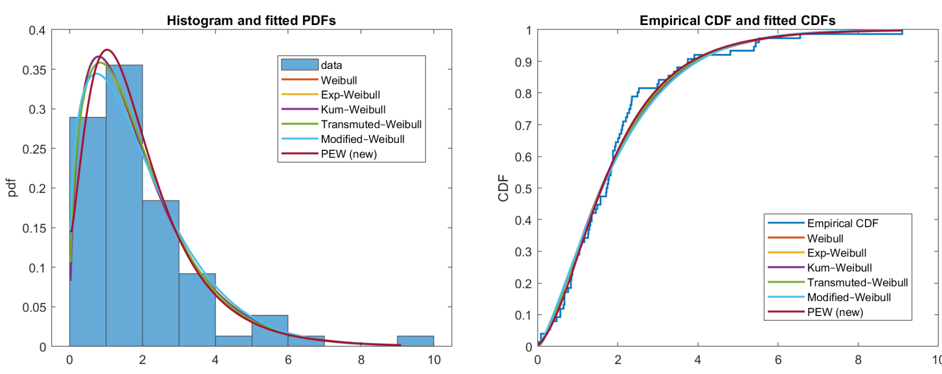

FIGURE 5. Fitted PEW and competing models to the first dataset

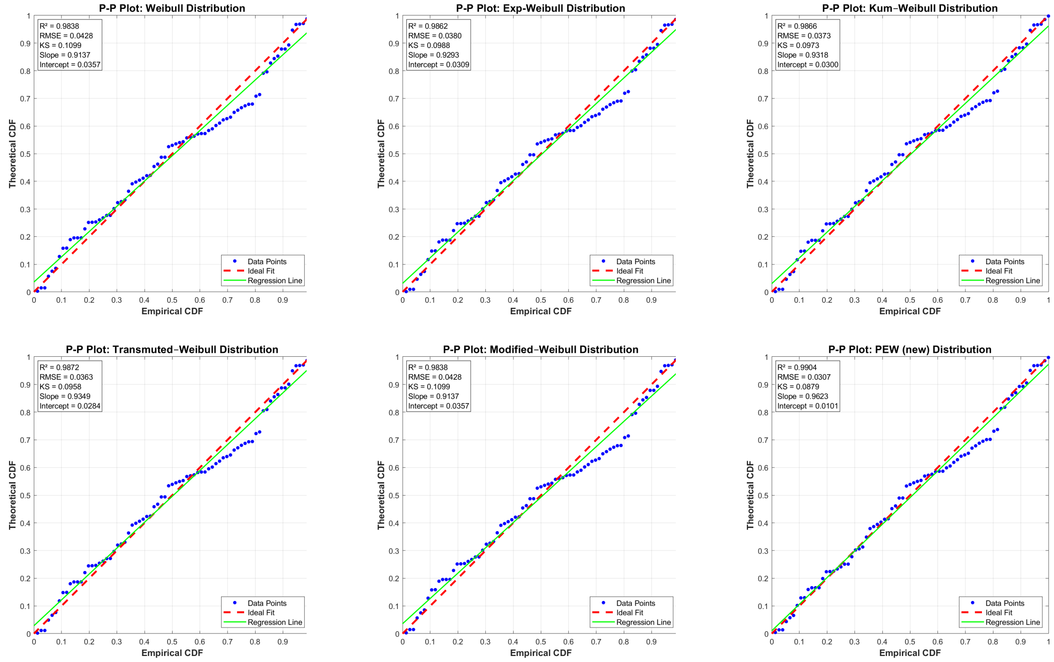

Figure 6. Comparison of P–P Plots for distributions in Modelling First Dataset

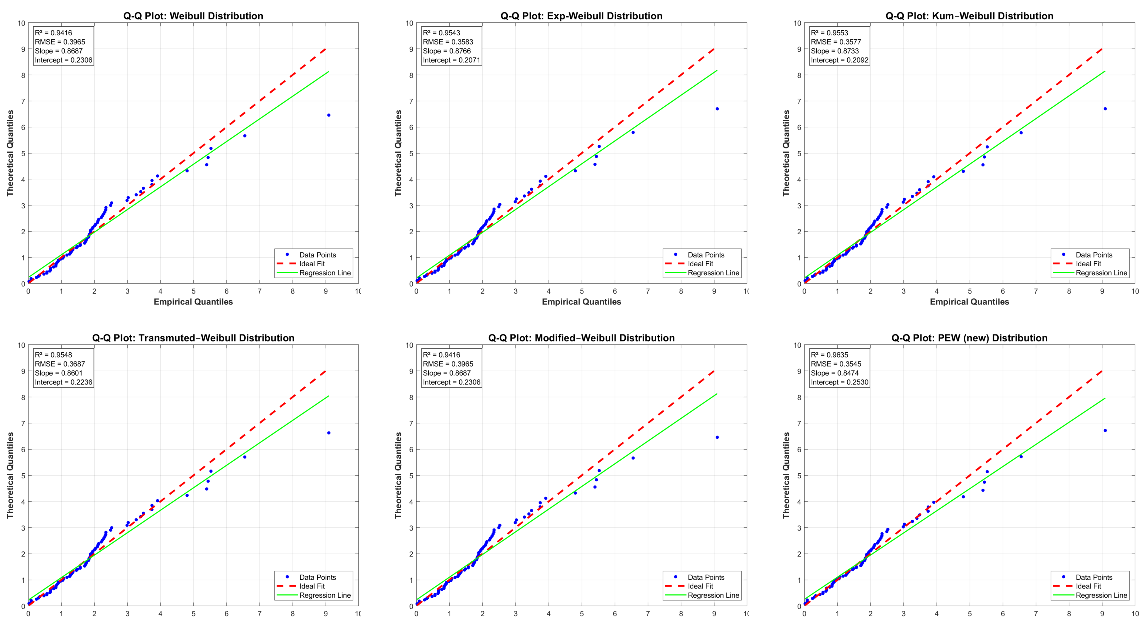

Figure 7. Comparison of Q-Q Plots for distributions in Modelling First Dataset

7.2. **Second Dataset.** The second dataset consists of 63 observations on the strength of 1.5 cm glass fibres, collected by staff at the UK National Physical Laboratory. The second dataset is [24]: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

Table 7 presents the maximum likelihood estimates of the parameters, along with their standard errors in parentheses, and Table 8 provides the criteria for comparison. Figure 8 represent the empirical density and the cumulative density of the data considered. Figures 9 and 10 represent the P-P plot and the Q-Q plot, respectively, for all distributions used for comparison at the second dataset.

