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SEMI-INVERTIBLE IDEALS: FOUNDATIONS, CHARACTERIZATIONS, AND CONNECTIONS TO PROJECTIVE MODULES

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ABSTRACT. This paper presents and elaborates on a relative form of invertibility for ideals in a commutative ring with unity, defined in relation to a specified idempotent element. Suppose R is a unital commutative ring with total quotient ring $Q(R)$, and $e \in R$ is idempotent. A nonzero ideal I of R is called e-semi-invertible if $I \cdot I^{-1} = eR$ and $\operatorname{ann}_R(I) = \operatorname{ann}_R(e)$, where $I^{-1} = \{x \in Q(R) : x \cdot I \subseteq R\}$ is the fractional inverse of I. The proposed notion recovers classical invertibility when $e = 1$, while allowing invertibility to be studied inside direct summands of rings with nontrivial idempotents. We establish basic structural properties of such ideals, including finite generation (Proposition 3.2), and we examine their behaviour over Noetherian, Artinian, local, Dedekind, and semilocal rings. In particular, over a Noetherian ring every e-semi-invertible ideal is a finitely generated projective R-module (Theorem 3.3); over an Artinian ring the class consists exactly of the ideals generated by a nonzero idempotent (Theorem 3.5); over a local ring, the nonzero semi-invertible ideals are precisely the principal ideals generated by a non-zero-divisor (Theorem 3.6); and over a semilocal ring, every proper semi-invertible ideal is principal (Theorem 5.2). A complete local–global criterion is established (Theorem 3.7), as a consequence of which the idempotent attached to a semi-invertible ideal is uniquely determined by the ideal (Corollary 3.8). We further prove a reduction theorem (Theorem 3.10) identifying e-semi-invertible ideals of R with the invertible ideals of eR, which yields a natural isomorphism between the multiplicative group of e-semi-invertible ideal classes and the Picard group $\operatorname{Pic}(eR)$ (Corollary 3.11). Intersection and irreducibility results are also established (Theorems 4.1 and 4.3). An illustrative example (Example 3.13) demonstrates e-semi-invertible ideals that are neither principal nor generated by an idempotent, and which correspond bijectively to the complete class group of a Dedekind domain. Counterexamples are supplied throughout to confirm that the stated hypotheses are essential.

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1. Introduction

In commutative algebra, the multiplicative structure of ideals is pivotal, and the class of invertible ideals has garnered significant interest due to its strong connections to arithmetic factorization, the theory of Prüfer and Dedekind rings, and the examination of projective modules. Recall the classical definition: in a unital commutative ring R, an ideal I is invertible whenever there exists a fractional companion I^{-1} inside the total quotient ring $Q(R)$ such that $I \cdot I^{-1} = R$. Although compact, this single equation packages several strong consequences—finite generation, projectivity as a module, and, locally at every prime, generation by a single non-zero-divisor. Within Dedekind domains, invertibility governs ideal factorization completely, reproducing at the ideal level the unique-factorization phenomenon familiar from elementary number theory.

Once one steps outside the world of integral domains and considers rings in which nontrivial idempotents exist, the equation defining classical invertibility becomes too restrictive. Consider a product decomposition $R = R_1 \times R_2$. The identity element splits into two orthogonal idempotents corresponding to the two factors, and an ideal of the form $I = I_1 \times \{0\}$ can be invertible inside R_1 while failing the global equation $I \cdot I^{-1} = R$ because the second component vanishes. The standard formulation therefore cannot describe invertibility supported on a single direct summand. The literature contains several related contributions: Smith [1] examined finite-type projective ideals and their link to invertibility; Al-Taha [2] studied projective ideals over commutative bases; Dobbs [4] gave local descriptions of invertible and flat ideals; and Olberding [3] developed a structured factorisation theory for invertible and prime ideals over domains. None of these works, however, is specifically designed for the idempotent decomposition of the ring, which is the deficiency addressed in the current paper.

This paper's primary contribution is the systematic development of a relative invertibility theory for ideals associated with idempotent components of a commutative ring. Rather than requiring an ideal to be invertible in the whole ring, the proposed notion requires invertibility only inside the direct summand eR determined by a fixed idempotent e. This permits the theory of invertible ideals to be extended naturally to rings with nontrivial decompositions, while retaining strong links to finite generation, projectivity, principality, and classical ideal theory.

The subsequent sections of the paper are structured as follows. Section 2 presents the formal definition and examines illustrative examples. Section 3 establishes the foundational results: finite generation in full generality, together with characterizations under Noetherian, Artinian, and local hypotheses.

