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GLOBAL DYNAMICS OF AN EULERIAN MULTI-PATCH EPIDEMIC MODEL IN A PERIODIC ENVIRONMENT

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Abstract. We develop a multi-patch SIS compartmental epidemic model that captures the dynamics of an infectious disease in a population of individuals who can travel among n patches. The model is formulated as a non-autonomous system of ordinary differential equations, with periodic terms accounting for general incidence, recovery, birth, death, and travel between cities. Through theories of dynamical systems, we compute the basic reproduction number, $\mathcal{R}_0$ of the model and we establish that it is the threshold between the persistence and the extinction of the epidemic. We further derive some conditions under which these periodic steady states are globally asymptotically stable.

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1. Introduction

The spatial spread of infectious diseases is a phenomenon that involves many different factors. Modelling this spread is a complex task. Spatial heterogeneity can be accounted in two ways: either by formulating muti-patch models using ordinary differential equations, or by introducing spatial variation through the use of partial differential equations. However, there are, cases where the latter spatial approach may not be appropriate [1–3]. In the case of a human specific disease that is transmitted through direct contact, in a large country with a small number of potentially significant cities and well-developed transportation system, travel from one city to another is rapid and the spread on an epidemic occurs only at the destination. In this context, the movement of individuals between distinct discrete geographical regions can play an important role in the spread of the disease.

To incorporate this spatial heterogeneity, one approach is to use a patch model with different classes of individuals. In such model that includes humans or/and vectors mobility, we need to specify how

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exactly, individuals move. Here, we have adopted a Eulerian approach where individuals are not assigned to a particular patch, but instead simply move around between the various patches at certain rates [4,5]. Another more sophisticated patch model which captures the mobility of individuals is a Lagrangian approach [6–8]. A fascinating question is whether a disease could remain endemic by travelling geographically around a specific patch. The main disadvantage of this approach lies in the high dimensionality of the resulting models [2,9]. Consequently, multi-patch epidemic models give rise to large systems of non-linear differential equations and establishing their global dynamics can be a mathematical challenge [10,11]. Thus, these models are often studied through computer simulations. Some compartmental models with Eulerian and Lagrangian approach have been formulated and discussed by several authors [2,3,12–15]. However, these formulated models give rise to autonomous systems differential equations which are not applicable in environments with variable climates.

Indeed, mathematical models and field observations show that the intensity and mechanisms of seasonality can influence the spread and persistence of infectious diseases [16,17]. From a practical perspective, understanding the mechanisms and effects of seasonality provides valuable insights into the functioning of parasite-host systems, how and when control measures should be implemented, and how health risks are evolving in the face of shifting seasonal patterns [18,19]. Seasonal variations in temperature, precipitation, resource availability, contact rates, population birth and death rates, and immune defences are ubiquitous and can exert significant impact on population dynamics. Empirical data highlight several biologically distinct mechanisms through which seasonality can influence host-pathogen interactions [20]. From a practical standpoint, a better understanding of the mechanisms linking seasonal environmental changes to disease dynamics could help to predict long-term health risks, implement effective public health programs, set goals and use limited resources more effectively. [21].

To model infectious diseases, the number of individuals contacted by an infective per unit of time is called a contact rate of the infection. However, in most studies, an important parameter often is neglected or arbitrary used by authors: the incidence of the disease. The incidence of a disease is the total new infective infected by all individuals in the infected compartment per unit of time. In the literature, there exists several incidences for direct or indirect transmission, but the choice of the incidence to use depends on many factors such as toxicity of parasite and the characteristics of the environment [22,23]. Hence, the suitable incidence to use when studying a specific epidemic model is a real challenge; particularly when investigating the spread of an infection in metapopulation, because each patch is an environment with its own characteristics [24–26]. So, it is unrealistic to consider the same type of incidence of infection accross all the patches.

To contribute to this challenge, we consider the model of Fang *et al.* [27] with periodic general incidence function which includes the common properties of particular incidence functions. Then, the model reads as follows

$$(1)\qquad \begin{cases} \dot{u}_i = b_i(t,h_i)h_i - \mu_i(t)u_i - f_i(t,u_i,v_i) + \gamma_i(t)v_i + \sum_{j=1}^{n} a_{ij}(t)u_j, \;\; 1 \le i \le n, \\ \dot{v}_i = f_i(t,u_i,v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \sum_{j=1}^{n} b_{ij}(t)v_j, \;\; 1 \le i \le n, \end{cases}$$

where in each patch $i = 1,\ldots,n$, the whole population, h_i is grouped into two classes: susceptible individuals, u_i and infectious individuals v_i; namely $h_i = u_i + v_i$. Susceptible individuals become infective after contact with infectious individuals through the incidence function, $f_i(t,u_i,v_i)$ [28]. Infective individuals return to susceptible class after recovering at rate $\gamma_i(t)$. While individuals are recruited with function $b_i(t,h_i)$, function $\mu_i(t)$ denotes the natural death rate and $-a_{ii}(t)$, $-b_{ii}(t)$ capture the emigration rate of susceptible and infectious individuals respectively in the patch i. For all $i \neq j$, $a_{ij}(t)$, $b_{ij}(t)$ represent respectively the ω-periodic immigration rate of susceptible and infectious individuals from jth patch to ith.

The main purpose of this paper is to analyse the global dynamics of the complex non-linear system (1). Consequently, we organize the paper as follows: in next section, we compute the threshold dynamics of the system and recall some important preliminaries results used in our analysis. In Section 3, we focus on the global behaviour of the system around the basic reproduction rate, $\mathcal{R}_0$. In the last section, we conclude.

2. Preliminaries

For $i = 1,\ldots,n$, let $u = (u_1,u_2,\ldots,u_n)^T$, $v = (v_1,v_2,\ldots,v_n)^T$, $x_i := (u_i,v_i)^T$ and $g_i := (g_{i1},g_{i2})^T$. Then, we write the system (1) as follows:

$$(2)\qquad \dot{x}_i = g_i(t,u_i,v_i), \forall i = 1,\ldots,n$$

where

$$g_{i1}(t,u_i,v_i) := b_i(t,h_i)h_i - \mu_i(t)u_i - f_i(t,u_i,v_i) + \gamma_i(t)v_i + \sum_{j=1}^{n} a_{ij}(t)u_j,$$

and

$$g_{i2}(t,u_i,v_i) := f_i(t,u_i,v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \sum_{j=1}^{n} b_{ij}(t)v_j.$$

Hence, for all positive initial conditions $x_i(0)$, the Cauchy's problem associated to (3) reads as follows:

$$(3)\qquad \begin{cases} \dot{x}_i = g_i(t,u_i,v_i), \\ \qquad\qquad\qquad\qquad \forall i = 1,\ldots,n. \\ x_i(0) = x_i^0, \end{cases}$$

2.1. **Definitions, notations and assumptions.** To make sense biologically, we assume throughout this paper that all parameters of system (3) are positive ω-periodic functions. In addition, we assume that during the dispersal process, hosts death and birth rates are neglected. That is

$$\sum_{j=1}^{n} a_{ji}(t) = 0, \sum_{j=1}^{n} b_{ji}(t) = 0, \forall i = 1, \dots, n \text{ and } t \in [0, \omega], \tag{4}$$

and the travelling matrix of susceptible, $(a_{ij}(t))$ and infectious, $(b_{ij}(t))$ hosts are supposed to be irreducible.

Moreover, we further assume that, $\forall i = 1, \dots, n$ the functions f_i and b_i satisfy the following conditions:

(A1): f_i and b_i are respectively non-negative and positive ω-periodic functions, that is:

$$f_i(t+\omega, u_i, v_i) = f_i(t, u_i, v_i) \text{ and } b_i(t+\omega, h_i) = b_i(t, h_i), \forall t \in [0, \omega],$$

(A2): $f_i(t, 0, v_i) = f_i(t, u_i, 0) = 0,\ \forall u_i, v_i \geq 0,$
(A3): f_i is continuously differentiable and we note $\dfrac{\partial f_i}{\partial v_i}(t, u_i, 0) := p_i(t, u_i), \forall u_i \geq 0.$
(A4): there exists $\epsilon_1 > 0$ such that when $0 < v_i < \epsilon_1$,

$$f_i(t, u_i, v_i) \geq p_i(t, u_i)v_i + \frac{1}{2}\epsilon_1 v_i \frac{\partial^2 f}{\partial v_i^2}(t, u_i, v_i),$$

(A5): b_i is continuously differentiable with $\dfrac{\partial b_i}{\partial h_i}(t, h_i) < 0,\ \forall h_i \in (0, \infty).$

Let $X = \mathbb{R}_+^q$ and $(\mathbb{R}^q, X)$ be the standard ordered q-dimensional Euclidean space with a norm $\|.\|$. For $z, t \in X$, we denote $z \geq t,$ if $z - t \in X,\ z > t,$ if $z - t \in X \setminus \{0\}, z \gg t,$ if $z - t \in \text{int}(X)$.

