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ON HESITANT INTUITIONISTIC FUZZY SOFT SUBALGEBRAS, IDEALS AND DEDUCTIVE SYSTEMS OVER HILBERT ALGEBRAS

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Abstract. In this paper, we introduce the notion of hesitant intuitionistic fuzzy soft sets over Hilbert algebras and develop the notions of hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems. We investigate fundamental properties of these structures and establish several relationships among them. In particular, we show that every hesitant intuitionistic fuzzy soft ideal is both a hesitant intuitionistic fuzzy soft subalgebra and a deductive system, while the converses do not necessarily hold. Furthermore, we introduce upper and lower hesitant level subsets and use them to characterize hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems in terms of their corresponding crisp structures. We also study structural properties such as closure under intersection and provide constructions of associated subalgebras and ideals derived from hesitant intuitionistic fuzzy soft sets. These results extend the theory of hesitant fuzzy structures and provide a new framework for studying uncertainty in Hilbert algebras.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [25]. Since then, fuzzy set theory has been widely studied due to its applications in many real-life problems. After its introduction, many generalizations of fuzzy sets have been proposed. In particular, the integration of fuzzy sets with other uncertainty models such as soft sets and rough sets has been investigated in [1,3,7].

An important extension of fuzzy sets is the notion of intuitionistic fuzzy sets, introduced by Atanassov [2], which characterizes uncertainty by considering both membership and non-membership degrees. This concept has been widely applied in algebraic structures and decision-making problems.

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In 2009–2010, Torra and Narukawa [23,24] introduced the concept of hesitant fuzzy sets, which are defined as functions from a reference set into the power set of the unit interval. This concept provides a flexible way to represent situations where membership degrees are not given by a single value but by a set of possible values. Since then, hesitant fuzzy set theory has been developed further and applied in various fields of mathematics and related areas.

Following this development, several extensions and applications of hesitant fuzzy sets have been studied in different algebraic structures. For example, Zhu et al. [27] introduced dual hesitant fuzzy sets in 2012. In 2014, Jun et al. [16] studied hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [18] investigated hesitant fuzzy filters and MV-filters of MTL-algebras, and later [19] introduced hesitant fuzzy prefilters and filters in EQ-algebras. In addition, Jun and Ahn [15] studied hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras. Mosrijai et al. [21] introduced hesitant fuzzy sets on Hilbert algebras, and further developments on hesitant fuzzy structures in Hilbert algebras have been discussed in [13].

On the other hand, Hilbert algebras have been extensively studied in algebra and logic. Diego [9] showed that Hilbert algebras form a locally finite variety. Further studies on Hilbert algebras and their deductive systems were carried out by Busneag [5,6], Jun [14], and Dudek [10]. The fuzzification of algebraic structures in Hilbert algebras has also been investigated, for instance, by Dudek and Jun [11] and Kim [20] in the study of fuzzy and t-fuzzy ideals. Moreover, intuitionistic fuzzy approaches to Hilbert algebras have been explored by Zhan and Tan [26], particularly in the context of intuitionistic fuzzy deductive systems. However, the study of hesitant intuitionistic fuzzy soft structures in Hilbert algebras, particularly in relation to subalgebras, ideals, and deductive systems, has not yet been fully developed.

Motivated by these developments, in this paper we introduce the concept of hesitant intuitionistic fuzzy soft sets over Hilbert algebras and develop and systematically study the notions of hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems. We investigate their fundamental properties and establish relationships among these structures. In particular, we show that hesitant intuitionistic fuzzy soft ideals generate both subalgebras and deductive systems. Furthermore, we introduce upper and lower hesitant level subsets and use them to characterize these structures. We also study closure properties and provide several related results.

2. Preliminaries

In this section, we recall some basic definitions and properties of Hilbert algebras that will be used throughout the paper. We also present the fundamental concepts related to hesitant intuitionistic fuzzy soft sets, which form the main framework of this study. These preliminary results provide the necessary background for developing hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems in the subsequent sections.

Definition 2.1 ([9]). *A Hilbert algebra is a structure $X = (X, *, 1)$ consisting of a nonempty set X, a binary operation $*$ on X, and a distinguished element $1 \in X$, satisfying the following axioms:*

$$(\forall x, y \in X)\,\big(x * (y * x) = 1\big), \tag{2.1}$$

$$(\forall x, y, z \in X)\,\big((x * (y * z)) * ((x * y) * (x * z)) = 1\big), \tag{2.2}$$

$$(\forall x, y \in X)\,\big((x * y = 1 \text{ and } y * x = 1) \Rightarrow x = y\big). \tag{2.3}$$

The following basic properties will be used throughout the paper.

Lemma 2.1 ([10]). *Let $X = (X, *, 1)$ be a Hilbert algebra. Then the following statements hold:*

(1) $(\forall x \in X)\,(x * x = 1)$;
(2) $(\forall x \in X)\,(1 * x = x)$;
(3) $(\forall x \in X)\,(x * 1 = 1)$;
(4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))$;
(5) $(\forall x, y, z \in X)\,\big((x * z) * ((z * y) * (x * y)) = 1\big)$.

For a Hilbert algebra $X = (X, *, 1)$, define a binary relation $\leqslant$ on X by

$$(\forall x, y \in X)\,(x \leqslant y \iff x * y = 1).$$

Then $\leqslant$ is a partial order on X, and 1 is the greatest element of X with respect to $\leqslant$.

We recall three important types of subsets in a Hilbert algebra, namely subalgebras, ideals, and deductive systems, which play a fundamental role in the study of its algebraic structure.

Definition 2.2 ([17]). *Let $X = (X, *, 1)$ be a Hilbert algebra. A nonempty subset S of X is called a subalgebra of X if*

$$(\forall x, y \in S)\,(x * y \in S).$$

Definition 2.3 ([8]). *Let $X = (X, *, 1)$ be a Hilbert algebra. A nonempty subset S of X is called an ideal of X if the following conditions hold:*

(1) $1 \in S$;
(2) $(\forall x, y \in X)\,(y \in S \Rightarrow x * y \in S)$;
(3) $(\forall x, y_1, y_2 \in X)\,\big((y_1 \in S \text{ and } y_2 \in S) \Rightarrow (y_1 * (y_2 * x)) * x \in S\big)$.

Definition 2.4 ([12]). *Let $X = (X, *, 1)$ be a Hilbert algebra. A nonempty subset S of X is called an deductive system of X if the following conditions hold:*

(1) $1 \in S$;

(2) $(\forall x \in S, \forall y \in X)(x \cdot y \in S \Rightarrow y \in S)$.

Definition 2.5 ([4]). *Let X be a reference set. A hesitant intuitionistic fuzzy set (HIFS) on X is defined by two functions*

$$h, k : X \to P([0,1]),$$

where for each $x \in X$, $h(x)$ and $k(x)$ denote the sets of possible membership and non-membership degrees of x, respectively, satisfying

$$\sup h(x) + \sup k(x) \leqslant 1.$$

Here, $HIF(X)$ denotes the family of all hesitant intuitionistic fuzzy sets on X.