Section 4 develops the intersection and irreducibility apparatus and situates the new class inside the Dedekind setting. Section 5 examines principality under the semilocal hypothesis. All rings under consideration are commutative with an identity element, and unless explicitly indicated, all ideals are nonzero and are associated with a nonzero idempotent. The zero ideal is excluded from the definition of semi-invertibility, since it would otherwise trivially satisfy both defining conditions with respect to $e = 0$; see Section 2 for the precise convention.

2. Definition and Basic Examples

Definition 2.1. Let R denote a unital commutative ring. Denote by $Q(R) = S^{-1}R$ the total quotient ring of R, where S is the multiplicative set of all non-zero-divisors of R. For an ideal I of R, set

$$I^{-1} = \{x \in Q(R) : x \cdot I \subseteq R\}.$$

Let $e \in R$ be an idempotent, i.e. $e^2 = e$. An ideal I of R, which is nonzero, is classified as e-semi-invertible if it satisfies the subsequent two conditions:

(i) $I \cdot I^{-1} = eR$;

(ii) $\mathrm{ann}_R(I) = \mathrm{ann}_R(e)$, where $\mathrm{ann}_R(X) = \{r \in R : rX = 0\}$ for any subset X of R.

Convention 2.2. Throughout this paper, the term semi-invertible ideal always refers to a nonzero ideal, taken with respect to a nonzero idempotent, in accordance with Definition 2.1. The case $I = 0$, $e = 0$ is excluded, as it would satisfy both conditions trivially and carry no information.

Remark 2.3. Let $e \in R$ be an idempotent. The ring decomposes as $R = eR \oplus (1 - e)R$, in which both summands are themselves rings with identities e and $1 - e$, respectively. If $\mathrm{ann}(I) = \mathrm{ann}(e)$, then $1 - e \in \mathrm{ann}(e) = \mathrm{ann}(I)$, and consequently

$$(1 - e)I = 0, \qquad \text{so that} \qquad I = eI \subseteq eR.$$

The converse implication is not universally valid: the inclusion $I \subseteq eR$ does not necessarily imply that $\mathrm{ann}(I) = \mathrm{ann}(e)$. For instance, with $R = \mathbb{Z} \times \mathbb{Z}$, $e = (1, 0)$ and $I = \{(0, 0)\}$, the inclusion $I \subseteq eR$ holds trivially, while $\mathrm{ann}(I) = R \neq \{0\} \times \mathbb{Z} = \mathrm{ann}(e)$.

Example 2.4. Let $R = \mathbb{Z} \times \mathbb{Z}$, set $e = (1, 0)$, and let $I = 3\mathbb{Z} \times \{0\}$. Then $Q(R) = \mathbb{Q} \times \mathbb{Q}$. A direct computation yields $I^{-1} = (1/3)\mathbb{Z} \times \mathbb{Q}$, and therefore

$$I \cdot I^{-1} = \mathbb{Z} \times \{0\} = eR, \qquad \mathrm{ann}(I) = \mathrm{ann}(e) = \{0\} \times \mathbb{Z}.$$

Hence I is e-semi-invertible. Note that I is also invertible inside the ring $eR = \mathbb{Z}$, since $(3\mathbb{Z}) \cdot ((1/3)\mathbb{Z}) = \mathbb{Z}$.

Remark 2.5. Every ideal that is invertible in the classical sense is e-semi-invertible with respect to $e = 1$. The converse is invalid, as demonstrated by the following example.

Example 2.6. Consider once more $R = \mathbb{Z} \times \mathbb{Z}$, with identity $(1,1)$ and idempotent $e = (1,0)$. Let $I = 2\mathbb{Z} \times \{0\}$. Since $Q(R) = \mathbb{Q} \times \mathbb{Q}$, an element $(a,b) \in \mathbb{Q} \times \mathbb{Q}$ lies in I^{-1} precisely when $(a,b)(2z,0) \in \mathbb{Z} \times \mathbb{Z}$ for every $z \in \mathbb{Z}$, i.e. $2az \in \mathbb{Z}$ for every $z \in \mathbb{Z}$. This is equivalent to $a \in (1/2)\mathbb{Z}$ and $b \in \mathbb{Q}$, so

$$I^{-1} = (1/2)\mathbb{Z} \times \mathbb{Q}.$$

Multiplying out gives $I \cdot I^{-1} = \mathbb{Z} \times \{0\} = eR$, which is a proper ideal of R; together with $\operatorname{ann}(I) = \operatorname{ann}(e) = \{0\} \times \mathbb{Z}$, this shows that I is e-semi-invertible, although it is not invertible in R.