Definition 1. *A $q \times q$ matrix function is cooperative if all its off-diagonal entries are non-negative. It is said to be irreducible if its index set $\{1, 2, \dots, n\}$ cannot be split into two complementary sets $\{n_1, n_2, \dots, n_q\}$ and $\{m_1, m_2, \dots, m_l\}$ such that $c_{m_r n_d} = 0, \forall r = 1, 2, \dots, l$ and $d = 1, 2, \dots, q$.*

Definition 2. *Let $x(t, y)$ be the unique solution of system (3). For all $t \geq 0$, a family of mappings $\Phi(t) : X \to X$, such that $\Phi(t)y = x(t, y), \forall y \in X$ is called periodic semi-flow generated by (3) if $\Phi(0) = id_X$; $\Phi(t+\omega) = \Phi(t) \circ \Phi(\omega)$ and $\Phi(t)y$ is continuous in $\mathbb{R}_+ \times X$.*

If $t = \omega$, $P := \Phi(\omega)$ is called Poincaré map associated to system (3); satisfying $P^n(x^0) = x(n\omega, x^0), \forall n \geq 1$.

Definition 3. *Let U be a non-empty, closed, and order convex set and $\kappa : U \longrightarrow U$ be a continuous map. κ is said to be strictly sub-homogeneous if $\kappa(\alpha a) > \alpha\kappa(a)$ with $a \gg 0$ and $\alpha \in (0, 1)$.*

Let us consider the following linear ordinary differential system:

$$\dot{\mathcal{Z}}(t) = \mathcal{A}(t)\mathcal{Z}(t), \tag{5}$$

where $\mathcal{A}(t)$ is a continuous, cooperative, irreducible and ω-periodic $n \times n$ matrix function. Let $\Phi_{\mathcal{A}}(t)$ be the fundamental solution matrix of (5) and $\rho\big(\Phi_{\mathcal{A}}(\omega)\big)$ be the spectral radius of the matrix $\Phi_{\mathcal{A}}(\omega)$. By the Perron-Frobenius theorem (see [29], Theorem A.3), $\rho\big(\Phi_{\mathcal{A}}(\omega)\big)$ is the principal eigenvalue of $\Phi_{\mathcal{A}}(\omega)$ which is associated to the positive eigenvector. The following result is useful for our subsequent comparison arguments.

Lemma 2.1. *Let $\eta = \dfrac{1}{\omega}\ln \rho(\Phi_{\mathcal{A}}(\omega))$, then there exists a positive ω-periodic function $\theta(t)$ such that $\theta(t)e^{\eta t}$ is a solution of (5).*

Proof. For an positive eigenvector ϱ associated with the principal eigenvalue $\rho(\Phi_{\mathcal{A}}(\omega))$, introducing the new variable, $\tilde{\mathcal{Z}}(t) = \theta(t)e^{\eta t}$; the linear system (5) leads to:

$$\dot{\theta}(t) = \big(\mathcal{A}(t) - \eta I_q\big)\theta(t). \tag{6}$$

Therefore, $\theta(t) := \Phi_{\mathcal{A}(t)-\eta I_q}(t)\varrho$ is a positive solution of (6). Then, one can easily see that $e^{\eta t}\Phi_{\mathcal{A}(t)-\eta I_q}(t) = \Phi_{\mathcal{A}(.)}(t)$. So, $\theta(\omega) = \Phi_{\mathcal{A}(t)-\eta I_q}(\omega)\varrho = e^{-\eta\omega}\Phi_{\mathcal{A}(.)}(\omega)\varrho = \varrho = \theta(0)$. It then follows that $\theta(t)$ is a positive ω-periodic solution of (6). Thus, $\tilde{\mathcal{Z}}(t)$ is a solution of (5). □

2.2. Existence and characterization of solutions.

Proposition 2.1. *For any positive initial condition x_i^0, for $i = 1, \dots, n$, problem (1) has a unique positive solution for all $t \geq 0$. Moreover, there is an $h^\star > 0$ such that every forward orbit in $\mathbb{R}_+^{2n}$ of (1) enters into*

$$\mathcal{G} := \Big\{(u, v) \in \mathbb{R}_+^{2n} : \sum_{i=1}^{n} h_i \leq h^\star\Big\}, \tag{7}$$

and $\mathcal{G}$ is positively invariant for model (1).

Theorem 2.1. *Let $M(t, .)$ be the periodic function matrix defined by*

$$M(t,.) = \begin{pmatrix} b_1(t,.) - \mu_1(t) + a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & b_2(t,.) - \mu_2(t) + a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & b_n(t,.) - \mu_n(t) + a_{nn}(t) \end{pmatrix}.$$

If $\rho(\Phi_{M(.,0)}(\omega)) > 1$, then system (1) admits a unique positive periodic disease-free equilibrium.

Proof. To prove the existence of periodic disease-free equilibrium, we consider the following equation:

$$\dot{u}_i := g_{1i}(t, u_i, 0) = b_i(t, h_i)h_i - \mu_i(t)u_i + \sum_{j=1}^{n} a_{ij}(t)u_j, \;\; i = 1, \dots, n; \tag{8}$$

which we write as follows: $\dot{u} = M(t,\lambda)u$; with

$$M(t,\lambda) = \begin{pmatrix} b_1(t,\lambda) - \mu_1(t) + a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & b_2(t,\lambda) - \mu_2(t) + a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & b_n(t,\lambda) - \mu_n(t) + a_{nn}(t) \end{pmatrix}.$$

We note that matrix $M(t,\lambda)$ is irreducible with non-negative off-diagonal elements and the RHS, of equation (8) satisfies the following conditions:

C1) $g_{1i}(t,u,0) \geq 0$, for every $u \geq 0$.
C2) $\dfrac{\partial g_{1i}}{\partial u_j}(t,u,0) \geq 0, \forall i \neq j$ and $D_u g_{1i}(t,0,0)$ is irreducible.
C3) $g_{1i}(t,.,0)$ is strictly sub-homogeneous on $\mathbb{R}^n_+$.
C4) $g_{1i}(t,0,0) = 0$ and $g_{1i}(t,u,0) < D_u g_{1i}(t,0,0)u, \forall t \geq 0$ and $u \gg 0$.

Since the non-linear system (8) is dominated by linear system: $\dot{u} \leq D_u g_{1i}(t,0,0)u$, then, for any initial condition $u^0 \in \mathbb{R}^n_+$, the global existence and stability, of the unique solution $u(t,u^0),\ \forall t \geq 0$ is given by Corollary 3.2 in [30]. Now, it remains to prove that $u(t,u^0)$ is positive and bounded. Indeed, let P_u be the Poincaré map associated with (8). Let λ^* be the sufficient positive large real number such that $\int_0^\omega (b_i(t,\lambda^*) - \mu_i(t))dt < 0$. Hence, thanks to Lemme 2.1, there exists $\bar{\theta}(t)$ such that $\bar{\theta}(t)e^{\bar{\eta}t}$ is a solution of $\dot{u}(t) = M(t,\lambda^*)u(t)$ with $\bar{\eta} = \dfrac{1}{\omega}\rho(\Phi_{M(.,\lambda^*)}(\omega))$. Moreover, let $\bar{I}(t) = e^{\bar{\eta}t}\sum_{i=1}^n \bar{\theta}_i(t)$ and $L(t) = \max\{b_i(t,\lambda^*) - \mu_i(t) : 1 \leq i \leq n\}$. It yields that $\dot{I}(t) \leq L(t)I(t)$, and then $I(t) \to 0$ as $t \to \infty$. Therefore $\rho(\Phi_{M(.,\lambda^*)}(\omega)) < 1$. Thus, for $\bar{l} > 0$ large enough such that $\bar{l}\bar{\theta}(t) > \lambda^*$ for all $t \in [0,\omega]$. Hence, thanks to standard comparison principle, it follows that $\forall \bar{m} \geq 0$:

$$0 < P_u^{\bar{m}}(\bar{l}\bar{\theta}(0)) \leq \Phi_{M(.,\lambda^*)}(\bar{m}\omega)l\bar{\theta}(0) = \rho(\Phi_{M(.,\lambda^*)}(\bar{m}\omega))l\theta(0) = \rho(\Phi_{M(.,\lambda^*)}(\omega))^{\bar{m}}\bar{l}\bar{\theta}(0) < \bar{l}\bar{\theta}(0).$$

