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A PROPOSED FRACTIONAL DERIVATIVE AND ITS APPLICATIONS TO FRACTIONAL DIFFERENTIAL MODELS

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Abstract. This paper introduced a new fractional derivative defined by the Mittag-Leffler function to overcome the limitations of classical operators . The proposed operator provided a flexible memory structure while preserving consistency with the classical integer-order derivative. The main analytical properties of the operator were established, including its corresponding fractional integral and Laplace transform. The Banach fixed-point theorem proved The existence and uniqueness of solutions for the associated Cauchy-type problems. Furthermore, approximate solutions were constructed using the Adomian decomposition method. Numerical simulations showed that the proposed operator exhibited more stable dynamic behavior, reduced oscillatory effects, and faster convergence compared to classical fractional derivatives such as the Riemann–Liouville, Caputo–Fabrizio, and Atangana–Baleanu operators. These results indicated that the proposed fractional derivative provided an effective and mathematically consistent framework for modeling nonlocal systems with memory effects, offering improved analytical tractability and enhanced modeling performance.

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1. Introduction

Fractional calculus is now an important mathematical tool for modeling systems that have memory, hereditary properties, and interactions that aren't local. In many areas, including diffusion processes, viscoelasticity, control theory, and heat transfer, fractional differential equations have been used successfully. The Riemann-Liouville and Caputo operators are two of the most well-known classical definitions of fractional derivatives. This is because they have strong analytical foundations. But these operators have singular kernels, which can make some applications hard to analyze and calculate [1], [2], [3].

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To address these constraints, numerous fractional derivatives featuring non-singular kernels have been introduced in recent years. The Caputo-Fabrizio derivative, which is based on an exponential kernel, and the Atangana-Baleanu derivative, which uses the Mittag-Leffler function, are two well-known examples [4], [5], [6], [7], [8]. While these operators enhance the modeling of memory effects and circumvent kernel singularities, challenges persist concerning analytical tractability, mathematical structure, and the development of cohesive frameworks for fractional operators [9], [10], [11], [12], [13].

Motivated by these observations, this paper introduces a new fractional derivative defined through the generalized exponential function associated with the Mittag–Leffler function. The proposed operator is constructed to provide an alternative framework for describing memory effects while preserving consistency with the classical derivative as the fractional order approaches an integer value. Several fundamental analytical properties of the new operator are investigated, including its associated fractional integral and Laplace transform. Furthermore, the existence and uniqueness of solutions for the corresponding Cauchy-type problems are established by means of the Banach fixed-point theorem.

The rest of this document is structured as follows: The required mathematical preliminary is presented in Section 2; the proposed derivative is formulated and its analytical properties are investigated in Section 3; the existence and uniqueness results for the associated fractional differential equations are established in Section 4; the analytical and numerical solutions, along with comparative simulations, are provided in Section 5; and the paper is concluded and potential future research directions are outlined in Section 6.

2. Preliminaries

Definition 1. [14], [15] *The Mittag-Leffler function, which is written as $E_\gamma(z)$, is a function that generalizes the exponential function. It is very important in fractional calculus and solving fractional differential equations. It is defined by the following power series for complex number z:*

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma+1)}, (Re(\gamma) > 0). \tag{2.1}$$

A more general two-parameter form is:

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma+\beta)}, (Re(\gamma) > 0, \beta \in \mathbb{C}). \tag{2.2}$$

The γ-Exponential function is defined as follows:

$$e_\gamma^{\vartheta z} = \exp_\gamma(\vartheta z) = z^{\gamma-1} E_{\gamma,\gamma}\left(\vartheta z^\gamma\right) = \sum_{k=0}^{\infty} \frac{\vartheta^k z^{(k+1)\gamma-1}}{\Gamma((k+1)\gamma)}, \tag{2.3}$$

this results in $exp_1^{\vartheta z} = e^{\vartheta z}$, where $z \in \mathbb{C}/\{0\}, \mathrm{Re}(\gamma) > 0$, and $\vartheta \in \mathbb{C}$.

Definition 2. [16] *The Laplace transform of* $\mathfrak{f}(x)$ *is defined by*

$$\hat{\mathfrak{f}}(s) = \mathcal{L}\{\mathfrak{f}(x)\} = \int_0^\infty e^{-sx}\mathfrak{f}(x)dx, \tag{2.4}$$

where $x \geq 0$ *and* $\mathfrak{f}(x)$ *is an integrable function. We say that* $\mathfrak{f}(x) = \mathcal{L}^{-1}\{\hat{\mathfrak{f}}(s)\}$ *is the unique inverse Laplace transform of* $\hat{\mathfrak{f}}(s)$*.*

The following is a Laplace transform of some functions, where ϑ is a constant [17]

$$\mathcal{L}\{\delta(t-\tau)\} = \exp(-\tau s), \tag{2.5}$$

$$\mathcal{L}\left\{\exp_\gamma(\theta x)\right\} = \frac{1}{s^\gamma - \vartheta}. \tag{2.6}$$

Definition 3. [18] *Let* $\mathfrak{f} \in \mathfrak{L}_1[\mathfrak{a},\mathfrak{b}], \mathfrak{g} \in AC^n[\mathfrak{a},\mathfrak{b}]$ *and* $\mathfrak{a} < \mathfrak{b}$*, the Riemann-Liouville fractional integral and derivative of order* $\gamma > 0$ *are defined by*

$$\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \mathfrak{f}_\gamma(t) = \frac{1}{\Gamma(\gamma)}\int_{\mathfrak{a}}^t \frac{\mathfrak{f}(\tau)d\tau}{(t-\tau)^{1-\gamma}}, (t > \mathfrak{a}; \gamma > 0), \tag{2.8}$$

$$\mathfrak{D}^\gamma_{\mathfrak{a},t}\mathfrak{g}(t) = \mathfrak{g}^{(n)}_{n-\gamma}(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_{\mathfrak{a}}^t (t-\tau)^{n-\gamma-1}\mathfrak{g}(\tau)d\tau, \tag{2.9}$$

respectively.