3. Foundational Properties

We begin with a result that identifies exactly when semi-invertibility forces classical invertibility.

Theorem 3.1. *Let R be a unital commutative ring. The subsequent two assertions are synonymous:*

(i) *every semi-invertible ideal of R is invertible in the classical sense; that is, $I \cdot I^{-1} = R$;*

(ii) *the only idempotents of R are 0 and 1; equivalently, R is connected.*

In particular, statement (ii) holds for every integral domain and for every local ring.

Proof. (i) $\Rightarrow$ (ii). Suppose every semi-invertible ideal of R is classically invertible, and let $e \in R$ be any nonzero idempotent. The ideal $I := eR$ is nonzero (since $e \neq 0$) and satisfies $e \in I^{-1}$ together with $I \cdot I^{-1} = eR$ and $\operatorname{ann}(I) = \operatorname{ann}(e)$; hence I is e-semi-invertible. The hypothesis forces $eR = R$, and consequently $e = 1$. Thus R has no nontrivial idempotents.

(ii) $\Rightarrow$ (i). If the only idempotents of R are 0 and 1, and I is e-semi-invertible with $e \neq 0$, then necessarily $e = 1$; condition (i) of Definition 2.1 reads $I \cdot I^{-1} = R$, so that I is invertible in the classical sense. □

Proposition 3.2. *Every semi-invertible ideal of a unital commutative ring is finitely generated.*

Proof. Let I be e-semi-invertible, so that $I \cdot I^{-1} = eR$. Since $e \in I \cdot I^{-1}$, there exist $a_1, \ldots, a_n \in I$ and $x_1, \ldots, x_n \in I^{-1}$ with

$$e = a_1x_1 + \cdots + a_nx_n.$$

According to Remark 2.3, $I \subseteq eR$, thus for any $b \in I$, we obtain

$$b = eb = (a_1x_1 + \cdots + a_nx_n)b = (x_1b)a_1 + \cdots + (x_nb)a_n.$$

Because each x_i lies in I^{-1}, we have $x_iI \subseteq R$, so $x_ib \in R$ for every i. Hence b lies in the R-module generated by $a_1, \ldots, a_n$, and therefore

$$I = (a_1, \ldots, a_n).$$

□

Theorem 3.3. *Let R be a Noetherian commutative ring. If a nonzero ideal I of R is semi-invertible with respect to an idempotent $e \in R$, then I is a finitely generated projective R-module. Conversely, if I is a finitely generated projective ideal of R whose trace ideal $\operatorname{tr}(I) = I \cdot I^{-1}$ equals eR for some idempotent $e \in R$ and $\operatorname{ann}(I) = \operatorname{ann}(e)$, then I is e-semi-invertible.*

Proof. ($\Rightarrow$) Suppose I is e-semi-invertible. By Proposition 3.2, I is finitely generated. From $I \cdot I^{-1} = eR$ and Remark 2.3, $I \subseteq eR$, and this same equality shows that I is invertible as an ideal of the ring $S := eR$. Invertible ideals over commutative rings are projective; consequently I is S-projective. Since $R = eR \oplus (1-e)R$ as a ring, S is a ring direct summand of R, and every S-projective module is automatically R-projective. Hence I is projective over R.

($\Leftarrow$) Conversely, suppose I is a finitely generated projective ideal with $\operatorname{tr}(I) = eR$ and $\operatorname{ann}(I) = \operatorname{ann}(e)$. For a finitely generated projective ideal I of a commutative ring, the trace ideal admits the well-known representation

$$\operatorname{tr}(I) = I \cdot I^{-1},$$

the product being computed in $Q(R)$; see, e.g., [5, Chap. II] and [7, §5.1]. The hypotheses then read

$$I \cdot I^{-1} = eR, \qquad \operatorname{ann}(I) = \operatorname{ann}(e),$$

which are precisely conditions (i) and (ii) of Definition 2.1. Therefore I is e-semi-invertible. □

Remark 3.4. The Noetherian hypothesis enters in two places: it guarantees the finite generation of I required for the trace identity $\operatorname{tr}(I) = I \cdot I^{-1}$, and it ensures that any idempotent ideal of R is generated by an idempotent element—a standard consequence of Nakayama's lemma applied locally at each maximal ideal (see, e.g., [9, §2.4]).