That means that $P_u^{\bar{m}}(\bar{l}\bar{\theta}(0))$ is bounded. Since, $\rho(\Phi_{M(.,0)}(\omega)) > 1$, it comes from ([31], see Theorem 2.1.2) that the Poincaré map P_u has a unique positive fixed point $\bar{u}(0)$ which is globally attractive. Therefore, system (1) admits a unique positive periodic disease-free equilibrium, $\mathcal{E}^0(t) := (\bar{u}(t), \mathbf{0})$. □

2.3. **Thresholds dynamics.** In this part of the paper, we compute the thresholds dynamics of system (1) according to the theory developed by Wang and Zhao [32] for periodic systems. Linearising system (1) at the periodic disease-free equilibrium, $\mathcal{E}^0(t)$ we obtain the following system:

$$\dot{v}(t) = (F(t) - V(t))v(t), \tag{9}$$

with

$$F(t) = \text{diag}\{p_1(t,u), p_2(t,u), \dots, p_n(t,u)\}$$

and

$$V(t) = -\begin{pmatrix} -\mu_1(t) - \gamma_1(t) + b_{11}(t) & b_{12}(t) & \dots & b_{1n}(t) \\ b_{21}(t) & -\mu_2(t) - \gamma_2(t) + b_{22}(t) & \dots & b_{2n}(t) \\ \dots & \dots & \dots & \dots \\ b_{n1}(t) & b_{n2}(t) & \dots & -\mu_n(t) - \gamma_n(t) + b_{nn}(t) \end{pmatrix}.$$

For all $t \geq s$, let $\mathbb{W}(t,s)$ be the evolution operator of the linear periodic system

$$\dot{w} = -V(t)w.$$

For each $s \in \mathbb{R}$, the $n \times n$ matrix $\mathbb{W}(t,s)$ satisfies the equation

$$\dot{\mathbb{W}}(t,s) = -V(t)\mathbb{W}(t,s), \forall t \geq s, \quad \mathbb{W}(s,s) = I, \tag{10}$$

where I is the $n \times n$ identity matrix.

Let C_ω be the ordered Banach space of all ω-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ which is equipped with the maximum norm $\|.\|$ and the positive cone

$$\mathrm{C}_\omega^+ := \big\{\phi \in \mathrm{C}_\omega : \phi(t) \geq 0, \forall t \in \mathbb{R}\big\}.$$

Suppose $\phi(s) \in \mathrm{C}_\omega$ is the initial distribution of infectious individuals in the periodic habitat. Then $F(s)\phi(s) \in \mathrm{C}_\omega$ is the distribution of newly infected individuals at time t produced by infected individuals introduced at time $s \leq t$. Hence, $\mathbb{W}(t,s)F(s)\phi(s)$ represents the distribution of those infected individuals who were newly infected at time s and remain in the compartment at time s. Thus,

$$\psi(t) = \int_{-\infty}^{t} \mathbb{W}(t,s)F(s)\phi(s)ds = \int_0^\infty \mathbb{W}(t,t-a)F(t-a)\phi(t-a)da, \ a \in [0,\infty)$$

is the distribution of accumulative new infections at time t, produced by all those infected $\phi(s)$ introduced at the previous time.

Let $\mathcal{L} : \mathrm{C}_\omega \longrightarrow \mathrm{C}_\omega$ be the linear operator defined by

$$(\mathcal{L}\phi)(t) = \int_0^\infty \mathbb{W}(t,t-a)F(t-a)\phi(t-a)da, \quad \forall t \in \mathbb{R}, \ \phi \in \mathrm{C}_\omega.$$

Thus, the basic reproduction ratio, $\mathcal{R}_0$ of system (1) is given by the spectral radius of $\mathcal{L}$: $\mathcal{R}_0 = \rho(\mathcal{L})$.

Remark 2.1.

i) *If only susceptible hosts can travel, namely $b_{ij}(t) = 0, \forall 1 \leq i,j \leq n$, then $V(t) = diag(\mu_1(t) + \gamma_1(t), \mu_2(t) + \gamma_2(t), \dots, \mu_n(t) + \gamma_n(t))$. Thus, thanks to Lemma 2.2 from* [32],

$$\mathcal{R}_0 = \max_{1 \leq i \leq n} \{\mathcal{R}_{0,i}\} \tag{11}$$

where $[Z(t)] = \dfrac{1}{\omega}\displaystyle\int_0^\omega Z(t)dt$ and $\mathcal{R}_{0,i} = \dfrac{[p_i(t,\bar{u})]}{[\mu_i(t) + \gamma_i(t)]}$ is a specific basic reproduction rate of patch i when the network is deconnected.

ii) *If matrices $F(t)$ and $V(t)$ are both constant, then $\mathcal{R}_0 = \rho(FV^{-1})$. Hence, we derive Salmani [13] inegality type as follows:*

$$\min_{1\le i\le n} \mathcal{R}_{0,i} \le \rho(FV^{-1}) \le \max_{1\le i\le n} \mathcal{R}_{0,i},\ \textit{if}\ \mu_i = \mu_j = \mu, \gamma_i = \gamma_j = \gamma,\ \forall i \neq j.$$

3. Main results

The following result will be useful in the rest of the paper.

Lemma 3.1. *The following statements hold:*

i) $\mathcal{R}_0 = 1$ *if and only if* $\rho(\Phi_{F-V}(\omega)) = 1$.

ii) $\mathcal{R}_0 < 1$ *if and only if* $\rho(\Phi_{F-V}(\omega)) < 1$.

iii) $\mathcal{R}_0 > 1$ *if and only if* $\rho(\Phi_{F-V}(\omega)) > 1$.

3.1. Dependence of $\mathcal{R}_0$ on incidence function and matrix movement.

Theorem 3.1. *If there exists a positive real τ such that for all $1 \le i \le n$, $\hat{f}_i(t, u_i, v_i) = \tau f_i(t, u_i, v_i)$, then the basic reproduction number, $\mathcal{R}_0^\diamondsuit$ associated to this incidence function is computed as follows:*

$$\mathcal{R}_0^\diamondsuit = \tau \mathcal{R}_0.$$

Proof. Let us consider the following linear ω-periodic system:

$$\dot{w}(t) \;=\; \left[\frac{1}{\xi}F(t) - V(t)\right] w(t),\ \forall t \in \mathbb{R}_+,\ \xi \in (0, \infty), \tag{12}$$

with $\mathbb{W}(t, s, \xi), t \ge s, s \in \mathbb{R}$, the associated evolution operator on $\mathbb{R}^{2n}$. It is obvious that

$$\mathbb{W}(t, 0, 1) = \Phi_{F-V}(t),\ \forall\, t \ge 0.$$

If $\hat{f}_i(t, u_i, v_i) = \tau f_i(t, u_i, v_i)$ then, the linear system (12) becomes:

$$\dot{w}(t) = \left[\frac{\tau}{\xi}F(t) - V(t)\right] w(t),\ \ \forall t \in \mathbb{R}_+, \xi \in (0, \infty).$$

Let $\hat{F}(t) = \tau F(t)$, $\hat{V}(t) = V(t)$ and $\hat{\mathbb{W}}(\omega, 0, \xi)$ the monodromy matrix of the following system:

$$\dot{\hat{w}}(t) = \left[\frac{1}{\xi}\hat{F}(t) - \hat{V}(t)\right] \hat{w}(t),\ \ \forall t \in \mathbb{R}_+, \xi \in (0, \infty).$$

Since

$$\hat{\mathbb{W}}(\omega, 0, \xi) = \mathbb{W}\left(\omega, 0, \frac{\xi}{\tau}\right).$$

Then, thanks to the Theorem 2.1 in [30], we have

$$\rho(\hat{\mathbb{W}}(\omega, 0, \mathcal{R}_0^\diamondsuit)) = 1 \Longleftrightarrow \rho\left(\mathbb{W}\left(\omega, 0, \frac{\mathcal{R}_0^\diamondsuit}{\tau}\right)\right) = 1.$$

Hence, $\mathcal{R}_0^\diamondsuit = \tau \mathcal{R}_0$. □

Theorem 3.2. *$\mathcal{R}_0$ is a non-decreasing function with respect to infected individuals movement matrix, $\mathcal{B} = (b_{ij})$.*