Table 7. Parameter estimates with standard errors (The second dataset).

Model	λ	β	θ	a
W	1.6281 (0.0371)	5.7807 (0.5761)		
EW	1.7181 (0.0861)	7.2846 (1.7069)	0.6712 (0.2489)	
KW	1.3665 (0.0544)	6.6178 (0.2069)	0.6468 (0.0609)	0.2569 (0.0699)
TW	1.5484 (0.0565)	5.1501 (0.6683)	-0.5011 (0.2745)	
MW	0.0311 (0.0437)	6.3808 (0.9753)	0.0407 (0.0249)	
PEW	1.4298 (0.0875)	4.4461 (0.7215)	14.4444 (15.3454)	

Table 8. Model comparison (The second dataset).

Model	logLik	AIC	BIC	CAIC	HQC	KS	p_{KS}	AD	p_{AD}	W^2
W	-15.2068	34.4137	38.7000	40.7000	36.0995	0.1522	0.0969	1.2408	0.0031	0.2151
EW	-14.6755	35.3510	41.7804	44.7804	37.8798	0.1462	0.1221	1.0866	0.0075	0.1976
KW	-14.0788	36.1576	44.7301	48.7301	39.5292	0.1307	0.2129	0.9773	0.0140	0.1707
TW	-14.3360	34.6720	41.1014	44.1014	37.2007	0.1374	0.1689	1.0359	0.0101	0.1691
MW	-14.8947	35.7893	42.2187	45.2187	38.3181	0.1331	0.1959	0.8350	0.0314	0.1540
PEW	-13.2400	32.4800	38.9094	41.9094	35.0087	0.1202	0.2980	0.8136	0.0355	0.1253