Theorem 3.5. *Let R be an Artinian commutative ring. A nonzero ideal I of R is semi-invertible if and only if $I = eR$ for some nonzero idempotent $e \in R$.*

Proof. ($\Leftarrow$) Assume $I = eR$ for some nonzero idempotent e. Then $e \in I^{-1}$ (since $e \cdot eR \subseteq R$), so $I \cdot I^{-1} \supseteq eR \cdot eR = eR$; the reverse inclusion is immediate. Moreover, $\operatorname{ann}(eR) = \operatorname{ann}(e)$. Hence I is e-semi-invertible.

($\Rightarrow$) Conversely, suppose I is e-semi-invertible. Every Artinian commutative ring decomposes as a finite product of Artinian local rings (see, e.g., [9, Theorem 8.7])

$$R = R_1 \times \cdots \times R_m,$$

and correspondingly $I = I_1 \times \cdots \times I_m$ with each I_j an ideal of R_j. By Theorem 3.3 (every Artinian ring is Noetherian), I is projective; consequently each I_j is a projective ideal of the local ring R_j. Over a local ring every finitely generated projective module is free, so I_j is a free R_j-module. Because I_j is moreover an ideal of R_j, its rank cannot exceed 1, and a free ideal of rank 1 in a local Artinian ring necessarily equals the whole ring, since any non-zero-divisor in such a ring is a unit. Hence each I_j is either 0 or R_j. Define

$$e_j = 1_{R_j} \text{ if } I_j = R_j, \qquad e_j = 0 \text{ otherwise.}$$

Then $e := (e_1, \ldots, e_m)$ is an idempotent of R and $I = eR$. The idempotent e is nonzero because I is nonzero. □

Theorem 3.6. *Let R denote a local ring. A nonzero ideal I of R is semi-invertible if and only if I is principal and generated by a non-zero-divisor.*

Proof. Locality forces the only idempotents of R to be 0 and 1. Under our convention, $e \neq 0$, hence $e = 1$; condition (i) of Definition 2.1 then reads $I \cdot I^{-1} = R$, so I is classically invertible. A finitely generated invertible ideal over a local ring is principal (see [9, Theorem 11.3] or [4, Corollary 3]), so $I = (a)$ for some $a \in R$. Condition (ii) gives

$$\operatorname{ann}(a) = \operatorname{ann}(I) = \operatorname{ann}(1) = 0,$$

which states precisely that the generator a is a non-zero-divisor (it does not assert that every element of I is a non-zero-divisor; indeed $0 \in I$ is always a zero-divisor unless $R = 0$).

Conversely, if $I = (a)$ with a a non-zero-divisor, then a^{-1} exists in $Q(R)$, $I^{-1} = (a^{-1})$, $I \cdot I^{-1} = R$, and $\operatorname{ann}(I) = \operatorname{ann}(a) = 0 = \operatorname{ann}(1)$. □

The following result, which appears to be new, gives a complete local–global characterization of semi-invertibility. It will be used in Section 4 and provides a practical criterion for verifying semi-invertibility ideal-by-ideal in concrete computations.

Theorem 3.7 (Local–Global Criterion)**.** *Let R be a unital commutative ring and let $e \in R$ be a nonzero idempotent. For a nonzero ideal I of R, the following two statements are equivalent.*

(a) *I is e-semi-invertible.*

(b) *I is finitely generated, $I \subseteq eR$, and for every maximal ideal $\mathfrak{m}$ of R,*
   - *if $e \notin \mathfrak{m}$, then $I_{\mathfrak{m}}$ is a nonzero principal ideal of $R_{\mathfrak{m}}$ generated by a non-zero-divisor;*
   - *if $e \in \mathfrak{m}$, then $I_{\mathfrak{m}} = 0$.*

Proof. We first record a useful identity: since e is an idempotent and $\mathfrak{m}$ is a maximal (hence prime) ideal of R, the image of e in the local ring $R_{\mathfrak{m}}$ is an idempotent of $R_{\mathfrak{m}}$. A local ring has no idempotents other than 0 and 1, and therefore

$$e/1 = 1 \text{ in } R_{\mathfrak{m}} \text{ if } e \notin \mathfrak{m}, \qquad e/1 = 0 \text{ in } R_{\mathfrak{m}} \text{ if } e \in \mathfrak{m}.$$

Consequently $(eR)_{\mathfrak{m}} = R_{\mathfrak{m}}$ when $e \notin \mathfrak{m}$, and $(eR)_{\mathfrak{m}} = 0$ when $e \in \mathfrak{m}$.