Proof. Let $\mathcal{B}$ et $\tilde{\mathcal{B}}$ be two infected individuals movement matrix such that $\mathcal{B} \leq \bar{\mathcal{B}}$, that is all entries b_{ij} and $\bar{b}_{ij}$ respectively of $\mathcal{B}$ and $\bar{\mathcal{B}}$ satisfy $b_{ij}(t) \leq \bar{b}_{ij}(t),\ \forall i, j = 1, 2, \ldots, n$. Then, $\forall t \in \mathbb{R},\ \phi \in \mathrm{C}_\omega$.

$$\int_0^\infty \mathbb{W}(t, t-a)F(t-a)\phi(t-a)da \leq \int_0^\infty \bar{\mathbb{W}}(t, t-a)F(t-a)\phi(t-a)da, \tag{13}$$

where $\bar{\mathbb{W}}(t, s)$ satisfies

$$\dot{\bar{\mathbb{W}}}(t, s) = -\bar{V}(t)\bar{\mathbb{W}}(t, s), \forall t \geq s, \tag{14}$$

with

$$\bar{V}(t) = -\begin{pmatrix} -\mu_1(t) - \gamma_1(t) + \bar{b}_{11}(t) & \bar{b}_{12}(t) & \ldots & \bar{b}_{1n}(t) \\ \bar{b}_{21}(t) & -\mu_2(t) - \gamma_2(t) + \bar{b}_{22}(t) & \ldots & \bar{b}_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ \bar{b}_{n1}(t) & \bar{b}_{n2}(t) & \ldots & -\mu_n(t) - \gamma_n(t) + \bar{b}_{nn}(t) \end{pmatrix}.$$

It yields that $(\mathcal{L}\phi)(t) \leq (\bar{\mathcal{L}}\phi)(t), \forall t \in \mathbb{R},\ \phi \in \mathrm{C}_\omega$. Hence $\rho(\mathcal{L}) \leq \rho(\bar{\mathcal{L}})$. So, $\mathcal{R}_0(\mathcal{B}) \leq \mathcal{R}_0(\bar{\mathcal{B}})$. □

3.2. **Epidemic extinction.** Here, we focus on the global dynamics of the model near of the periodic disease-free equilibrium, $\mathcal{E}^0(t)$.

Theorem 3.3. *If $\mathcal{R}_0 < 1$ and $a_{ij}(t) = b_{ij}(t)$ for all $i, j = 1, \ldots, n$ then $\mathcal{E}^0(t)$ is globally for $(u^0, v^0) \in \mathbb{R}^n_+ \setminus \{0\} \times \mathbb{R}^n_+$.*

Proof. If $a_{ij}(t) = b_{ij}(t)$ for all $i, j = 1, \ldots, n$, then we have:

$$\dot{h}_i = b(t, h_i)h_i - (\mu_i + \gamma_i)h_i + \sum_{j=1}^n b_{ij}(t)h_j, i = 1, \ldots, n. \tag{15}$$

Since (15) admits a unique positive equilibrium $\bar{h}(t)$ which is globally asymptotically stable for $h^0 \in \mathbb{N}^n_+ \setminus \{0\}$, there exists t_ε such that

$$\forall t \geq t_\varepsilon, h_i(t) < \bar{u}_i(t) + \varepsilon, i = 1, \ldots, n. \tag{16}$$

Moreover, assuming (A6):

$$f_i(t, u_i, v_i) < p_i(t, u)v_i,\ \forall u_i, v_i \geq 0,\ i = 1, \ldots, n.$$

Then, we have:

$$\dot{v}_i < p_i(t, \bar{u} + \varepsilon)v_i - (\mu_i + \gamma_i)v_i + \sum_{j=1}^n b_{ij}(t)v_j,\ \ 1 \leq i \leq n. \tag{17}$$

Let us consider the following auxiliary system:

$$\dot{\tilde{v}}(t) = (F_\varepsilon(t) - V(t))\tilde{v}(t),\ \tilde{v}(t) = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)^T,\ \text{with } F_\varepsilon = \mathrm{diag}\big(p_i(t, \bar{u} + \varepsilon)\big)_{1 \leq i \leq n}. \tag{18}$$

By the continuity of the spectral radius, if $\mathcal{R}_0 < 1$, we can fix ε small enough such that $\rho(\Phi_{F_\varepsilon - V}(\omega)) < 1$. Thus, thanks to Lemma 2.1, there exists a positive ω-periodic function $\theta_1(t)$ such that $\theta_1(t)e^{\eta t}$ is a solution of (18); with $\eta = \dfrac{1}{\omega}\ln \rho(\Phi_{F_\varepsilon - V}(\omega))$. Since θ_1 is bounded and $\eta < 0$, then standard comparison theorem indicates that solutions of (18) tend to zero as t goes to infinity. Therefore, for any initial condition $(u^0, v^0) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, with $u^0 \neq 0$, we have $h^0 \in \mathbb{R}^n_+ \setminus \{0\}$. By the global attractivity of $\bar{h}(t)$, it then follows that

$$\lim_{t\to+\infty}(u(t) - \bar{u}(t)) = \lim_{t\to+\infty}(h(t) - v(t) - \bar{u}(t)) = 0.$$

□

Theorem 3.4. *If $\mathcal{R}_0 < 1$ then there exist $\xi > 0$ such that for every $x^0 \in \mathcal{G}$, with $v_i^0 < \xi, \;\; i = 1, \ldots, n$, the solution $x(t, x^0)$ of system (1) satisfies: $\lim_{t\to\infty}(u(t), v(t)) = (\bar{u}(t), 0)$.*

Proof. Let us define the following auxiliary system:

$$u_i = b_i(t, u_i)u_i - \mu_i(t)u_i + (b_i(t, 0+) + \gamma_i)\epsilon + \sum_{j=1}^{n} a_{ij}(t)u_j, \;\; i = 1, \ldots, n, \tag{19}$$

with ϵ a small constant to be determined. From above, we can select ϵ small enough such that (19) admits a unique positive equilibrium $\bar{u}(\epsilon)$ which is globally asymptotically stable.

Let $u_\epsilon(t, \bar{h})$ be the solution of (19) through $\bar{h}$ at $t = 0$ and $T_\epsilon = m_1\omega > 0$ such that

$$u_\epsilon(t, \bar{h}) < \bar{u}(\epsilon) + \varepsilon, \forall t \geq T_\epsilon, \;\; \varepsilon = (\epsilon, \ldots, \epsilon), \text{and } m_1 \in \mathbb{N}^*. \tag{20}$$

Let $F^\epsilon = \mathrm{diag}\big(p_i(t, \bar{u} + \varepsilon)\big)_{1\leq i\leq n}$ with $F^0 = F$. By the continuity of the spectral radius, if $\mathcal{R}_0 < 1$ then, we can restrict ϵ small enough such that $\rho(\Phi_{F^\epsilon - V}(\omega)) < 1$. Thanks to Lemma 2.1, there exists a positive ω-periodic function $\theta_2(t)$ such that $\theta_2(t)e^{\eta_2 t}$ is a solution of $\dot{\hat{v}}(t) = (F^\epsilon - V)\hat{v}(t)$; with $\eta_2 = \dfrac{1}{\omega}\ln \rho(\Phi_{F^\epsilon - V}(\omega))$. Let choose $k > 0$ such that $k\theta_2(t) < \varepsilon, \forall t \in [0, \omega]$. Let $\hat{v}(t, \delta)$ be the solution through δ at $t = 0$ of the following auxiliary system:

$$\dot{\hat{v}}_i = p_i(t, \bar{h})\hat{v}_i - (\mu_i + \gamma_i)\hat{v}_i + \sum_{j=1}^{n} b_{ij}(t)\hat{v}_j, \; i = 1, \ldots, n \tag{21}$$

such that for all $\delta > 0$ small enough, we have

$$\hat{v}(t, \delta) < k\theta_2(t) < \varepsilon, \forall t \in [0, T_\epsilon]. \tag{22}$$

Lemma 3.2. *For all initial condition $x^0 \in \mathcal{G}$, with $v_i^0 < \delta$, $i = 1, \ldots, n$, the non negative solution $x(t, x^0)$ of system (1) satisfies, $v(t) \leq k\theta_2(t)e^{\eta_2 t}, \forall t \geq 0$.*