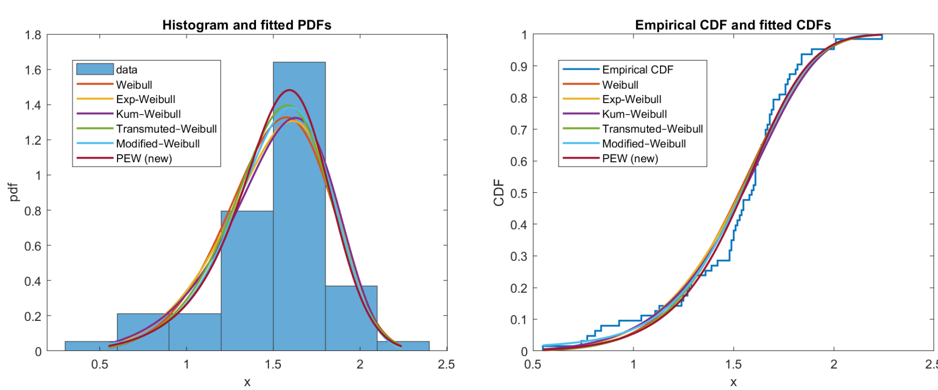

Figure 8. Fitted PEW and competing models to the second dataset

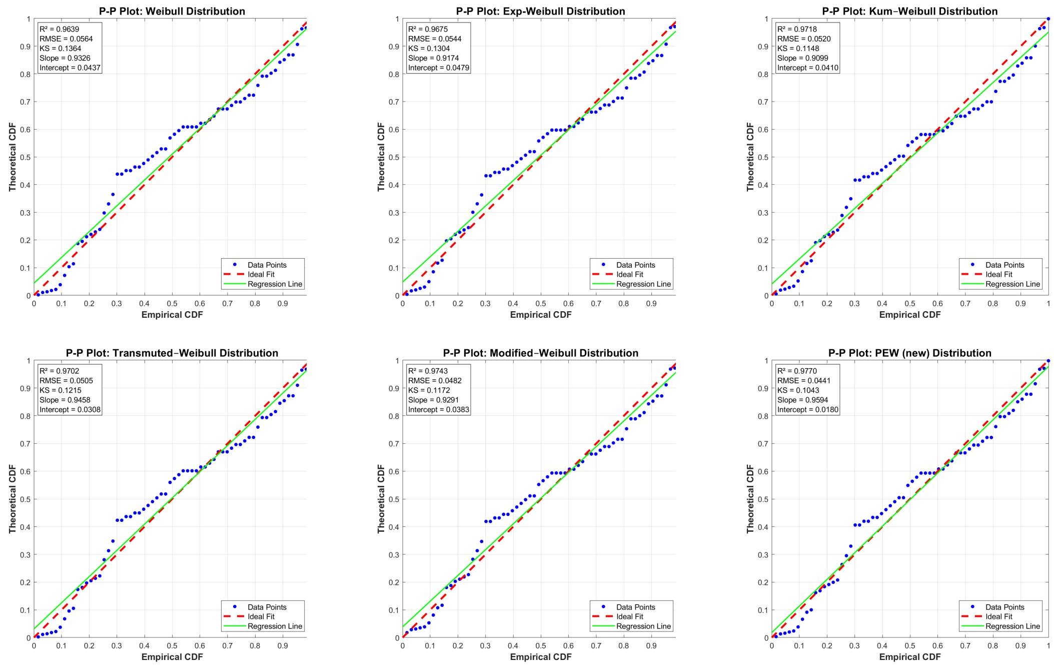

FIGURE 9. Comparison of P–P Plots for distributions in Modelling Second Dataset

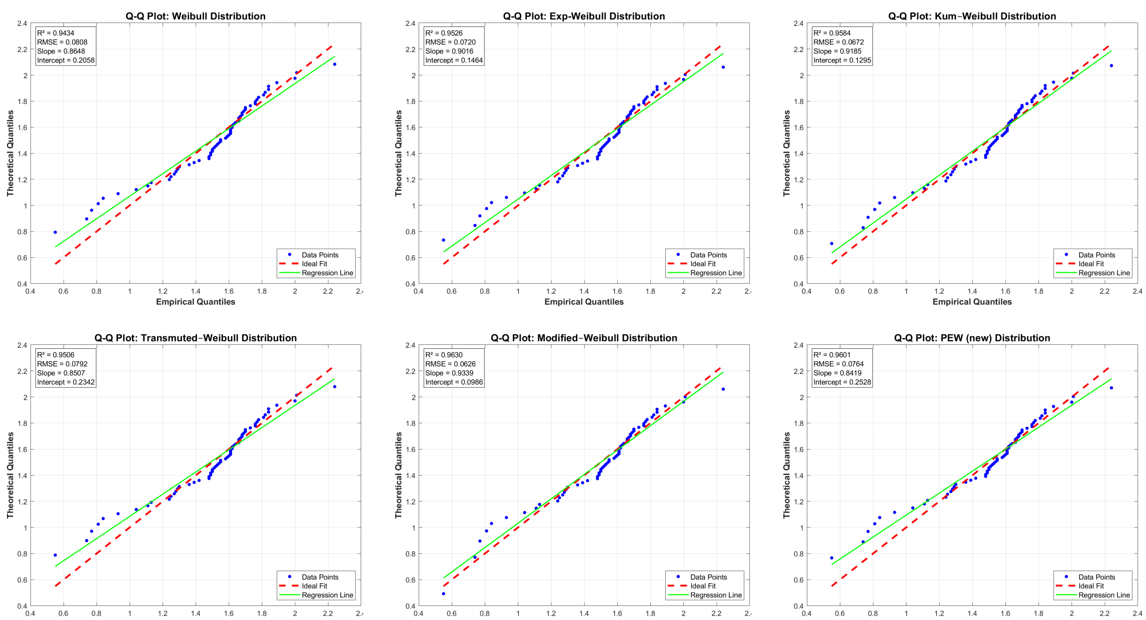

FIGURE 10. Comparison of Q-Q Plots for distributions in Modelling Second Dataset

7.3. **Third Dataset.** The third Dataset is the survival times (in days) of 72 guinea pigs that had been infected with virulent tubercle bacilli. The observations are recorded and presented. For previous studies of this dataset we refer you to [25]. The third Dataset is presented as: 10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

Table 9 presents the maximum likelihood estimates of the parameters, along with their standard errors in parentheses, and Table 10 provides the criteria for comparison. Figure 11 represent the empirical density and the cumulative density of the data considered. Figures 12 and 13 represent the P-P plot and the Q-Q plot, respectively, for all distributions used for comparison at the third dataset.