Definition 2.6 ([22]). *Let X be a universe and Y be a set of parameters. A hesitant intuitionistic fuzzy soft set $(\widetilde{H}, Y)$ over X is defined by a mapping*

$$\widetilde{H} : Y \to HIF(X),$$

where for each $p \in Y$, $\widetilde{H}(p)$ is a hesitant intuitionistic fuzzy set on X, that is,

$$h_{\widetilde{H}(p)}, k_{\widetilde{H}(p)} : X \to P([0,1]),$$

satisfying

$$\sup h_{\widetilde{H}(p)}(x) + \sup k_{\widetilde{H}(p)}(x) \leqslant 1 \quad \text{for all } x \in X.$$

3. Hesitant Intuitionistic Fuzzy Soft Subalgebras and Ideals

In this section, we introduce the concepts of hesitant intuitionistic fuzzy soft subalgebras and hesitant intuitionistic fuzzy soft ideals in Hilbert algebras. We investigate their basic properties and establish several fundamental relationships between them. In particular, we show that every hesitant intuitionistic fuzzy soft ideal is a hesitant intuitionistic fuzzy soft subalgebra. Furthermore, we study level subsets of hesitant intuitionistic fuzzy soft sets and provide characterizations of these structures in terms of their upper and lower π-level subsets.

Definition 3.1. *Let $X = (X, *, 1)$ be a Hilbert algebra and $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft set on X, where for each $p \in Y$, $\widetilde{H}(p) = \{(x, h_{\widetilde{H}(p)}(x), k_{\widetilde{H}(p)}(x)) \mid x \in X\}$.*

Then $(\widetilde{H}, Y)$ is called a hesitant intuitionistic fuzzy soft subalgebra of X if, for all $x, y \in X$ and $p \in Y$, the following conditions hold:

$$\begin{aligned} h_{\widetilde{H}(p)}(x * y) &\supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y), \\ k_{\widetilde{H}(p)}(x * y) &\subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y). \end{aligned} \tag{3.1}$$

Example 3.1. *Let $X = \{1, a, b, c, d\}$ be a set equipped with the binary operation $*$ given in Table 1.*

Table 1. Cayley table for the $*$-operation

$*$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Then $X = (X, *, 1)$ is a Hilbert algebra. Let $Y = \{p_1, p_2, p_3\}$. Define a hesitant intuitionistic fuzzy soft set $(\widetilde{H}, Y)$ on X as follows.

Table 2. Values of $h_{\widetilde{H}(p)}$

$h_{\widetilde{H}(p)}$	1	a	b	c	d
p_1	$\{0.6, 0.7\}$	$\{0.5, 0.6\}$	$\{0.6\}$	$\{0.6\}$	$\{0.6\}$
p_2	$\{0.5, 0.6\}$	$\{0.5\}$	$\{0.5\}$	$\{0.6\}$	$\{0.5\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

Table 3. Values of $k_{\widetilde{H}(p)}$

$k_{\widetilde{H}(p)}$	1	a	b	c	d
p_1	$\{0.3\}$	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.4\}$	$\{0.4, 0.5\}$
p_2	$\{0.4\}$	$\{0.5\}$	$\{0.5\}$	$\{0.4\}$	$\{0.5\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

It is straightforward to verify that for each $p \in Y$ and $x \in X$,

$$\sup h_{\widetilde{H}(p)}(x) + \sup k_{\widetilde{H}(p)}(x) \leqslant 1.$$

Moreover, for all $x, y \in X$ and $p \in Y$, we have

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y), \quad k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y).$$

Hence, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X.

Proposition 3.1. *Every hesitant intuitionistic fuzzy soft subalgebra $(\widetilde{H}, Y)$ of a Hilbert algebra X satisfies*

$$(\forall x \in X)(\forall p \in Y) \begin{cases} h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x). \end{cases} \tag{3.2}$$

Proof. Let $x \in X$ and $p \in Y$. Since X is a Hilbert algebra, we have $x * x = 1$. Then, by (3.1),

$$h_{\widetilde{H}(p)}(1) = h_{\widetilde{H}(p)}(x * x) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(x),$$
$$k_{\widetilde{H}(p)}(1) = k_{\widetilde{H}(p)}(x * x) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(x).$$

□

Proposition 3.2. *If $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of a Hilbert algebra X, then*

$$(\forall x \in X)(\forall p \in Y)\begin{cases} h_{\widetilde{H}(p)}(1 * x) \supseteq h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(1 * x) \subseteq k_{\widetilde{H}(p)}(x). \end{cases} \tag{3.3}$$

Proof. Let $x \in X$ and $p \in Y$. Since $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra, by (3.1) we have

$$h_{\widetilde{H}(p)}(1 * x) \supseteq h_{\widetilde{H}(p)}(1) \cap h_{\widetilde{H}(p)}(x), \quad k_{\widetilde{H}(p)}(1 * x) \subseteq k_{\widetilde{H}(p)}(1) \cup k_{\widetilde{H}(p)}(x).$$

From Proposition 3.1, we obtain

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x), \quad k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x).$$

Hence,

$$h_{\widetilde{H}(p)}(1 * x) \supseteq h_{\widetilde{H}(p)}(x), \quad k_{\widetilde{H}(p)}(1 * x) \subseteq k_{\widetilde{H}(p)}(x).$$

□

Definition 3.2. *Let $X = (X, *, 1)$ be a Hilbert algebra and $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft set on X. Then $(\widetilde{H}, Y)$ is called a hesitant intuitionistic fuzzy soft ideal of X if, for all $x, y, y_1, y_2 \in X$ and $p \in Y$, the following conditions hold:*

$$\begin{cases} h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x), \end{cases} \tag{3.4}$$

$$\begin{cases} h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(y), \\ k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(y), \end{cases} \tag{3.5}$$

$$\begin{cases} h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \supseteq h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2), \\ k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \subseteq k_{\widetilde{H}(p)}(y_1) \cup k_{\widetilde{H}(p)}(y_2). \end{cases} \tag{3.6}$$

Example 3.2. *Let $X = \{1, x, y, z, 0\}$ be a set equipped with the binary operation $*$ given in Table 4.*

TABLE 4. Cayley table for the $*$-operation

$*$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then $X = (X, *, 1)$ is a Hilbert algebra. Let $Y = \{p_1, p_2, p_3\}$. Define a hesitant intuitionistic fuzzy soft set $(\widetilde{H}, Y)$ on X as follows.