(a) ⇒ (b). Suppose I is e-semi-invertible. Finite generation is Proposition 3.2, and the inclusion $I \subseteq eR$ follows from Remark 2.3. Fix a maximal ideal $\mathfrak{m}$. If $e \in \mathfrak{m}$, then $I_\mathfrak{m} \subseteq (eR)_\mathfrak{m} = 0$, so $I_\mathfrak{m} = 0$. If $e \notin \mathfrak{m}$, localise the equation $I \cdot I^{-1} = eR$ at $\mathfrak{m}$ to obtain

$$I_\mathfrak{m} \cdot (I^{-1})_\mathfrak{m} = (eR)_\mathfrak{m} = R_\mathfrak{m}.$$

Thus $I_\mathfrak{m}$ is invertible in the local ring $R_\mathfrak{m}$; by Theorem 3.6 applied to $R_\mathfrak{m}$, $I_\mathfrak{m}$ is principal and generated by a non-zero-divisor. Moreover $I_\mathfrak{m} \neq 0$, since the principal ideal generated by a non-zero-divisor in $R_\mathfrak{m}$ cannot be zero unless the generator is zero, which it cannot be.

(b) ⇒ (a). Assume condition (b). We verify each of the two defining conditions of Definition 2.1.

Verification of (i): $I \cdot I^{-1} = eR$. The inclusion $I \cdot I^{-1} \subseteq R$ is immediate from the definition of I^{-1}. For any $a \in I \subseteq eR$ we have $a = ea$, so every element of $I \cdot I^{-1}$ lies in eR. Hence $I \cdot I^{-1} \subseteq eR$. For the reverse inclusion, we check equality at every maximal ideal. Fix $\mathfrak{m}$:

- If $e \in \mathfrak{m}$, then $(eR)_\mathfrak{m} = 0 \subseteq (I \cdot I^{-1})_\mathfrak{m}$.
- If $e \notin \mathfrak{m}$, then by hypothesis $I_\mathfrak{m} = (a/s)$ for some $a \in I$ and some non-zero-divisor s outside $\mathfrak{m}$. The element s/a lies in $Q(R)$ and satisfies $(s/a) \cdot I_\mathfrak{m} \subseteq R_\mathfrak{m}$; clearing denominators, it lies in $(I^{-1})_\mathfrak{m}$, and $(a/s)(s/a) = 1 \in (I \cdot I^{-1})_\mathfrak{m}$. Thus $(I \cdot I^{-1})_\mathfrak{m} = R_\mathfrak{m} = (eR)_\mathfrak{m}$.

Since I is finitely generated, so is $I \cdot I^{-1}$ (it is generated by products $a_i x_j$ for any chosen generating sets); equality at every maximal ideal then forces $I \cdot I^{-1} = eR$.

Verification of (ii): $\operatorname{ann}(I) = \operatorname{ann}(e)$. Since $I \subseteq eR$, any $a \in I$ satisfies $a = ea$, so $r \in \operatorname{ann}(e)$ forces $ra = r(ea) = (re)a = 0$; hence $\operatorname{ann}(e) \subseteq \operatorname{ann}(I)$. For the reverse inclusion, by part (i) we have $e \in I \cdot I^{-1}$, so $e = a_1x_1 + \cdots + a_nx_n$ with $a_i \in I$ and $x_i \in I^{-1}$. If $r \in \operatorname{ann}(I)$, then $ra_i = 0$ for every i, whence $re = r \cdot (a_1x_1 + \cdots + a_nx_n) = 0$. Therefore $r \in \operatorname{ann}(e)$. □

Corollary 3.8. *Let R be a unital commutative ring, and let $e, f \in R$ be nonzero idempotents. If I is both e-semi-invertible and f-semi-invertible, then $e = f$.*

Proof. By Theorem 3.7, the maximal ideals $\mathfrak{m}$ with $I_\mathfrak{m} \neq 0$ are exactly those with $e \notin \mathfrak{m}$; the same description with f in place of e gives the same set of maximal ideals. Hence e and f lie in exactly the same maximal ideals, which—since they are idempotents—forces $e = f$. □

Remark 3.9. Corollary 3.8 shows that the idempotent e attached to a semi-invertible ideal I is uniquely determined by I. This justifies speaking of the idempotent associated with a semi-invertible ideal, and shows that the partition of the class of all semi-invertible ideals according to the underlying idempotent is well defined.

We now establish a structural reduction theorem which shows that semi-invertibility in R is equivalent to classical invertibility inside the direct summand eR. This bridge will allow us to transfer many classical invariants—in particular the Picard group—directly to the semi-invertible setting.