TABLE 9. Parameter estimates with standard errors (Third Dataset).

Model	λ	β	θ	a
W	199.6021 (13.6302)	1.8254 (0.1587)		
EW	112.8998 (46.3016)	1.1604 (0.3090)	2.6537 (1.5361)	
KW	130.4724 (119.5130)	0.9912 (1.0443)	3.1103 (3.8684)	1.7319 (5.5274)
TW	244.7427 (29.2898)	1.9791 (0.1732)	0.6412 (0.2951)	
MW	0.0057 (0.0007)	0.3051 (0.0000)	0.0000 (0.0000)	
PEW	78.1176 (15.9400)	1.0065 (0.1284)	199.9998 (231.8206)	

TABLE 10. Model comparison (Third Dataset).

Model	logLik	AIC	BIC	CAIC	HQC	KS	p_{KS}	AD	p_{AD}	W^2
W	-427.3621	858.7241	863.2775	865.2775	860.5368	0.1048	0.3814	1.0072	0.0118	0.1680
EW	-425.6561	857.3122	864.1422	867.1422	860.0312	0.0891	0.5854	0.5382	0.1678	0.0863
KW	-425.6379	859.2757	868.3824	872.3824	862.9011	0.0890	0.5876	0.5354	0.1705	0.0858
TW	-426.3163	858.6325	865.4625	868.4625	861.3516	0.0978	0.4678	0.7752	0.0441	0.1265
MW	-444.6132	895.2264	902.0564	905.0564	897.9455	0.2946	0.0000	7.2686	0.0000	1.4050
PEW	-425.0133	856.0265	862.8565	865.8565	858.7456	0.0830	0.6724	0.5442	0.1621	0.0890