TABLE 5. Values of $h_{\widetilde{H}(p)}$

$h_{\widetilde{H}(p)}$	1	x	y	z	0
p_1	$\{0.6, 0.7\}$	$\{0.5, 0.6\}$	$\{0.5\}$	$\{0.5\}$	$\{0.4\}$
p_2	$\{0.6\}$	$\{0.5\}$	$\{0.5\}$	$\{0.4\}$	$\{0.4\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

TABLE 6. Values of $k_{\widetilde{H}(p)}$

$k_{\widetilde{H}(p)}$	1	x	y	z	0
p_1	$\{0.2, 0.3\}$	$\{0.3, 0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.5\}$
p_2	$\{0.3\}$	$\{0.4\}$	$\{0.4\}$	$\{0.5\}$	$\{0.5\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

It is straightforward to verify that for each $x \in X$ and $p \in Y$,

$$\sup h_{\widetilde{H}(p)}(x) + \sup k_{\widetilde{H}(p)}(x) \leqslant 1.$$

Moreover, for all $x, y, y_1, y_2 \in X$ and $p \in Y$, the conditions (3.4)–(3.6) are satisfied. Hence, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of X.

Proposition 3.3. *If $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of a Hilbert algebra X, then*

$$(\forall x, y \in X)(\forall p \in Y) \begin{cases} h_{\widetilde{H}(p)}((y * x) * x) \supseteq h_{\widetilde{H}(p)}(y), \\ k_{\widetilde{H}(p)}((y * x) * x) \subseteq k_{\widetilde{H}(p)}(y). \end{cases} \tag{3.7}$$

Proof. Let $x, y \in X$ and $p \in Y$. By (3.6), taking $y_1 = y$ and $y_2 = 1$, we obtain

$$h_{\widetilde{H}(p)}((y * (1 * x)) * x) \supseteq h_{\widetilde{H}(p)}(y) \cap h_{\widetilde{H}(p)}(1),$$

$$k_{\widetilde{H}(p)}((y * (1 * x)) * x) \subseteq k_{\widetilde{H}(p)}(y) \cup k_{\widetilde{H}(p)}(1).$$

Since $1 * x = x$ in a Hilbert algebra, it follows that

$$h_{\widetilde{H}(p)}((y * x) * x) \supseteq h_{\widetilde{H}(p)}(y),$$
$$k_{\widetilde{H}(p)}((y * x) * x) \subseteq k_{\widetilde{H}(p)}(y).$$

□

Lemma 3.1. *If $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of a Hilbert algebra X, then for all $x, y \in X$ and $p \in Y$,*

$$x \leqslant y \;\Rightarrow\; \begin{cases} h_{\widetilde{H}(p)}(x) \subseteq h_{\widetilde{H}(p)}(y), \\ k_{\widetilde{H}(p)}(x) \supseteq k_{\widetilde{H}(p)}(y). \end{cases} \tag{3.8}$$

Proof. Let $x, y \in X$ and $p \in Y$ be such that $x \leqslant y$. Then $x * y = 1$. Since $1 * y = y$, we have

$$y = (1 * y) = ((x * y) * (x * y)) * y.$$

By (3.6), taking $y_1 = x * y$ and $y_2 = x$, we obtain

$$\begin{aligned} h_{\widetilde{H}(p)}(y) &= h_{\widetilde{H}(p)}(((x * y) * (x * y)) * y) \\ &\supseteq h_{\widetilde{H}(p)}(x * y) \cap h_{\widetilde{H}(p)}(x) \\ &= h_{\widetilde{H}(p)}(1) \cap h_{\widetilde{H}(p)}(x) \supseteq h_{\widetilde{H}(p)}(x), \end{aligned}$$

where the last inclusion follows from (3.4). Hence $h_{\widetilde{H}(p)}(x) \subseteq h_{\widetilde{H}(p)}(y)$.

Similarly,

$$\begin{aligned} k_{\widetilde{H}(p)}(y) &= k_{\widetilde{H}(p)}(((x * y) * (x * y)) * y) \\ &\subseteq k_{\widetilde{H}(p)}(x * y) \cup k_{\widetilde{H}(p)}(x) \\ &= k_{\widetilde{H}(p)}(1) \cup k_{\widetilde{H}(p)}(x) \subseteq k_{\widetilde{H}(p)}(x), \end{aligned}$$

by (3.4). Hence $k_{\widetilde{H}(p)}(x) \supseteq k_{\widetilde{H}(p)}(y)$. □

Theorem 3.1. *Every hesitant intuitionistic fuzzy soft ideal of a Hilbert algebra X is a hesitant intuitionistic fuzzy soft subalgebra of X.*

Proof. Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft ideal of X. Let $x, y \in X$ and $p \in Y$.

Since $y * (x * y) = 1$, we have $y \leqslant x * y$. By Lemma 3.1, we have

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x * y), \quad k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x * y).$$

On the other hand, by (3.5),

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(y), \quad k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(y).$$

Hence,

$$h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(y), \quad k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(y).$$

Therefore,

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y), \quad k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y),$$

and so $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X. □

The following example shows that the converse of Theorem 3.1 does not hold in general.

Example 3.3. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ given in Table 4. Let $Y = \{p\}$. Define a hesitant intuitionistic fuzzy soft set $(\widetilde{H}, Y)$ on X as follows:*

Table 7. Values of $h_{\widetilde{H}(p)}$ and $k_{\widetilde{H}(p)}$

x	1	x	y	z	0
$h_{\widetilde{H}(p)}(x)$	$\{0.6\}$	$\{0.4\}$	$\{0.5\}$	$\{0.5\}$	$\{0.3\}$
$k_{\widetilde{H}(p)}(x)$	$\{0.3\}$	$\{0.5\}$	$\{0.4\}$	$\{0.4\}$	$\{0.6\}$

It is easy to verify that $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X.

However, $(\widetilde{H}, Y)$ is not a hesitant intuitionistic fuzzy soft ideal of X. Indeed, take $x = x$, $y = y$, and $z = z$. Then

$$z * x = 1, \quad y * (z * x) = y, \quad (y * (z * x)) * x = y * x = x.$$

Hence,

$$h_{\widetilde{H}(p)}((y * (z * x)) * x) = h_{\widetilde{H}(p)}(x) = \{0.4\},$$

while

$$h_{\widetilde{H}(p)}(y) \cap h_{\widetilde{H}(p)}(z) = \{0.5\}.$$

Thus,

$$\{0.4\} \nsupseteq \{0.5\},$$

which violates condition (3.6). Therefore, $(\widetilde{H}, Y)$ is not a hesitant intuitionistic fuzzy soft ideal of X.

Theorem 3.2. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft subalgebra of a Hilbert algebra X. Then, for each $p \in Y$, the sets*

$$X_{h_{\widetilde{H}(p)}} = \{x \in X : h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1)\}$$

and

$$X_{k_{\widetilde{H}(p)}} = \{x \in X : k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1)\}$$

are subalgebras of X.

Proof. Let $p \in Y$.

First, let $x, y \in X_{h_{\widetilde{H}(p)}}$. Then

$$h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1) = h_{\widetilde{H}(p)}(y).$$

Since $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra, we have

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y) = h_{\widetilde{H}(p)}(1).$$

On the other hand, by Proposition 3.1,

$$h_{\widetilde{H}(p)}(x * y) \subseteq h_{\widetilde{H}(p)}(1).$$

Hence,

$$h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1),$$

and so $x * y \in X_{h_{\widetilde{H}(p)}}$.