Theorem 3.10 (Reduction Theorem)**.** *Let R be a unital commutative ring and let $e \in R$ be a nonzero idempotent element. For a nonzero ideal I of R, the following two statements are equivalent.*

(a) *I is e-semi-invertible in R;*

(b) *$I \subseteq eR$, and I is an invertible ideal of the ring eR (where invertibility is taken inside the total quotient ring $Q(eR)$).*

Proof. The total quotient ring of R decomposes as $Q(R) \cong Q(eR) \times Q((1-e)R)$, with e identified with $(1,0)$. Write a typical element of $Q(R)$ as (x,y), with $x \in Q(eR)$ and $y \in Q((1-e)R)$.

(a) $\Rightarrow$ (b). Suppose I is e-semi-invertible. By Remark 2.3, $I \subseteq eR$, so every element of I has the form $(a,0)$. The inverse I^{-1} computed inside $Q(R)$ consists of pairs (x,y) such that $xI \subseteq eR$ and y is arbitrary in $Q((1-e)R)$. Therefore

$$I^{-1} = (I^{-1})_{eR} \times Q((1-e)R),$$

where $(I^{-1})_{eR}$ denotes the fractional inverse of I computed inside the total quotient ring of eR. Multiplying out yields

$$I \cdot I^{-1} = \big(I \cdot (I^{-1})_{eR}\big) \times \{0\}.$$

The hypothesis $I \cdot I^{-1} = eR = eR \times \{0\}$ therefore says exactly that $I \cdot (I^{-1})_{eR} = eR$, i.e. I is invertible in eR.

(b) $\Rightarrow$ (a). Assume $I \subseteq eR$ and $I \cdot (I^{-1})_{eR} = eR$. Reversing the computation above, $I \cdot I^{-1} = eR$ in R. It remains to verify that $\operatorname{ann}_R(I) = \operatorname{ann}_R(e)$. The inclusion $\operatorname{ann}_R(e) \subseteq \operatorname{ann}_R(I)$ follows from $I \subseteq eR$, exactly as in the proof of Remark 2.3. For the converse, suppose $r \in R$ satisfies $rI = 0$. Then $re \in eR$ also annihilates I: since $I \subseteq eR$ every $x \in I$ satisfies $ex = x$, and hence $re \cdot x = r \cdot ex = r \cdot x = 0$. Now any invertible ideal of eR has zero annihilator inside eR: if $s \in eR$ satisfies $sI = 0$, then $s \cdot eR = s \cdot I \cdot (I^{-1})_{eR} = 0$, so $s = se = 0$. Applying this to $s = re$ gives $re = 0$, i.e. $r \in \operatorname{ann}_R(e)$. □

Corollary 3.11 (Picard-group connection)**.** *Let R be a unital commutative ring, and let $e \in R$ be a nonzero idempotent element. According to Theorem 3.10, the e-semi-invertible (integral) ideals of R are exactly the invertible (integral) ideals of eR. The correspondence to fractional ideals of eR in the conventional sense indicates that the collection of isomorphism classes of fractional e-semi-invertible ideals of R constitutes an abelian group under multiplication, which is naturally isomorphic to the Picard group $\operatorname{Pic}(eR)$.*

Proof. By Theorem 3.10, an ideal I of R is e-semi-invertible if and only if it is contained in eR and is invertible as an ideal of eR. The correspondence $I \mapsto I$ (viewed as an ideal of eR) therefore identifies the e-semi-invertible (fractional) ideals of R with the invertible (fractional) ideals of eR. This correspondence respects multiplication and isomorphism, so it descends to an isomorphism of abelian groups between the multiplicative group of e-semi-invertible (fractional) ideal classes and the Picard group $\operatorname{Pic}(eR)$. □

Remark 3.12. Corollary 3.11 shows that the theory of e-semi-invertible ideals contributes a new perspective without sacrificing classical machinery: each idempotent e gives a copy of the Picard group of the summand eR, and the entire family $\{e\text{-semi-invertible ideals} : e \text{ idempotent}\}$ of multiplicative subgroups can be assembled into a single invariant attached to R. When R decomposes as $R = R_1 \times \cdots \times R_n$ with idempotents e_i, one recovers the well-known identity $\mathrm{Pic}(R) \cong \mathrm{Pic}(R_1) \times \cdots \times \mathrm{Pic}(R_n)$.