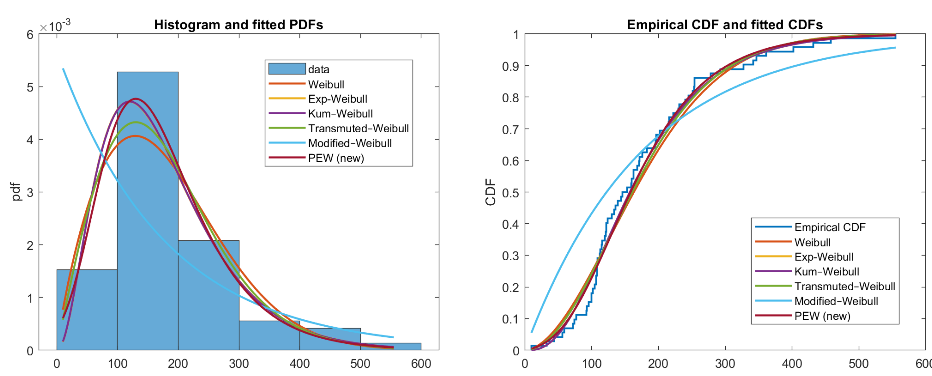

FIGURE 11. Fitted PEW and competing models to the third dataset

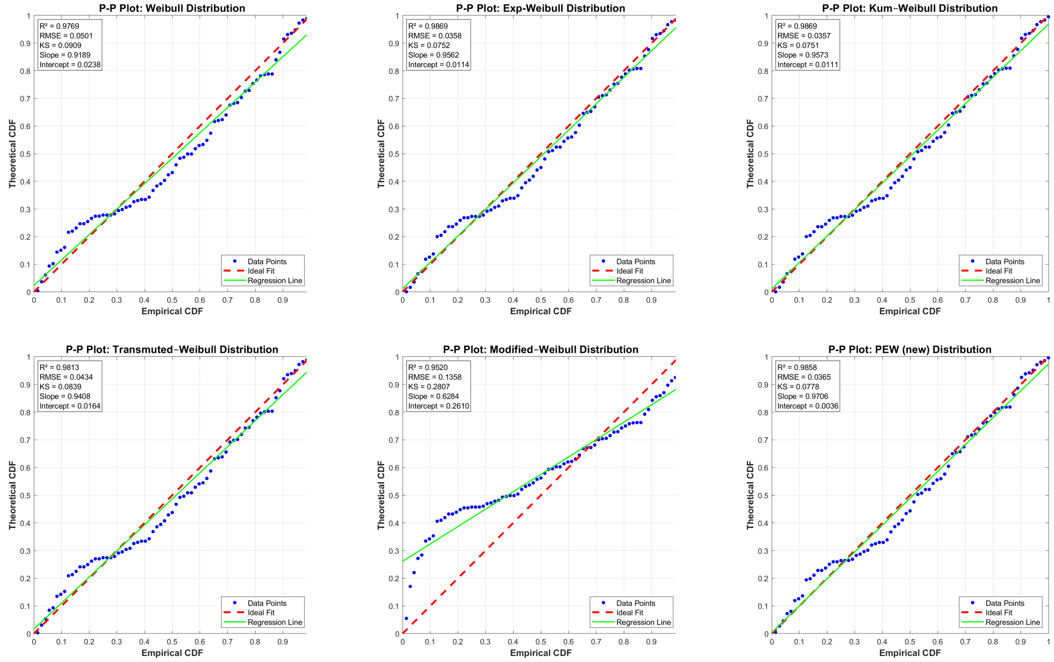

Figure 12. Comparison of P–P Plots for distributions in Modelling Third Dataset

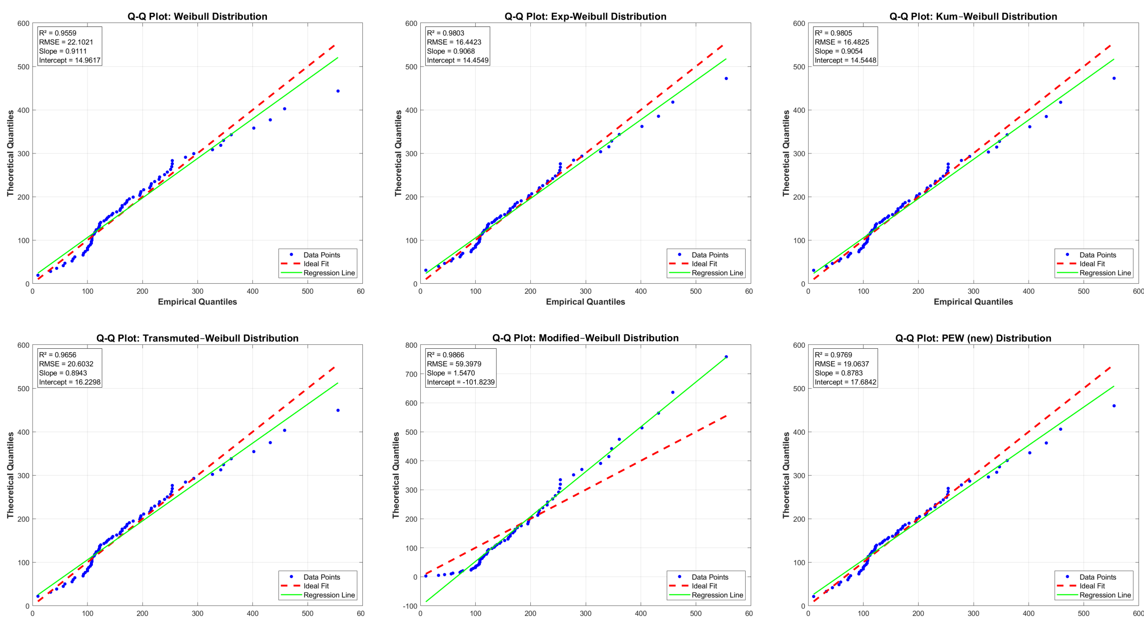

Figure 13. Comparison of Q-Q Plots for distributions in Modelling Third Dataset

7.4. **Conclusions of Applications.** The empirical analysis study with three real life data provides a better fit of **Proposed Power Exponential Weibull (PEW)** distribution. The PEW model generally showed the smallest values of information criteria (AIC, BIC, CAIC and HQC) and goodness-of-fit measures (Kolmogorov-Smirnov, Anderson-Darling, Cram'er-von Mises) over a number of competing models. The results provide evidence that the PEW distribution is more flexible and a better fit, especially at tails, thus it would be an excellent choice and reliable tool for lifetime reliability modeling.

8. Conclusions

In this article, we proposed a new and very flexible family of continuous probability distributions namely the Power Exponential–G (PE-G) family. The proposed family extends any baseline distribution by introducing a shape parameter θ via the power-exponential transformation and can therefore better fit different structures of data. We established several statistical properties such as moments, moment generating function, quantile function, entropy and some reliability measures.

One particular case of this family is studied in details, that is the Power Exponential Weibull (PEW) distribution, to illustrate how our model can work. It is observed that the relative risk model peak gives more flexible and enriched forms of hazard functions for lifetime data and reliability study, with greater adaptability than the classical Weibull models. The parameters were consistently and efficiently estimated using the maximum likelihood approach, and the accuracy and robustness of these estimators were confirmed through simulation studies.

Applications to three real data sets further confirmed that the PEW distribution offers superior performance over several well-known competing models, as evidenced by lower values of AIC, BIC, and other goodness-of-fit statistics. These results highlight the practical importance and modeling versatility of the proposed distribution family.

Future research directions may include Bayesian estimation procedures, multivariate extensions, regression-type modelling under the PE-G framework, and applications in other fields such as finance, climatology, and biostatistics. Overall, the Power Exponential–G family provides a valuable addition to the toolbox of modern statistical modelling.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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