Next, let $x, y \in X_{k_{\widetilde{H}(p)}}$. Then

$$k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1) = k_{\widetilde{H}(p)}(y).$$

Thus,

$$k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y) = k_{\widetilde{H}(p)}(1).$$

Again, by Proposition 3.1,

$$k_{\widetilde{H}(p)}(x * y) \supseteq k_{\widetilde{H}(p)}(1).$$

Hence,

$$k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1),$$

and so $x * y \in X_{k_{\widetilde{H}(p)}}$.

Therefore, both $X_{h_{\widetilde{H}(p)}}$ and $X_{k_{\widetilde{H}(p)}}$ are subalgebras of X. □

Theorem 3.3. *Let* $(\widetilde{H}, Y)$ *be a hesitant intuitionistic fuzzy soft ideal of a Hilbert algebra* X. *Then, for each* $p \in Y$, *the sets*

$$X_{h_{\widetilde{H}(p)}} = \{x \in X : h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1)\}$$

and

$$X_{k_{\widetilde{H}(p)}} = \{x \in X : k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1)\}$$

are ideals of X.

Proof. Let $p \in Y$. Clearly, $1 \in X_{h_{\widetilde{H}(p)}} \cap X_{k_{\widetilde{H}(p)}}$.

First, we show that $X_{h_{\widetilde{H}(p)}}$ is an ideal. Let $x, y \in X$ with $y \in X_{h_{\widetilde{H}(p)}}$. Then

$$h_{\widetilde{H}(p)}(y) = h_{\widetilde{H}(p)}(1).$$

By (3.5),

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(y) = h_{\widetilde{H}(p)}(1).$$

Together with (3.4), we obtain

$$h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1),$$

and hence $x * y \in X_{h_{\widetilde{H}(p)}}$.

Next, let $x, y_1, y_2 \in X$ with $y_1, y_2 \in X_{h_{\widetilde{H}(p)}}$. Then

$$h_{\widetilde{H}(p)}(y_1) = h_{\widetilde{H}(p)}(y_2) = h_{\widetilde{H}(p)}(1).$$

By (3.6),

$$h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \supseteq h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2) = h_{\widetilde{H}(p)}(1).$$

Using (3.4), we get

$$h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) = h_{\widetilde{H}(p)}(1),$$

and so $(y_1 * (y_2 * x)) * x \in X_{h_{\widetilde{H}(p)}}$. Thus $X_{h_{\widetilde{H}(p)}}$ is an ideal of X.

Similarly, we show that $X_{k_{\widetilde{H}(p)}}$ is an ideal. Let $x, y \in X$ with $y \in X_{k_{\widetilde{H}(p)}}$. Then

$$k_{\widetilde{H}(p)}(y) = k_{\widetilde{H}(p)}(1).$$

By (3.5),

$$k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(y) = k_{\widetilde{H}(p)}(1).$$

Together with (3.4), we obtain

$$k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1),$$

and hence $x * y \in X_{k_{\widetilde{H}(p)}}$.

Next, let $x, y_1, y_2 \in X$ with $y_1, y_2 \in X_{k_{\widetilde{H}(p)}}$. Then

$$k_{\widetilde{H}(p)}(y_1) = k_{\widetilde{H}(p)}(y_2) = k_{\widetilde{H}(p)}(1).$$

By (3.6),

$$k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \subseteq k_{\widetilde{H}(p)}(y_1) \cup k_{\widetilde{H}(p)}(y_2) = k_{\widetilde{H}(p)}(1).$$

Using (3.4), we get

$$k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) = k_{\widetilde{H}(p)}(1),$$

and so $(y_1 * (y_2 * x)) * x \in X_{k_{\widetilde{H}(p)}}$. Thus $X_{k_{\widetilde{H}(p)}}$ is an ideal of X. $\square$

Definition 3.3. *Let* $(\widetilde{H}, Y)$ *be a hesitant intuitionistic fuzzy soft set over* X*, and let* $p \in Y$*. For any* $\pi \subseteq [0,1]$*, define*

$$U(h_{\widetilde{H}(p)}, \pi) = \{x \in X \mid h_{\widetilde{H}(p)}(x) \supseteq \pi\}$$

and

$$L(k_{\widetilde{H}(p)}, \pi) = \{x \in X \mid k_{\widetilde{H}(p)}(x) \subseteq \pi\}.$$

Then $U(h_{\widetilde{H}(p)}, \pi)$ *and* $L(k_{\widetilde{H}(p)}, \pi)$ *are called the upper and lower hesitant* π*-level subsets of* $(\widetilde{H}, Y)$*, respectively.*

Theorem 3.4. *Let* $(\widetilde{H}, Y)$ *be a hesitant intuitionistic fuzzy soft set on a Hilbert algebra* X*. Then* $(\widetilde{H}, Y)$ *is a hesitant intuitionistic fuzzy soft subalgebra of* X *if and only if, for each* $p \in Y$ *and for all* $\pi \subseteq [0,1]$*, the nonempty sets* $U(h_{\widetilde{H}(p)}, \pi)$ *and* $L(k_{\widetilde{H}(p)}, \pi)$ *are subalgebras of* X*.*

Proof. $(\Rightarrow)$ Assume that $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X, and let $p \in Y$.

Let $\pi \subseteq [0,1]$ be such that $U(h_{\widetilde{H}(p)}, \pi) \neq \varnothing$. Take $x \in U(h_{\widetilde{H}(p)}, \pi)$. Then

$$h_{\widetilde{H}(p)}(x) \supseteq \pi.$$

By Proposition 3.1,

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x) \supseteq \pi,$$

so $1 \in U(h_{\widetilde{H}(p)}, \pi)$.

If $x, y \in U(h_{\widetilde{H}(p)}, \pi)$, then

$$h_{\widetilde{H}(p)}(x) \supseteq \pi, \quad h_{\widetilde{H}(p)}(y) \supseteq \pi.$$

Thus,

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y) \supseteq \pi,$$

and hence $x * y \in U(h_{\widetilde{H}(p)}, \pi)$. Therefore, $U(h_{\widetilde{H}(p)}, \pi)$ is a subalgebra.

Similarly, let $\pi \subseteq [0,1]$ be such that $L(k_{\widetilde{H}(p)}, \pi) \neq \varnothing$, and let $x \in L(k_{\widetilde{H}(p)}, \pi)$. Then

$$k_{\widetilde{H}(p)}(x) \subseteq \pi.$$

By Proposition 3.1,

$$k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x) \subseteq \pi,$$

so $1 \in L(k_{\widetilde{H}(p)}, \pi)$.

If $x, y \in L(k_{\widetilde{H}(p)}, \pi)$, then

$$k_{\widetilde{H}(p)}(x) \subseteq \pi, \quad k_{\widetilde{H}(p)}(y) \subseteq \pi.$$

Hence,

$$k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y) \subseteq \pi,$$

so $x * y \in L(k_{\widetilde{H}(p)}, \pi)$. Thus, $L(k_{\widetilde{H}(p)}, \pi)$ is a subalgebra.