Proof. By contradiction, suppose not. Then, there exist q, $1 \leq q \leq n$ and $T' > T_\epsilon$ such that:

$$\begin{cases} v(t) \leq k\theta_2(t)e^{\eta_2 t} & \text{for } t \in [0, T'], \\ v_q(T') = k(e^{\eta_2 T'}\theta_2(T'))_q, & \\ v_q(t) > k(e^{\eta_2 T'}\theta_2(T'))_q & \text{for } 0 < t - T' \ll 1. \end{cases} \tag{23}$$

If $t \in [0, T']$, then we have

$$\dot{u}_i < b_i(t, u_i)u_i - \mu_i(t)u_i + (b_i(t, 0+) + \gamma_i)\epsilon + \sum_{j=1}^{n} a_{ij}(t)u_j, \;\; i = 1, \ldots, n. \tag{24}$$

Thus, it follows from the comparison principle that $u(T') < \bar{u}(T', \epsilon) + \varepsilon$. Hence, for $0 < t - T' \ll 1$, it implies that

$$\dot{v}_i < p_i(t, \bar{u}(\epsilon) + \varepsilon)v_i - (\mu_i + \gamma_i)v_i + \sum_{j=1}^{n} b_{ij}(t)v_j, \; i = 1, \ldots, n. \tag{25}$$

Since $v(T') \leq k\theta_2(T')e^{\eta_2 T'}$, the comparison principle implies that

$$v(t) < k\theta_2(t)e^{\eta_2 t} \text{ for } 0 < t - T' \ll 1. \tag{26}$$

So

$$v_q(t) < k(\theta_2(t)e^{\eta_2 t})_q \text{ for } 0 < t - T' \ll 1, \tag{27}$$

which lead to a contradiction. □

Consequently,

$$u(t) < \bar{u}(\epsilon) + \varepsilon, \forall t \geq T_\epsilon \text{ and } v(t) < k\theta_2(t)e^{\eta_2 t}, \forall t > T_\epsilon. \tag{28}$$

It then follows that $v(t) \to 0$ as $t \to \infty$.

Let P be the Poincaré map associated to system (1). Given the initial condition $x^0 \in \mathcal{G}$ with $u^0 \neq 0$ and $v_i^0 < \delta$, $i = 1, \ldots, n$, it easily follows that $u(t) \in int(\mathbb{R}_+^n), \forall t > 0$. Let $\mathcal{O}' = \mathcal{O}'(x^0)$ be the omega limit set of x^0 for P. Since, $v(t) \to 0$ as $t \to \infty$, then $\mathcal{O}' = \bar{\mathcal{O}}' \times \{0\}$ holds. Hence, we claim that $\bar{\mathcal{O}}' \neq \{0\}$. Let assume that, by contradiction, $\bar{\mathcal{O}}' = \{0\}$. It then follows that $\lim\limits_{t\to\infty} x(t, x^0) = \lim\limits_{m\to\infty} P^m(x^0) = (0, 0)$. Since $\rho(\Phi_{M(.,0)}) > 1$, we can choose a small $\ell > 0$ such that $\rho(\Phi_{M(.,0)-\ell \mathrm{diag}(1,\ldots,1)}(\omega)) > 1$. Thus, there exists $t' > 0$ such that

$$\dot{u}_i(t) > (b_i(t, 0+) - \ell)u_i - \mu_i u_i + \sum_{j=1}^{n} a_{ij}(t)u_j, \; \forall t \geq t', i = 1, \ldots, n.$$

Let $\eta_4 = \frac{1}{\omega} \ln \rho(\Phi_{M(.,0)-\ell \mathrm{diag}(1,\ldots,1)})(\omega))$ and $\theta_4(t)$ be a positive ω-periodic function such that $\theta_4(t)e^{\eta_4 t}$ is a solution of the following auxiliary linear solution:

$$\dot{u}_i(t) = (b_i(t, 0+) - \ell)u_i - \mu_i u_i + \sum_{j=1}^{n} a_{ij}(t)u_i, \; \forall, i = 1, \ldots, n. \tag{29}$$

Let select a small number d such that $u(t') > d\theta_4(0)$. Hence, thanks to the comparison theorem, we have:

$$u(t) \geq de^{\eta_4(t-t')} \min_{t-t'\geq 0} \theta(t - t'), \; \forall t \geq t'. \tag{30}$$

Thus, $\lim_{t\to\infty} S(t) = \infty$. which leads to a contradiction.

Moreover, we note that for all $u^0 \in \mathbb{R}^n_+, m \geq 0$, $P^m(u^0, 0) = (P^m_u(u^0), 0)$. Hence by Lemma 2.1, $\mathcal{O}'$ is an internal chain transitive set for P. So $\bar{\mathcal{O}}'$ is an internal chain transitive set for P_u. Since, $\bar{\mathcal{O}}' \neq 0$ and $\bar{u}(0)$ is globally asymptotically stable for (1) in $\mathbb{R}^n_+ \setminus \{0\}$, we have $\bar{\mathcal{O}}' \cap W^s(\bar{u}(0)) \neq \emptyset$. It then follows that $\bar{\mathcal{O}}' = \bar{u}(0)$. Thus, $\mathcal{O}' = \{(\bar{u}(0), 0)\}$. Consequently, $(u(t), v(t)) \to (\bar{u}(t), 0)$ as $t \to \infty$. □

3.3. Existence of positive ω-periodic solutions.

Theorem 3.5. *Let $\mathcal{R}_0 > 1$. Then, system (1) is uniformly persistent and there exists at least one positive equilibrium.*

Proof. Let us consider the following sets:

$$\begin{aligned} X &:= \{(u, v) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+\}, \\ X_0 &:= \{(u, v) \in X : u > 0\}, \\ \partial X_0 &:= X \setminus X_0. \end{aligned}$$

It then suffices to show that system (1) is uniformly persistent with respect to $(X_0, \partial X_0)$. Note that X and X_0 are positively invariant and ∂X_0 is relatively closed in X. Furthermore, system (1) is point dissipative.

We consider the set: $M_\partial = \{x^0 : P^m(x^0) \in \partial X_0, \forall m \geq 0\}$ and claim that:

$$M_\partial = \{(u, 0) : u \geq 0\}. \tag{31}$$

Assume $x^0 \in M_\partial$. Let us show that $v(t) = 0, \forall t \geq 0$. Suppose not. Then, there exist $1 \leq i_0 \leq n$ and $t_0 \geq 0$ such that $v_{i0}(t_0) > 0$. Define a partition of $\{1, \ldots, n\}$ into two sets J_1 and J_2 such that $v_i(t_0) = 0, \forall i \in J_1$ and $v_i(t_0) > 0, \forall i \in J_2$. Hence,

$$\dot{v}_i(t_0) \geq b_{ji_0} v_{i0}(t_0) > 0, \tag{32}$$

and there exist $\epsilon_0 > 0$ such that $v_j(t) > 0, \forall j \in J_1; t \in]t_0, t_0 + \epsilon_0[$. Thus, we can restrict ϵ_0 small enough such that $I_i(t) > 0, \forall i \in J_2; t \in]t_0, t_0 + \epsilon_0[$. Thus $x^0 \notin M_\partial$; so (31) holds. Moreover, we note that the Poincaré map P has two fixed points in M_∂; that is $\mathcal{M}_0 = (0, 0)$ and $\mathcal{M}_1 = (\bar{u}(t), 0)$.

For ϵ_1 small enough, let us consider the following perturbed system:

$$\dot{u}_i = b_i(t, u_i + \epsilon_1)u_i - \mu_i(t)u_i - f_i(t, u_i, \epsilon_1) + \sum_{j=1}^{n} a_{ij}(t)u_j, \ i = 1, \ldots, n. \tag{33}$$

Since (33) admits a unique positive equilibrium $\bar{u}_{\epsilon_1}(t)$ which is globally asymptotically stable, by continuity, we can restrict ϵ_1 such that $\bar{u}_{\epsilon_1}(t) > \bar{u}(t) - \bar{\xi}$, with $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n)$.