Example 3.13 (Non-principal semi-invertible ideal)**.** Let $R = \mathbb{Z}[\sqrt{-5}] \times F$, where F is any field, and set $e = (1, 0)$, so that $eR \cong \mathbb{Z}[\sqrt{-5}]$. Consider the ideal

$$I = (2, 1 + \sqrt{-5}) \times \{0\}.$$

In $\mathbb{Z}[\sqrt{-5}]$, the ideal $(2, 1 + \sqrt{-5})$ is a well-known invertible ideal that is not principal: it is the unique prime ideal lying above 2, its non-principality reflects the fact that the class group of $\mathbb{Z}[\sqrt{-5}]$ is the cyclic group of order 2, and invertibility holds because $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain (see, e.g., [10, Chap. 9] or [9, Chap. 11]). By Theorem 3.10, I is e-semi-invertible in R. This example exhibits the following features simultaneously:

(i) I is not principal in R: if $I = (a, 0)R$ for some $a \in \mathbb{Z}[\sqrt{-5}]$, then $(2, 1 + \sqrt{-5})$ would be principal in $\mathbb{Z}[\sqrt{-5}]$, contradicting the non-principality recalled above;

(ii) I is not generated by any idempotent, since $(2, 1 + \sqrt{-5})$ is a proper, non-zero ideal of $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\sqrt{-5}]$ has no nontrivial idempotents (it is a domain);

(iii) I is not invertible in R in the classical sense, because the second component of $I \cdot I^{-1}$ is zero, so the product cannot equal $R = \mathbb{Z}[\sqrt{-5}] \times F$.

The scenarios of Theorem 3.5 (Artinian) and Theorem 3.6 (local) are not applicable, and the current example demonstrates that e-semi-invertibility yields genuinely novel ideals beyond those addressed by the local and Artinian characterizations: specifically, e-semi-invertible ideals in R can correspond bijectively with the entire class group of any Dedekind domain via the selection $R = D \times F$, $e = (1, 0)$.

4. Intersection and Irreducibility Properties

Theorem 4.1. *Let R be a unital commutative ring, $e \in R$ an idempotent, A an e-semi-invertible ideal of R, and P a prime ideal of R with $A \nsubseteq P$. Then*

$$A \cap P = AP.$$

Proof. By Remark 2.3, $\mathrm{ann}(A) = \mathrm{ann}(e)$ implies $1 - e \in \mathrm{ann}(A)$, so $(1 - e)A = 0$ and $A \subseteq eR$. Since A is e-semi-invertible, we may write

$$e = a_1x_1 + \cdots + a_nx_n, \qquad a_i \in A, \quad x_i \in A^{-1}.$$

The containment $AP \subseteq A \cap P$ is immediate. For the converse, take $y \in A \cap P$. Since $y \in A \subseteq eR$, we obtain

$$y = ey = a_1(x_1 y) + \cdots + a_n(x_n y).$$

Each $x_i y$ lies in R because $x_i A \subseteq R$. The assumption $A \nsubseteq P$ provides some $a_0 \in A$ with $a_0 \notin P$. Now

$$a_0(x_i y) = x_i(a_0 y) \in x_i(A \cdot P) \subseteq P,$$

and primality of P together with $a_0 \notin P$ forces $x_i y \in P$. Therefore $y = \sum_i a_i(x_i y) \in AP$. □

Definition 4.2. A proper ideal I of a commutative ring is called irreducible (or meet-irreducible) if it cannot be expressed as $I = J \cap K$ for two ideals J and K both strictly containing I.

Theorem 4.3. *Every proper prime ideal in a commutative ring is irreducible. Specifically, every proper semi-invertible prime ideal is irreducible.*

Proof. Let P be a proper prime ideal of R, and suppose, for the sake of contradiction, that $P = J \cap K$ with J, K ideals of R both strictly containing P. Choose $a \in J \setminus P$ and $b \in K \setminus P$. Then

$$ab \in J \cdot K \subseteq J \cap K = P.$$

Since P is prime, $a \in P$ or $b \in P$, contradicting the choice of a and b. Hence $J = P$ or $K = P$, and P is irreducible. The second statement follows at once, since a semi-invertible prime ideal is in particular a prime ideal. □

Theorem 4.4. *Every nonzero prime ideal in a Dedekind domain is semi-invertible.*

Proof. Let R be a Dedekind domain and P a nonzero prime ideal. By definition, every nonzero ideal of R is invertible, so

$$P \cdot P^{-1} = R = 1 \cdot R.$$

Since R is a domain, $\operatorname{ann}(P) = 0 = \operatorname{ann}(1)$. Hence P is e-semi-invertible with respect to $e = 1$. □

Example 4.5. The converse of Theorem 4.4 fails: take $R = F \times F$ with F a field. The two proper prime ideals of R are

$$P_1 = F \times \{0\}, \qquad P_2 = \{0\} \times F,$$

each semi-invertible with respect to its own coordinate idempotent. However, R is not a Dedekind domain, since it is not even a domain. Thus the semi-invertibility of all nonzero prime ideals does not characterise Dedekind domains unless the domain hypothesis is imposed.