($\Leftarrow$) Assume that for each $p \in Y$ and each $\pi \subseteq [0,1]$, the nonempty sets $U(h_{\widetilde{H}(p)}, \pi)$ and $L(k_{\widetilde{H}(p)}, \pi)$ are subalgebras.

Let $x, y \in X$ and $p \in Y$. Set

$$\pi = h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y).$$

Then $x, y \in U(h_{\widetilde{H}(p)}, \pi)$, and hence

$$x * y \in U(h_{\widetilde{H}(p)}, \pi),$$

which implies

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y).$$

Next, set

$$\pi_1 = k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y).$$

Then $x, y \in L(k_{\widetilde{H}(p)}, \pi_1)$, and hence

$$x * y \in L(k_{\widetilde{H}(p)}, \pi_1),$$

which implies

$$k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y).$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X. □

Theorem 3.5. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft set on a Hilbert algebra X. Then $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of X if and only if, for each $p \in Y$ and for all $\pi \subseteq [0,1]$, the nonempty sets $U(h_{\widetilde{H}(p)}, \pi)$ and $L(k_{\widetilde{H}(p)}, \pi)$ are ideals of X.*

Proof. ($\Rightarrow$) Assume that $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of X, and let $p \in Y$.

Let $\pi \subseteq [0,1]$ be such that $U(h_{\widetilde{H}(p)}, \pi) \neq \varnothing$. Take $x \in U(h_{\widetilde{H}(p)}, \pi)$. Then

$$h_{\widetilde{H}(p)}(x) \supseteq \pi.$$

By (3.4),

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x) \supseteq \pi,$$

so $1 \in U(h_{\widetilde{H}(p)}, \pi)$.

If $x \in X$ and $y \in U(h_{\widetilde{H}(p)}, \pi)$, then

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(y) \supseteq \pi,$$

so $x * y \in U(h_{\widetilde{H}(p)}, \pi)$.

If $y_1, y_2 \in U(h_{\widetilde{H}(p)}, \pi)$, then

$$h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \supseteq h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2) \supseteq \pi,$$

so $(y_1 * (y_2 * x)) * x \in U(h_{\widetilde{H}(p)}, \pi)$. Hence $U(h_{\widetilde{H}(p)}, \pi)$ is an ideal.

Similarly, let $\pi \subseteq [0,1]$ be such that $L(k_{\widetilde{H}(p)}, \pi) \neq \varnothing$, and let $x \in L(k_{\widetilde{H}(p)}, \pi)$. Then

$$k_{\widetilde{H}(p)}(x) \subseteq \pi.$$

By (3.4),

$$k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x) \subseteq \pi,$$

so $1 \in L(k_{\widetilde{H}(p)}, \pi)$.

If $x \in X$ and $y \in L(k_{\widetilde{H}(p)}, \pi)$, then

$$k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(y) \subseteq \pi,$$

so $x * y \in L(k_{\widetilde{H}(p)}, \pi)$.

If $y_1, y_2 \in L(k_{\widetilde{H}(p)}, \pi)$, then

$$k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \subseteq k_{\widetilde{H}(p)}(y_1) \cup k_{\widetilde{H}(p)}(y_2) \subseteq \pi,$$

so $(y_1 * (y_2 * x)) * x \in L(k_{\widetilde{H}(p)}, \pi)$. Hence $L(k_{\widetilde{H}(p)}, \pi)$ is an ideal.

($\Leftarrow$) Assume that for each $p \in Y$ and each $\pi \subseteq [0,1]$, the nonempty sets $U(h_{\widetilde{H}(p)}, \pi)$ and $L(k_{\widetilde{H}(p)}, \pi)$ are ideals.

Let $x \in X$. Take $\pi = h_{\widetilde{H}(p)}(x)$. Then $x \in U(h_{\widetilde{H}(p)}, \pi)$, so $1 \in U(h_{\widetilde{H}(p)}, \pi)$ and hence

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x).$$

Let $x, y \in X$. Take $\pi = h_{\widetilde{H}(p)}(y)$. Then $y \in U(h_{\widetilde{H}(p)}, \pi)$, so

$$x * y \in U(h_{\widetilde{H}(p)}, \pi),$$

and hence

$$h_{\widetilde{H}(p)}(x * y) \supseteq h_{\widetilde{H}(p)}(y).$$

Let $x, y_1, y_2 \in X$. Take $\pi = h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2)$. Then

$$(y_1 * (y_2 * x)) * x \in U(h_{\widetilde{H}(p)}, \pi),$$

so

$$h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \supseteq h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2).$$

Similarly, using $L(k_{\widetilde{H}(p)}, \pi)$, we obtain

$$k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x), \quad k_{\widetilde{H}(p)}(x * y) \subseteq k_{\widetilde{H}(p)}(y),$$

and

$$k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) \subseteq k_{\widetilde{H}(p)}(y_1) \cup k_{\widetilde{H}(p)}(y_2).$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of X. $\square$

Definition 3.4. *Let $\{(\widetilde{H}_\alpha, Y)\}_{\alpha\in\Delta}$ be a family of hesitant intuitionistic fuzzy soft sets over a reference set X. The hesitant intuitionistic fuzzy soft intersection of this family is defined as the hesitant intuitionistic fuzzy soft set*

$$(\widetilde{H}, Y) = \bigcap_{\alpha\in\Delta} (\widetilde{H}_\alpha, Y)$$

where $\widetilde{H} : Y \to HIF(X)$ is given by, for each $p \in Y$ and $x \in X$,

$$h_{\widetilde{H}(p)}(x) = \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}(p)}(x) = \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(x).$$

Proposition 3.4. *Let $\{(\widetilde{H}_\alpha, Y)\}_{\alpha\in\Delta}$ be a family of hesitant intuitionistic fuzzy soft ideals of a Hilbert algebra X. Then*

$$\bigcap_{\alpha\in\Delta} (\widetilde{H}_\alpha, Y)$$

is a hesitant intuitionistic fuzzy soft ideal of X.

Proof. Let $(\widetilde{H}, Y) = \bigcap_{\alpha\in\Delta}(\widetilde{H}_\alpha, Y)$, where for each $p \in Y$ and $x \in X$,

$$h_{\widetilde{H}(p)}(x) = \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}(p)}(x) = \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(x).$$

We verify conditions (3.4)–(3.6).