Proposition 3.1. *If $\mathcal{R}_0 > 1$, there exists $\zeta > 0$ such that for any initial condition $x^0 \in X_0$, when $\|x^0 - \mathcal{M}_i\| \leq \zeta$, one has $\limsup_{m \to \infty} d\big(P^m(x^0), \mathcal{M}_i\big) \geq \zeta$, $i = 0, 1$.*

Proof. Suppose by contradiction that $\limsup_{m\to\infty} d\big(P^m(x^0), \mathcal{M}_i\big) < \zeta$ for some $x^0 \in X_0$ and i. Then, it follows that there exists an integer $N \geq 1$ such that for all $m \geq N$ we have $d\big(P^m(x^0), \mathcal{M}_i\big) < \zeta$. By the continuity of solution $x(t, x^0)$, we have $\|x\big(t, P^m(x^0)\big) - x(t, \mathcal{M}_i)\| \leq \epsilon_1, \forall t \geq 0,\ \epsilon_1 > 0$. Let $t = m\omega + t'$, with $t' \in [0, \omega]$ and $m = \left[\dfrac{t}{\omega}\right]$. $\left[\dfrac{t}{\omega}\right]$ is the greatest integer less than or equal to $\dfrac{t}{\omega}$. Then, we have $\|x(t, x^0) - x(t, \mathcal{M}_i)\| = \|x\big(t', P^m(x^0)\big) - x(t', \mathcal{M}_i)\| < \epsilon_1$. Hence, $v_i(t) \leq \epsilon_1, \forall t \geq 0$ and $1 \leq i \leq n$. It then follows that:

$$\dot{u}_i \geq b_i(t, u_i + \epsilon_1)u_i - \mu_i(t)u_i - f_i(t, u_i, \epsilon_1) + \sum_{j=1}^{n} a_{ij}(t)u_j, 1 \leq i \leq n. \tag{34}$$

Consequently, there exists $T_1 > 0$ such that $u(t) \geq \bar{u}(t) - \bar{\xi}, \forall t \geq T_1$. Furthermore, based on assumptions (A4), it follows that:

$$f_i(t, \bar{u}(t) - \bar{\xi}, v_i) \geq p_i(t, \bar{u}(t) - \bar{\xi})v_i + \frac{1}{2}\epsilon_1 v_i \frac{\partial^2 f}{\partial v_i^2}(t, \bar{u}(t) - \bar{\xi}, v_i). \tag{35}$$

Hence

$$\dot{v}_i \geq p_i(t, \bar{u}(t) - \bar{\xi})v_i + \frac{1}{2}\epsilon_1 v_i \frac{\partial^2 f}{\partial v_i^2}(t, \bar{u}(t) - \bar{\xi}, v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \sum_{j=1}^{n} b_{ij}(t)v_j. \tag{36}$$

Since $\mathcal{R}_0 > 1$, then $\rho(\Phi_{F-V}(\omega)) > 1$. By the continuity of the spectral radius, we have $\lim_{\bar{\xi}\to 0^+} \rho\big(\Phi_{F_{\bar{\xi}}-V}(\omega)\big) = \rho\big(\Phi_{F-V}(\omega)\big) > 1$. Thus, there exists $\epsilon^* > 0$ such that $\rho\big(\Phi_{F_{\bar{\xi}}-V}(\omega)\big) > 1,\ \forall \bar{\xi} \in [0, \epsilon^*]$. Hence, there exists a positive ω-periodic function $\theta_3(t)$ such that $e^{rt}\theta_3(t)$ is a solution of RHS(36), with $r = \dfrac{1}{\omega}\ln \rho(\Phi_{F_\epsilon - V}(\omega))$. Since $\rho(\Phi_{F_{\bar{\xi}}-V}(\omega)) > 1$, then $r > 0$ and $\lim_{t\to\infty} v_i(t) = \infty$. which leads to a contradiction. □

Note that , $\bar{u}(t)$ is globally asymptotically stable in $\mathbb{R}^n_+ \setminus \{0\}$. By the afore-mentioned proposition, it follows that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isolated invariant sets in X, $\mathcal{W}^s((0,0)) \cap X_0 = \emptyset$ and $\mathcal{W}^s(\mathcal{M}_i) \cap X_0 = \emptyset$ for $i = 0, 1$. Every orbit in M_∂ converges to either $\mathcal{M}_0$ or $\mathcal{M}_1$. Further, $\mathcal{M}_0$ and $\mathcal{M}_1$ are acyclic in M_∂. By the acyclicity theorem on uniform persistence for maps, we conclude that system (1) is uniformly persistent with respect to $(X_0, \partial X_0)$ [33]. Then, model (1) admits a fixed point $(u^*(t), v^*(t)) \in X_0$ as an equilibrium. Furthermore, suppose that $u^*(t) = 0$. Then, we derive that

$$\sum_{i=1}^{n} u_i^*(t) = 0. \tag{37}$$

Hence, $u_i^*(t) = 0,\ \forall i = 1, \ldots, n$. Which leads to a contradiction. So, the model admits at least one positive ω-periodic solution, $x^*(t)$. □

Theorem 3.6. *If $\mathcal{R}_0 > 1$ and $a_{ij}(t) = b_{ij}(t)$, then the positive ω-periodic solution $x^*(t)$ is globally attractive in $\mathbb{R}^n_+ \times int(\mathbb{R}^n_+)$.*

Proof. If $a_{ij}(t) = b_{ij}(t)$, then system (1) reads as follows:

$$\text{(38)}\qquad \begin{cases} \dot{u}_i = b_i(t,h_i)h_i - \mu_i(t)u_i - f_i(t,u_i,v_i) + \gamma_i(t)v_i + \displaystyle\sum_{j=1}^{n} a_{ij}(t)u_j, \ \ 1 \le i \le n, \\ \dot{v}_i = f_i(t,u_i,v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \displaystyle\sum_{j=1}^{n} a_{ij}(t)v_j, \ \ 1 \le i \le n. \end{cases}$$

Let consider the following equation:

$$\text{(39)}\qquad \dot{h}(t) = b_i(t,h_i)h_i - \mu_i(t) + \sum_{j=1}^{n} a_{ij}(t)h_j, \ \ 1 \le i \le n.$$

Since, (39) admits a unique positive periodic solution which is globally asymptotically stable in $int(\mathbb{R}^n_+)$, then system (38) becomes:

$$\text{(40)}\qquad \begin{cases} \dot{h}(t) = b_i(t,h_i)h_i - \mu_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}(t)h_j,, \ \ 1 \le i \le n, \\ \dot{v}_i = f_i(t,h_i - v_i,v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \displaystyle\sum_{j=1}^{n} a_{ij}(t)v_j, \ \ 1 \le i \le n. \end{cases}$$

Finally we obtain the following limiting system:

$$\text{(41)}\qquad \dot{\hat{v}}_i = f_i(t,\bar{u}_i - \hat{v}_i,\hat{v}_i) - \mu_i(t)\hat{v}_i - \gamma_i(t)\hat{v}_i + \sum_{j=1}^{n} a_{ij}(t)\hat{v}_j, \ \ 1 \le i \le n.$$

Let $\tilde{P} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be the Poincaré map associated with (41). Now we show that (41) has a bounded positive solution. Indeed, let us define following de matrix function:

$$\tilde{M}(t,\tilde{\lambda}) = \begin{pmatrix} \tilde{a}_{11}(t,\tilde{\lambda}) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \tilde{a}_{22}(t,\tilde{\lambda}) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & \tilde{a}_{nn}(t,\tilde{\lambda}) \end{pmatrix}.$$

where $\tilde{a}_{ii}(t,\tilde{\lambda}) = f_i(t,\bar{u}_i - \tilde{\lambda},\hat{v}_i) - \mu_i(t) - \gamma_i(t) + a_{ii}(t)$. Therefore, let choose $\tilde{\lambda}^* > 0$ such that $\int_0^{\omega} \tilde{a}_{11}(t,\tilde{\lambda}^*)dt < 0$, for $i = 1,\dots,n$; then by Lemma 2.1 there is a positive ω-periodic function $\tilde{\theta}(t)$ such that $\tilde{\theta}(t)e^{\tilde{\eta}t}$ is a solution of $\dot{\hat{v}}_i = \tilde{M}(t,\tilde{\lambda}^*)\hat{v}_i$ with $\tilde{\eta} = \dfrac{1}{\omega}\ln\rho(\Phi_{M(.,\tilde{\lambda})}(\omega))$.