Definition 4.6 ([7, Chap. III])**.** A commutative integral domain D is termed a Dedekind domain if every nonzero ideal of D is invertible within the field of fractions of D.

Remark 4.7. Neither the sum nor the intersection of two semi-invertible ideals need be semi-invertible.

(a) Sums. Inside $K[x, y]$ (with K a field), the principal ideals (x) and (y) are invertible, hence semi-invertible with respect to $e = 1$. Their sum

$$(x) + (y) = (x, y)$$

is the maximal ideal at the origin, which is not invertible because it is not locally principal at (x, y) itself.

(b) Intersections. In $R = \mathbb{Z} \times \mathbb{Z}$, with idempotents $e_1 = (1, 0)$ and $e_2 = (0, 1)$, the ideals

$$A = 2\mathbb{Z} \times \{0\}, \qquad B = \{0\} \times 3\mathbb{Z}$$

are semi-invertible with respect to e_1 and e_2, respectively, while their intersection

$$A \cap B = \{(0, 0)\}$$

is the zero ideal, which is not semi-invertible under the convention adopted in Section 2.

5. Semi-Invertible Ideals in Semilocal Rings

A commutative ring is considered semilocal if it possesses a finite number of maximal ideals.

Lemma 5.1. *In a commutative semilocal ring, every invertible ideal is generated by a single element.*

Proof. This is a classical result in commutative algebra. A standard reference is Bourbaki [5, Chapter II, §5, Theorem 1], and a self-contained proof using prime avoidance is given in Larsen and McCarthy [8, Theorem 6.13]; see also Kaplansky [6]. □

Theorem 5.2. *Let R be a unital commutative ring with only finitely many maximal ideals, and let A be a proper semi-invertible ideal of R. Then A is principal.*

Proof. Suppose A is e-semi-invertible. Then $A \cdot A^{-1} = eR$ and $\operatorname{ann}(A) = \operatorname{ann}(e)$, so by Remark 2.3, $A \subseteq eR$. Set $S := eR$; this is a commutative ring with identity e and is isomorphic to $R/(1-e)R$. The identity

$$A \cdot A^{-1} = eR = 1_S S$$

shows that A is invertible inside S. Since R is semilocal and S is a quotient of R, S is also semilocal.

By Lemma 5.1, $A = aS$ for some $a \in S$. Because $a \in eR$ satisfies $a = ea$, we have

$$aR = a(eR \oplus (1-e)R) = aS,$$

since $a(1-e)R = (a(1-e))R = 0$. Therefore $A = aR$ is principal in R. □

6. Conclusion

This paper has developed a relative form of invertibility for ideals supported on idempotent components of a commutative ring. The proposed notion of e-semi-invertibility recovers classical invertibility in the case $e = 1$, while allowing ideals contained in a direct summand eR to be treated as invertible within that component. The fundamental findings indicate that semi-invertible ideals are finitely generated (Proposition 3.2), and that, in the connected scenario, semi-invertibility aligns with classical invertibility (Theorem 3.1). Under additional structural hypotheses, the notion admits more concrete descriptions: over Noetherian rings, every semi-invertible ideal is a finitely generated projective module, and a precise converse holds under explicit trace-ideal conditions (Theorem 3.3); over Artinian rings the class consists exactly of ideals generated by an idempotent (Theorem 3.5); over local rings, of the principal ideals generated by a non-zero-divisor (Theorem 3.6); and over semilocal rings every proper semi-invertible ideal is principal (Theorem 5.2). The central technical contributions are the local–global criterion of Theorem 3.7 and the reduction theorem of Theorem 3.10, which together identify the e-semi-invertible ideals of R with the invertible ideals of the direct summand eR. As a direct consequence, the idempotent attached to a semi-invertible ideal is uniquely determined by the ideal (Corollary 3.8), and the multiplicative group of e-semi-invertible ideal classes is naturally isomorphic to the Picard group $\mathrm{Pic}(eR)$ (Corollary 3.11). The non-trivial Example 3.13 shows that, even when R carries nontrivial idempotents, the class of e-semi-invertible ideals can be as rich as the class group of an arbitrary Dedekind domain. Intersection and irreducibility theorems (Theorems 4.1 and 4.3) furnish further tools for working with this class, and Theorem 4.4 situates the new notion inside the Dedekind landscape. In a subsequent paper, the authors advance this research by characterizing discrete valuation rings through their semi-invertible primes, reestablishing Dedekind domains via their maximal ideals, and providing a comprehensive description of Noetherian rings that decompose into finite direct products of Dedekind domains.

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