(1) Let $x \in X$ and $p \in Y$. Then, for each $\alpha \in \Delta$,

$$h_{\widetilde{H}_\alpha(p)}(1) \supseteq h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}_\alpha(p)}(1) \subseteq k_{\widetilde{H}_\alpha(p)}(x).$$

Hence,

$$h_{\widetilde{H}(p)}(1) = \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(1) \supseteq \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(x) = h_{\widetilde{H}(p)}(x),$$

and

$$k_{\widetilde{H}(p)}(1) = \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(1) \subseteq \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(x) = k_{\widetilde{H}(p)}(x).$$

(2) Let $x, y \in X$ and $p \in Y$. For each $\alpha \in \Delta$,

$$h_{\widetilde{H}_\alpha(p)}(x * y) \supseteq h_{\widetilde{H}_\alpha(p)}(y), \quad k_{\widetilde{H}_\alpha(p)}(x * y) \subseteq k_{\widetilde{H}_\alpha(p)}(y).$$

Thus,

$$h_{\widetilde{H}(p)}(x * y) = \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(x * y) \supseteq \bigcap_{\alpha\in\Delta} h_{\widetilde{H}_\alpha(p)}(y) = h_{\widetilde{H}(p)}(y),$$

and

$$k_{\widetilde{H}(p)}(x * y) = \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(x * y) \subseteq \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(y) = k_{\widetilde{H}(p)}(y).$$

(3) Let $x, y_1, y_2 \in X$ and $p \in Y$. For each $\alpha \in \Delta$,

$$h_{\widetilde{H}_\alpha(p)}((y_1 * (y_2 * x)) * x) \supseteq h_{\widetilde{H}_\alpha(p)}(y_1) \cap h_{\widetilde{H}_\alpha(p)}(y_2),$$

$$k_{\widetilde{H}_\alpha(p)}((y_1 * (y_2 * x)) * x) \subseteq k_{\widetilde{H}_\alpha(p)}(y_1) \cup k_{\widetilde{H}_\alpha(p)}(y_2).$$

Hence,

$$\begin{aligned} h_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) &= \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}((y_1 * (y_2 * x)) * x) \\ &\supseteq \bigcap_{\alpha \in \Delta} \left(h_{\widetilde{H}_\alpha(p)}(y_1) \cap h_{\widetilde{H}_\alpha(p)}(y_2)\right) \\ &= h_{\widetilde{H}(p)}(y_1) \cap h_{\widetilde{H}(p)}(y_2), \end{aligned}$$

and

$$\begin{aligned} k_{\widetilde{H}(p)}((y_1 * (y_2 * x)) * x) &= \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}((y_1 * (y_2 * x)) * x) \\ &\subseteq \bigcup_{\alpha \in \Delta} \left(k_{\widetilde{H}_\alpha(p)}(y_1) \cup k_{\widetilde{H}_\alpha(p)}(y_2)\right) \\ &= k_{\widetilde{H}(p)}(y_1) \cup k_{\widetilde{H}(p)}(y_2). \end{aligned}$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft ideal of X. □

Proposition 3.5. *Let $\{(\widetilde{H}_\alpha, Y)\}_{\alpha \in \Delta}$ be a family of hesitant intuitionistic fuzzy soft subalgebras of a Hilbert algebra X. Then*

$$\bigcap_{\alpha \in \Delta} (\widetilde{H}_\alpha, Y)$$

is a hesitant intuitionistic fuzzy soft subalgebra of X.

Proof. Let $(\widetilde{H}, Y) = \bigcap_{\alpha \in \Delta} (\widetilde{H}_\alpha, Y)$, where for each $p \in Y$ and $x \in X$,

$$h_{\widetilde{H}(p)}(x) = \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}(p)}(x) = \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}(x).$$

Let $x, y \in X$ and $p \in Y$. Since each $(\widetilde{H}_\alpha, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra, we have

$$h_{\widetilde{H}_\alpha(p)}(x * y) \supseteq h_{\widetilde{H}_\alpha(p)}(x) \cap h_{\widetilde{H}_\alpha(p)}(y),$$

$$k_{\widetilde{H}_\alpha(p)}(x * y) \subseteq k_{\widetilde{H}_\alpha(p)}(x) \cup k_{\widetilde{H}_\alpha(p)}(y)$$

for all $\alpha \in \Delta$.

Hence,

$$\begin{aligned} h_{\widetilde{H}(p)}(x * y) &= \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(x * y) \\ &\supseteq \bigcap_{\alpha \in \Delta} \left(h_{\widetilde{H}_\alpha(p)}(x) \cap h_{\widetilde{H}_\alpha(p)}(y)\right) \\ &= h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y), \end{aligned}$$

and

$$\begin{aligned} k_{\widetilde{H}(p)}(x*y) &= \bigcup_{\alpha\in\Delta} k_{\widetilde{H}_\alpha(p)}(x*y) \\ &\subseteq \bigcup_{\alpha\in\Delta} \big(k_{\widetilde{H}_\alpha(p)}(x) \cup k_{\widetilde{H}_\alpha(p)}(y)\big) \\ &= k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y). \end{aligned}$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft subalgebra of X. □

4. Hesitant Intuitionistic Fuzzy Soft Deductive Systems

In this section, we extend the study to hesitant intuitionistic fuzzy soft deductive systems in Hilbert algebras. We define the notion of hesitant intuitionistic fuzzy soft deductive systems and examine their fundamental properties. We also establish connections between hesitant intuitionistic fuzzy soft ideals and deductive systems. In addition, we provide characterizations of hesitant intuitionistic fuzzy soft deductive systems by means of their level subsets and investigate the behavior of these structures under intersection.

Definition 4.1. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft set on a Hilbert algebra X. Then $(\widetilde{H}, Y)$ is called a hesitant intuitionistic fuzzy soft deductive system of X if, for all $x, y \in X$ and $p \in Y$, the following conditions hold:*

$$(\forall x \in X)(\forall p \in Y) \begin{cases} h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x), \end{cases} \tag{4.1}$$

$$(\forall x, y \in X)(\forall p \in Y) \begin{cases} h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x*y) \cap h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x*y) \cup k_{\widetilde{H}(p)}(x). \end{cases} \tag{4.2}$$

Example 4.1. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ given in Table 4. Let $Y = \{p_1, p_2, p_3\}$. Define a hesitant intuitionistic fuzzy soft set $(\widetilde{H}, Y)$ on X as follows.*

Table 8. Values of $h_{\widetilde{H}(p)}$

$h_{\widetilde{H}(p)}$	1	x	y	z	0
p_1	$\{0.6, 0.7\}$	$\{0.6\}$	$\{0.5\}$	$\{0.5\}$	$\{0.4\}$
p_2	$\{0.6\}$	$\{0.5\}$	$\{0.5\}$	$\{0.4\}$	$\{0.4\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

Table 9. Values of $k_{\widetilde{H}(p)}$

$k_{\widetilde{H}(p)}$	1	x	y	z	0
p_1	$\{0.2\}$	$\{0.3\}$	$\{0.4\}$	$\{0.4\}$	$\{0.5\}$
p_2	$\{0.3\}$	$\{0.4\}$	$\{0.4\}$	$\{0.5\}$	$\{0.5\}$
p_3	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

It is straightforward to verify that for each $x \in X$ and $p \in Y$,

$$\sup h_{\widetilde{H}(p)}(x) + \sup k_{\widetilde{H}(p)}(x) \leqslant 1.$$

Moreover, for all $x, y \in X$ and $p \in Y$, the conditions (4.1) and (4.2) are satisfied. Hence, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X.

Proposition 4.1. *Every hesitant intuitionistic fuzzy soft ideal of a Hilbert algebra X is a hesitant intuitionistic fuzzy soft deductive system of X.*

Proof. Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft ideal of X. We verify conditions (4.1) and (4.2).