Let $I(t) = e^{\tilde{\eta}t}\sum_{i=1}^{n}\tilde{\theta}_i(t)$ and $N(t) = \max\{\tilde{a}_{ii}(t,\tilde{\lambda}^*) : 1 \le i \le n\}$. It yields that $\dot{I}(t) \le N(t)I(t)$, and then $I(t) \to 0$ as $t \to \infty$. Therefore $\rho(\Phi_{M(.,\tilde{\lambda}^*)}(\omega)) < 1$. Moreover, let $l > 0$ be large enough such that $l\tilde{\theta}(t) > \tilde{\lambda}^*$ for all $t \in [0,\omega]$. Hence, thanks to standard comparison principle, it follows that $\forall \tilde{m} \ge 0$:

$$0 < \tilde{P}^{\tilde{m}}(l\tilde{\theta}(0)) \le \Phi_{M(.,\tilde{\lambda}^*)}(\tilde{m}\omega)l\tilde{\theta}(0) = \rho(\Phi_{M(.,\tilde{\lambda}^*)}(\tilde{m}\omega))l\tilde{\theta}(0) = \rho(\Phi_{M(.,\tilde{\lambda}^*)}(\omega))^{\tilde{m}}l\tilde{\theta}(0) < l\tilde{\theta}(0).$$

That means that $\tilde{P}^{\tilde{m}}(l\tilde{\theta}(0))$ is bounded. Since, $\mathcal{R}_0 > 1$, it then follows that the Poincaré map $\tilde{P}$ has a unique positive fixed point $\hat{v}^*(0)$ which is globally attractive.Thus thanks to Theorem 3.5, the Poincaré map associated with (38) admits $(\bar{h}(0) - \hat{v}^*(0),\hat{v}^*(0)) := (\hat{u}^*(0),\hat{v}^*(0))$ has a unique positive fixed point.

Furthermore, let $\tilde{P}_1$ be the Poincaré map associated to system (40). Given the initial condition $(h^0, v^0) \in \mathbb{R}_+^{2n} \backslash \{0\}$, it easily follows that $(h(t), v(t)) \in int(\mathbb{R}_+^{2n}), \forall t > 0$. Let $\mathcal{O}' = \mathcal{O}'(h^0, v^0)$ be the omega limit set of (h^0, v^0) for P_1. Since, $v(t) \to 0$ as $t \to \infty$, then $\mathcal{O}' =: \tilde{\mathcal{O}} \times \{u^*(0)\}$ holds. Hence, we claim that $\tilde{\mathcal{O}} \neq \{0\}$. Assume by contradiction, $\tilde{\mathcal{O}} = \{0\}$. It then follows that $\lim_{m\to\infty} P_1^m(h^0, v^0) = (\bar{h}(0), 0)$. Since $\rho(\Phi_{M(.,0)}) > 1$, we can choose a small $\ell_1 > 0$ such that $\rho(\Phi_{M(.,0)-\ell_1 \text{diag}(1,\ldots,1)}(\omega)) > 1$. Thus, there exists $t'' > 0$ such that for $0 < v_i(t) < \epsilon_2$ and $\bar{\xi}_1$ small enough,

$$f_i(t, h_i - v_i, v_i) \geq p_i(t, \bar{u}(t) - \bar{\xi}_1)v_i + \frac{1}{2}\epsilon_2 v_i \frac{\partial^2 f}{\partial v_i^2}(t, \bar{u}(t) - \bar{\xi}_1, v_i), \ \forall t \geq t''.$$

Consequently, we have:

$$\dot{v}_i > p_i(t, \bar{u}(t) - \bar{\xi}_1)v_i + \frac{1}{2}\epsilon_2 v_i \frac{\partial^2 f}{\partial v_i^2}(t, \bar{u}(t) - \bar{\xi}_1, v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \sum_{j=1}^{n} a_{ij}(t)v_j \ \forall t \geq t''.$$

Let $\eta_5 = \frac{1}{\omega} \ln \rho(\Phi_{M(.,0)-\ell_1 \text{diag}(1,\ldots,1)})(\omega))$ and $\theta_5(t)$ be a positive ω-periodic function such that $\theta_5(t)e^{\eta_5 t}$ is a solution of the following auxiliary linear solution:

$$\dot{v}_i = p_i(t, \bar{u}(t) - \bar{\xi}_1)v_i + \frac{1}{2}\epsilon_2 v_i \frac{\partial^2 f}{\partial v_i^2}(t, \bar{u}(t) - \bar{\xi}_1, v_i) - \mu_i(t)v_i - \gamma_i(t)v_i + \sum_{j=1}^{n} a_{ij}(t)v_j, \ \forall, i = 1, \ldots, n.$$

Let select a small number $\bar{d}$ such that $v(t'') > \bar{d}\theta_5(0)$. Hence, thanks to the comparison theorem, we have:

$$v(t) \geq \bar{d}e^{\eta_5(t-t'')} \min_{t-t'' \geq 0} \theta_5(t - t''), \ \forall t \geq t''. \tag{42}$$

Thus, $\lim_{t\to\infty} v_i(t) = \infty, \ \forall 1 \leq i \leq n$; which leads to a contradiction.

Moreover, we note that for all $v^0 \in \mathbb{R}_+^n, m \geq 0$, $\tilde{P}^m(\bar{u}(0), v^0) = (\bar{u}(0), \tilde{P}_1^m(v^0))$. Hence, by Lemma 2.1 in [34], $\mathcal{O}'$ is an internal chain transitive set for $\tilde{P}$. So, $\tilde{\mathcal{O}}$ is an internal chain transitive set for $\tilde{P}_1$. Since, $\tilde{\mathcal{O}} \neq \{0\}$ and $v^*(0)$ is globally asymptotically stable, we have $\tilde{\mathcal{O}} \cap \left\{v^0 : \tilde{P}_1^m(v^0) = v^*(0)\right\} \neq \emptyset$. It then follows that $\tilde{\mathcal{O}} = \{v^*(0)\}$. Thus, $\mathcal{O}' = \{(\bar{u}(0), v^*(0)\}$. Consequently, $(u(t), v(t)) \to (u^*(t), v^*(t))$ as $t \to \infty$.

Now, we prove the stability of $(u^*(t), v^*(t))$ for system (40). The Jacobian matrix of (40) evaluated at $(h^*(t), v^*(t))$ give:

$$\mathcal{M}(t) = \begin{pmatrix} \mathcal{M}_1(t) & 0 \\ \mathcal{M}_2(t) & \mathcal{M}_3(t) \end{pmatrix} \tag{43}$$

where

$$\mathcal{M}_1(t) = \begin{pmatrix} \tilde{a}_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & \tilde{a}_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & \tilde{a}_{nn}(t) \end{pmatrix}, \mathcal{M}_2(t) = \text{diag}\left(\frac{\partial f_i}{\partial h_i}(t, h^* - v^*, v^*)\right)$$

and

$$\mathcal{M}_3(t) = \begin{pmatrix} \tilde{c}_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \tilde{c}_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & \tilde{c}_{nn}(t) \end{pmatrix}$$

with

$$\begin{aligned} \tilde{a}_{ii}(t) &= \frac{\partial b_i}{\partial h_i}(t, h_i^*) + b_i(t, h_i^*) - \mu_i(t) + a_{ii}(t), \\ \tilde{c}_{ii}(t) &= -\frac{\partial f_i}{\partial v_i}(t, h^* - v^*, v^*) - \mu_i(t) - \gamma_i(t) + a_{ii}(t). \end{aligned}$$

Since, $\tilde{a}_{ii}(t) < b_i(t, h_i^*) - \mu_i(t) + a_{ii}(t) := k_{ii}(t)$, then, trough the comparison principle, we have $\Phi_{\mathcal{M}_3(*)}(t) \leq \Phi_{\mathcal{K}(*)}(t)$ with

$$\mathcal{K}(t) = \begin{pmatrix} k_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & k_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & k_{nn}(t) \end{pmatrix}$$

Hence, $\rho(\Phi_{\mathcal{M}_3(*)}(\omega)) < \rho(\Phi_{\mathcal{K}(*)}(\omega)) = 1$. Moreover, since $\rho(\Phi_{\mathcal{M}_1(*)}(\omega)) < 1$, it yields that $\rho(\Phi_{\mathcal{M}(*)}(\omega)) = \max\{\rho(\Phi_{\mathcal{M}_3(*)}(\omega)), \rho(\Phi_{\mathcal{M}_1(*)}(\omega))\} < 1$. Consequently, $(h^*(t), v^*(t))$ is stable. □

4. Conclusion

In this paper, we have proposed a SIS epidemic model in the patchy environment trough Eulerian approach in order to capture the effect of individuals mobility on the dynamics of a disease influenced by climate variations. In order to integrated the heterogeneity of environment in disease transmission, we have considered a general incidence function which incorporates the effect of seasonality. Through rigorous analysis, we have computed the disease-free periodic equilibrium, the positive periodic solutions and the basic reproduction rate, $\mathcal{R}_0$ of the model.

Three main results emerge from our investigation. The first result has pointed out the dependence of $\mathcal{R}_0$ on the incidence function and the infected individuals matrix movement. The second result has shown that $\mathcal{R}_0$ is the threshold between the persistence and the extinction of the disease. Finally, under some hypothesis on incidence function, we have proved that the two periodic steady states of the system are globally stable if susceptible and infectious individuals, in each patch have the same dispersal rate. Our analysis have shown the incidence function has hight effect on the dynamics of a epidemic model in patchy environment. So, realistic model must take this factor into account.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. van den Driessche, Spatial Structure: Patch Models, in: F. Brauer, P. van den Driessche, J. Wu, (eds) Mathematical Epidemiology, Lecture Notes in Mathematics, Vol. 1945, Springer, Berlin, Heidelberg, (2008). https://doi.org/10.1007/978-3-540-78911-6_7.

[2] D. Bichara, A. Iggidr, Multi-Patch and Multi-Group Epidemic Models: a New Framework, J. Math. Biol. 77 (2017), 107–134. https://doi.org/10.1007/s00285-017-1191-9.

[3] H. Guo, M.Y. Li, Z. Shuai, Global Dynamics of a General Class of Multistage Models for Infectious Diseases, SIAM J. Appl. Math. 72 (2012), 261–279. https://doi.org/10.1137/110827028.

[4] L. Sattenspiel, K. Dietz, A Structured Epidemic Model Incorporating Geographic Mobility among Regions, Math. Biosci. 128 (1995), 71–91. https://doi.org/10.1016/0025-5564(94)00068-b.

[5] M. Drechsler, Predicting Metapopulation Lifetime from Macroscopic Network Properties, Math. Biosci. 218 (2009), 59–71. https://doi.org/10.1016/j.mbs.2008.12.004.

[6] D. Bichara, A. Iggidr, Multi-Patch and Multi-Group Epidemic Models: A New Framework, J. Math. Biol. 77 (2017), 107–134. https://doi.org/10.1007/s00285-017-1191-9.

[7] D. Bichara, A. Iggidr, L. Smith, Multi-Stage Vector-Borne Zoonoses Models: A Global Analysis, Bull. Math. Biol. 80 (2018), 1810–1848. https://doi.org/10.1007/s11538-018-0435-1.

[8] D. Bichara, C. Castillo-Chavez, Vector-Borne Diseases Models with Residence Times – A Lagrangian Perspective, Math. Biosci. 281 (2016), 128–138. https://doi.org/10.1016/j.mbs.2016.09.006.

[9] G.R. Phaijoo, D.B. Gurung, Mathematical Study of Dengue Disease Transmission in Multi-Patch Environment, Appl. Math. 07 (2016), 1521–1533. https://doi.org/10.4236/am.2016.714132.

[10] L. Meng, W. Zhu, Generalized SEIR Epidemic Model for COVID-19 in a Multipatch Environment, Discret. Dyn. Nat. Soc. 2021 (2021), 5401253. https://doi.org/10.1155/2021/5401253.

[11] W. Wang, Population Dispersal and Disease Spread, Discret. Contin. Dyn. Syst. Ser. B 4 (2004), 797–804. https://doi.org/10.3934/dcdsb.2004.4.797.

[12] Z. Shuai, P. van den Driessche, Modelling and Control of Cholera on Networks with a Common Water Source, J. Biol. Dyn. 9 (2015), 90–103. https://doi.org/10.1080/17513758.2014.944226.

[13] M. Salmani, P.V.D. Driessche, A Model for Disease Transmission in a Patchy Environment, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), 185–202. https://doi.org/10.3934/dcdsb.2006.6.185.

[14] F. Brauer, P. van den Driessche, L. Wang, Oscillations in a Patchy Environment Disease Model, Math. Biosci. 215 (2008), 1–10. https://doi.org/10.1016/j.mbs.2008.05.001.

[15] A. Akuno, L. Ramírez-Ramírez, C. Mehta, C. Krishnanunni, T. Bui-Thanh, et al., Multi-Patch Epidemic Models with Partial Mobility, Residency, and Demography, Chaos Solitons Fractals 173 (2023), 113690. https://doi.org/10.1016/j.chaos.2023.113690.

[16] N. Bacaër, S. Guernaoui, The Epidemic Threshold of Vector-Borne Diseases with Seasonality, J. Math. Biol. 53 (2006), 421–436. https://doi.org/10.1007/s00285-006-0015-0.

[17] Y. Lou, X. Zhao, A Theoretical Approach to Understanding Population Dynamics with Seasonal Developmental Durations, J. Nonlinear Sci. 27 (2016), 573–603. https://doi.org/10.1007/s00332-016-9344-3.

[18] D. Alonso, M.J. Bouma, M. Pascual, Epidemic Malaria and Warmer Temperatures in Recent Decades in an East African Highland, Proc. R. Soc. B 278 (2011), 1661–1669. https://doi.org/10.1098/rspb.2010.2020.

[19] Z. Bai, Y. Zhou, Global Dynamics of an SEIRS Epidemic Model with Periodic Vaccination and Seasonal Contact Rate, Nonlinear Anal. Real World Appl. 13 (2012), 1060–1068. https://doi.org/10.1016/j.nonrwa.2011.02.008.

[20] M.N. Bayoh, S.W. Lindsay, Effect of Temperature on the Development of the Aquatic Stages of Anopheles gambiae sensu stricto (Diptera: Culicidae), Bull. Entomol. Res. 93 (2003), 375–381. https://doi.org/10.1079/ber2003259.

[21] F.B. Agusto, A.B. Gumel, P.E. Parham, Qualitative Assessment of the Role of Temperature Variations on Malaria Transmission Dynamics, J. Biol. Syst. 23 (2015), 1550030. https://doi.org/10.1142/s0218339015500308.

[22] L. Wang, X. Zhang, Z. Liu, An SEIR Epidemic Model with Relapse and General Nonlinear Incidence Rate with Application to Media Impact, Qual. Theory Dyn. Syst. 17 (2017), 309–329. https://doi.org/10.1007/s12346-017-0231-6.

[23] W. Liu, H.W. Hethcote, S.A. Levin, Dynamical Behavior of Epidemiological Models with Nonlinear Incidence Rates, J. Math. Biol. 25 (1987), 359–380. https://doi.org/10.1007/bf00277162.

[24] M.C. Eisenberg, Z. Shuai, J.H. Tien, P. van den Driessche, A Cholera Model in a Patchy Environment with Water and Human Movement, Math. Biosci. 246 (2013), 105–112. https://doi.org/10.1016/j.mbs.2013.08.003.

[25] D. Posny, J. Wang, Modelling Cholera in Periodic Environments, J. Biol. Dyn. 8 (2014), 1–19. https://doi.org/10.1080/17513758.2014.896482.

[26] A. Guiro, D. Ngom, D. Ouedraogo, Stability Analysis for a Class of Discrete Schistosomiasis Models with General Incidence, Adv. Differ. Equ. 2017 (2017), 116. https://doi.org/10.1186/s13662-017-1174-6.

[27] F. Zhang, X. Zhao, A Periodic Epidemic Model in a Patchy Environment, J. Math. Anal. Appl. 325 (2007), 496–516. https://doi.org/10.1016/j.jmaa.2006.01.085.

[28] K. Cooke, P. van den Driessche, X. Zou, Interaction of Maturation Delay and Nonlinear Birth in Population and Epidemic Models, J. Math. Biol. 39 (1999), 332–352. https://doi.org/10.1007/s002850050194.

[29] H.L. Smith, P. Waltman, The Theory of the Chemostat, Cambridge University Press, 1995. https://doi.org/10.1017/cbo9780511530043.

[30] X.Q. Zhao, Z.J. Jing, Global Asymptotic Behavior in Some Cooperative Systems of Functional Differential Equations, Can. Appl. Math. Q. 4 (1996), 421–444.

[31] X.Q. Zhao, Dynamical Systems in Population Biology, Springer, New York, 2003. https://doi.org/10.1007/978-0-387-21761-1.

[32] W. Wang, X. Zhao, Threshold Dynamics for Compartmental Epidemic Models in Periodic Environments, J. Dyn. Differ. Equ. 20 (2008), 699–717. https://doi.org/10.1007/s10884-008-9111-8.

[33] X.Q. Zhao, Uniform Persistence and Periodic Coexistence States in Infinite-Dimensional Periodic Semiflows with Applications, Can. Appl. Math. Q. 3 (1995), 473–495.

[34] M.W. Hirsch, H.L. Smith, X.Q. Zhao, Chain Transitivity, Attractivity, and Strong Repellors for Semidynamical Systems, J. Dyn. Differ. Equ. 13 (2001), 107–131. https://doi.org/10.1023/a:1009044515567.