Condition (4.1) follows directly from (3.4).

Next, let $x, y \in X$ and $p \in Y$. Since $1 * y = y$, we have

$$y = 1 * y = ((x * y) * (x * y)) * y,$$

by Lemma 2.1.

By (3.6), taking $y_1 = x * y$ and $y_2 = x$, we obtain

$$h_{\widetilde{H}(p)}(y) = h_{\widetilde{H}(p)}(((x * y) * (x * y)) * y) \supseteq h_{\widetilde{H}(p)}(x * y) \cap h_{\widetilde{H}(p)}(x),$$

and

$$k_{\widetilde{H}(p)}(y) = k_{\widetilde{H}(p)}(((x * y) * (x * y)) * y) \subseteq k_{\widetilde{H}(p)}(x * y) \cup k_{\widetilde{H}(p)}(x).$$

Thus, condition (4.2) holds. Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X. $\square$

Lemma 4.1. *If $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of a Hilbert algebra X, then for all $x, y \in X$ and $p \in Y$,*

$$x \leqslant y \ \Rightarrow \begin{cases} h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x), \\ k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x). \end{cases} \tag{4.3}$$

Proof. Let $x, y \in X$ and $p \in Y$ be such that $x \leqslant y$. Then $x * y = 1$.

By (4.2), we have

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x * y) \cap h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1) \cap h_{\widetilde{H}(p)}(x).$$

Since $h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x)$ by (4.1), it follows that

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x).$$

Similarly, by (4.2),

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x * y) \cup k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1) \cup k_{\widetilde{H}(p)}(x).$$

Since $k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x)$ by (4.1), we obtain

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x).$$

□

Theorem 4.1. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft deductive system of a Hilbert algebra X. Then the following hold.*

(1) *For all $x, y \in X$ and $p \in Y$,*

$$h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1) \;\Rightarrow\; h_{\widetilde{H}(p)}(x) \subseteq h_{\widetilde{H}(p)}(y),$$

$$k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1) \;\Rightarrow\; k_{\widetilde{H}(p)}(x) \supseteq k_{\widetilde{H}(p)}(y).$$

(2) *For all $x, y, z \in X$ and $p \in Y$,*

$$x * (y * z) = 1 \;\Rightarrow\; \begin{cases} h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y) \subseteq h_{\widetilde{H}(p)}(z), \\ k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y) \supseteq k_{\widetilde{H}(p)}(z). \end{cases}$$

(3) *For all $x, y, z \in X$ and $p \in Y$,*

$$\begin{cases} h_{\widetilde{H}(p)}(x * z) \cap h_{\widetilde{H}(p)}(z * y) \subseteq h_{\widetilde{H}(p)}(x * y), \\ k_{\widetilde{H}(p)}(x * z) \cup k_{\widetilde{H}(p)}(z * y) \supseteq k_{\widetilde{H}(p)}(x * y). \end{cases}$$

Proof. (1) Let $x, y \in X$ and $p \in Y$ such that $h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1)$. By (4.2), we have

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(1).$$

Since $h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x)$, it follows that

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x).$$

Similarly, if $k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1)$, then

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(1).$$

Since $k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x)$, we obtain

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x).$$

(2) Let $x, y, z \in X$ and $p \in Y$ such that $x * (y * z) = 1$. Then $x \leqslant y * z$. By Lemma 4.1,

$$h_{\widetilde{H}(p)}(x) \subseteq h_{\widetilde{H}(p)}(y * z), \quad k_{\widetilde{H}(p)}(x) \supseteq k_{\widetilde{H}(p)}(y * z).$$

Hence,

$$h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(y) \subseteq h_{\widetilde{H}(p)}(y * z) \cap h_{\widetilde{H}(p)}(y) \subseteq h_{\widetilde{H}(p)}(z),$$
$$k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(y) \supseteq k_{\widetilde{H}(p)}(y * z) \cup k_{\widetilde{H}(p)}(y) \supseteq k_{\widetilde{H}(p)}(z),$$

by (4.2).

(3) Let $x, y, z \in X$ and $p \in Y$. By Lemma 2.1,

$$(x * z) * ((z * y) * (x * y)) = 1.$$

Applying part (2), we obtain

$$h_{\widetilde{H}(p)}(x * z) \cap h_{\widetilde{H}(p)}(z * y) \subseteq h_{\widetilde{H}(p)}(x * y),$$
$$k_{\widetilde{H}(p)}(x * z) \cup k_{\widetilde{H}(p)}(z * y) \supseteq k_{\widetilde{H}(p)}(x * y).$$

□

Theorem 4.2. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft deductive system of a Hilbert algebra X. Then for each $p \in Y$, the sets*

$$X_{h,p} = \{x \in X : h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1)\}$$

and

$$X_{k,p} = \{x \in X : k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1)\}$$

are deductive systems of X.

Proof. Fix $p \in Y$.

Clearly, $1 \in X_{h,p}$ and $1 \in X_{k,p}$.

First, we show that $X_{h,p}$ is a deductive system. Let $x, y \in X$ such that $x \in X_{h,p}$ and $x * y \in X_{h,p}$. Then

$$h_{\widetilde{H}(p)}(x) = h_{\widetilde{H}(p)}(1), \quad h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1).$$

By (4.2), we have

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x * y) = h_{\widetilde{H}(p)}(1).$$

Hence,

$$h_{\widetilde{H}(p)}(y) = h_{\widetilde{H}(p)}(1),$$

which implies that $y \in X_{h,p}$. Therefore, $X_{h,p}$ is a deductive system of X.

Next, we show that $X_{k,p}$ is a deductive system. Let $x, y \in X$ such that $x \in X_{k,p}$ and $x * y \in X_{k,p}$. Then

$$k_{\widetilde{H}(p)}(x) = k_{\widetilde{H}(p)}(1), \quad k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1).$$

By (4.2), we have

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(x * y) = k_{\widetilde{H}(p)}(1).$$

Hence,

$$k_{\widetilde{H}(p)}(y) = k_{\widetilde{H}(p)}(1),$$

which implies that $y \in X_{k,p}$. Therefore, $X_{k,p}$ is a deductive system of X. □

Theorem 4.3. *Let $(\widetilde{H}, Y)$ be a hesitant intuitionistic fuzzy soft set on a Hilbert algebra X. Then $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X if and only if, for each $p \in Y$ and for all $\pi \subseteq [0,1]$, the nonempty sets $U(h_{\widetilde{H}(p)}, \pi)$ and $L(k_{\widetilde{H}(p)}, \pi)$ are deductive systems of X.*

Proof. ($\Rightarrow$) Assume that $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X, and fix $p \in Y$.

Let $\pi \subseteq [0,1]$ be such that $U(h_{\widetilde{H}(p)}, \pi) \neq \varnothing$, and let $x \in U(h_{\widetilde{H}(p)}, \pi)$. Then

$$h_{\widetilde{H}(p)}(x) \supseteq \pi.$$

By (4.1),

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x) \supseteq \pi,$$

so $1 \in U(h_{\widetilde{H}(p)}, \pi)$.

Let $x, y \in X$ such that $x, x * y \in U(h_{\widetilde{H}(p)}, \pi)$. Then

$$h_{\widetilde{H}(p)}(x) \supseteq \pi, \quad h_{\widetilde{H}(p)}(x * y) \supseteq \pi.$$

By (4.2),

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x * y) \supseteq \pi,$$

so $y \in U(h_{\widetilde{H}(p)}, \pi)$. Hence $U(h_{\widetilde{H}(p)}, \pi)$ is a deductive system.

Similarly, let $\pi \subseteq [0,1]$ be such that $L(k_{\widetilde{H}(p)}, \pi) \neq \varnothing$, and let $x \in L(k_{\widetilde{H}(p)}, \pi)$. Then

$$k_{\widetilde{H}(p)}(x) \subseteq \pi.$$

By (4.1),

$$k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x) \subseteq \pi,$$

so $1 \in L(k_{\widetilde{H}(p)}, \pi)$.

If $x, x * y \in L(k_{\widetilde{H}(p)}, \pi)$, then

$$k_{\widetilde{H}(p)}(x) \subseteq \pi, \quad k_{\widetilde{H}(p)}(x * y) \subseteq \pi.$$

By (4.2),

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(x * y) \subseteq \pi,$$

so $y \in L(k_{\widetilde{H}(p)}, \pi)$. Hence $L(k_{\widetilde{H}(p)}, \pi)$ is a deductive system.

($\Leftarrow$) Assume that for each $p \in Y$ and each $\pi \subseteq [0,1]$, the nonempty sets $U(h_{\widetilde{H}(p)}, \pi)$ and $L(k_{\widetilde{H}(p)}, \pi)$ are deductive systems.

Let $x \in X$ and $p \in Y$. Take $\pi = h_{\widetilde{H}(p)}(x)$. Then $x \in U(h_{\widetilde{H}(p)}, \pi)$, so

$$1 \in U(h_{\widetilde{H}(p)}, \pi),$$

which implies

$$h_{\widetilde{H}(p)}(1) \supseteq h_{\widetilde{H}(p)}(x).$$

Let $x, y \in X$. Take

$$\pi = h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x * y).$$

Then $x, x * y \in U(h_{\widetilde{H}(p)}, \pi)$, and hence

$$y \in U(h_{\widetilde{H}(p)}, \pi),$$

which implies

$$h_{\widetilde{H}(p)}(y) \supseteq h_{\widetilde{H}(p)}(x) \cap h_{\widetilde{H}(p)}(x * y).$$

Similarly, using $L(k_{\widetilde{H}(p)}, \pi)$, we obtain

$$k_{\widetilde{H}(p)}(1) \subseteq k_{\widetilde{H}(p)}(x),$$

and

$$k_{\widetilde{H}(p)}(y) \subseteq k_{\widetilde{H}(p)}(x) \cup k_{\widetilde{H}(p)}(x * y).$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X. □

Proposition 4.2. *Let $\{(\widetilde{H}_\alpha, Y)\}_{\alpha \in \Delta}$ be a family of hesitant intuitionistic fuzzy soft deductive systems of a Hilbert algebra X. Then*

$$\bigcap_{\alpha \in \Delta} (\widetilde{H}_\alpha, Y)$$

is a hesitant intuitionistic fuzzy soft deductive system of X.

Proof. Let $(\widetilde{H}, Y) = \bigcap_{\alpha \in \Delta} (\widetilde{H}_\alpha, Y)$, where for each $p \in Y$ and $x \in X$,

$$h_{\widetilde{H}(p)}(x) = \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}(p)}(x) = \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}(x).$$

We verify conditions (4.1) and (4.2).

(1) Let $x \in X$ and $p \in Y$. For each $\alpha \in \Delta$,

$$h_{\widetilde{H}_\alpha(p)}(1) \supseteq h_{\widetilde{H}_\alpha(p)}(x), \quad k_{\widetilde{H}_\alpha(p)}(1) \subseteq k_{\widetilde{H}_\alpha(p)}(x).$$

Hence,

$$h_{\widetilde{H}(p)}(1) = \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(1) \supseteq \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(x) = h_{\widetilde{H}(p)}(x),$$

and

$$k_{\widetilde{H}(p)}(1) = \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}(1) \subseteq \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}(x) = k_{\widetilde{H}(p)}(x).$$

(2) Let $x, y \in X$ and $p \in Y$. For each $\alpha \in \Delta$,

$$h_{\widetilde{H}_\alpha(p)}(y) \supseteq h_{\widetilde{H}_\alpha(p)}(x * y) \cap h_{\widetilde{H}_\alpha(p)}(x),$$

$$k_{\widetilde{H}_\alpha(p)}(y) \subseteq k_{\widetilde{H}_\alpha(p)}(x * y) \cup k_{\widetilde{H}_\alpha(p)}(x).$$

Thus,

$$\begin{aligned} h_{\widetilde{H}(p)}(y) &= \bigcap_{\alpha \in \Delta} h_{\widetilde{H}_\alpha(p)}(y) \\ &\supseteq \bigcap_{\alpha \in \Delta} \left(h_{\widetilde{H}_\alpha(p)}(x * y) \cap h_{\widetilde{H}_\alpha(p)}(x) \right) \\ &= h_{\widetilde{H}(p)}(x * y) \cap h_{\widetilde{H}(p)}(x), \end{aligned}$$

and

$$\begin{aligned} k_{\widetilde{H}(p)}(y) &= \bigcup_{\alpha \in \Delta} k_{\widetilde{H}_\alpha(p)}(y) \\ &\subseteq \bigcup_{\alpha \in \Delta} \left(k_{\widetilde{H}_\alpha(p)}(x * y) \cup k_{\widetilde{H}_\alpha(p)}(x) \right) \\ &= k_{\widetilde{H}(p)}(x * y) \cup k_{\widetilde{H}(p)}(x). \end{aligned}$$

Therefore, $(\widetilde{H}, Y)$ is a hesitant intuitionistic fuzzy soft deductive system of X. □

5. Conclusion

In this paper, we introduced the concept of hesitant intuitionistic fuzzy soft sets over Hilbert algebras and developed the notions of hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems. We investigated their fundamental properties and established important relationships among these structures. In particular, it was shown that every hesitant intuitionistic fuzzy soft ideal induces both a hesitant intuitionistic fuzzy soft subalgebra and a deductive system, while the converses do not necessarily hold.

Furthermore, we introduced upper and lower hesitant level subsets and used them to characterize hesitant intuitionistic fuzzy soft subalgebras, ideals, and deductive systems in terms of their corresponding crisp structures. We also studied structural properties such as closure under intersection and provided constructions of associated subalgebras and ideals derived from hesitant intuitionistic fuzzy soft sets.

The results obtained in this paper extend existing studies on hesitant fuzzy structures and provide a new framework for analyzing uncertainty in Hilbert algebras. Future research may focus on investigating other algebraic systems under this framework and exploring potential applications in logic and decision-making models.

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