Definition 4. [19], [20] *Let* $0 < \gamma \leq 1$*,* $\mathfrak{a} < \mathfrak{b}$*, and* $\mathfrak{f}(t) \in H^1[\mathfrak{a},\mathfrak{b}]$*, then the Atangana-Baleanu fractional derivative and the Atangana-Baleanu fractional integral of order* γ *are given by*

$$^{AB}\mathfrak{D}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \frac{B(\gamma)}{1-\gamma}\frac{d}{dt}\int_{\mathfrak{a}}^t E_\gamma\left(-\frac{\gamma}{1-\gamma}(t-\tau)^\gamma\right)\mathfrak{f}(\tau)d\tau, \tag{2.10}$$

$$^{AB}\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \frac{1}{B(\gamma)}\left(1-\gamma+\gamma\mathfrak{T}^\gamma_{\mathfrak{a},t}\right)\mathfrak{f}(t), \tag{2.11}$$

respectively, the fractional derivative and integral of Caputo-Fabrizio with order γ *are given by*

$$^{CF}\mathfrak{D}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \frac{B(\gamma)}{1-\gamma}\frac{d}{dt}\int_{\mathfrak{a}}^t \exp\left(-\frac{\gamma}{1-\gamma}(t-\tau)\right)\mathfrak{f}(\tau)d\tau, \tag{2.12}$$

$$^{CF}\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \frac{1}{B(\gamma)}\left(1-\gamma+\gamma\mathfrak{T}_{\mathfrak{a},t}\right)\mathfrak{f}(t), \tag{2.13}$$

where $B(\gamma)$ *is a function that satisfies the condition* $B(0) = B(1) = 1$*.*

Lemma 1. [21] *If* $\gamma > 0$ $(n-1 < \gamma < n), \beta > 0$ $(m-1 < \beta < m)$*, and* $\mathfrak{f},\mathfrak{g} \in \mathfrak{L}_1(\mathfrak{a},\mathfrak{b})$ *such that* $\mathfrak{g}_{n-\gamma} \in AC^n[\mathfrak{a},\mathfrak{b}]$ *then*

$$\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{T}^\beta_{\mathfrak{a},t}\mathfrak{f}(t) = \mathfrak{T}^{\gamma+\beta}_{\mathfrak{a},t}\mathfrak{f}(t), \tag{2.14}$$

$$\mathfrak{D}^\gamma_{\mathfrak{a},t}\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{f}(t) = \mathfrak{g}(t), \tag{2.15}$$

$$\mathfrak{T}^\gamma_{\mathfrak{a},t}\mathfrak{D}^\gamma_{\mathfrak{a},t}\mathfrak{g}(t) = \mathfrak{g}(t) - \sum_{j=0}^{n-1}\frac{\mathfrak{g}^{(n-j-1)}_{n-\gamma}(\mathfrak{a})}{\Gamma(\gamma-j)}(t-\mathfrak{a})^{\gamma-j-1}. \tag{2.16}$$

3. The new fractional derivative

Definition 5. (*the new fractional derivative*) *Assume that* $t \in (\mathfrak{a}, \mathfrak{b}], \mathfrak{a} < \mathfrak{b}, \gamma > 0$, *and* $n-1 < \gamma \leq n$, *the left and right proposed fractional derivative is defined respectively as follows:*

$$^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a}+,t}\mathfrak{f}(t) = \frac{1}{\gamma-n+1}\frac{d^n}{dt^n}\int_{\mathfrak{a}}^{t} \exp_{n-\gamma}\left(-\frac{n-\gamma}{\gamma-n+1}(t-\tau)\right)\mathfrak{f}(\tau)d\tau, \tag{3.1}$$

$$^{*}\mathfrak{D}^{\gamma}_{t,\mathfrak{b}-}\mathfrak{f}(t) = \frac{(-1)^n}{\gamma-n+1}\frac{d^n}{dt^n}\int_{t}^{\mathfrak{b}} \exp_{n-\gamma}\left(-\frac{n-\gamma}{\gamma-n+1}(\tau-t)\right)\mathfrak{f}(\tau)d\tau, \tag{3.2}$$

where $\mathfrak{f} \in AC^n[\mathfrak{a}, \mathfrak{b}]$ (*see Theorem 1*).

Theorem 1. *Let* $\mathfrak{a} < \mathfrak{b}, n \in \mathbb{N}$, *and* $n-1 < \gamma \leq n$. *Then the two statements are equivalent:*
i. The left and right sides of proposed fractional derivative in (3.1) *and* (3.2) *respectively exists and is finite for every* $t \in (\mathfrak{a}, \mathfrak{b}]$.
ii. $\mathfrak{f} \in AC^n[\mathfrak{a}, \mathfrak{b}]$.

Proof. Sufficiency. Assume that $\mathfrak{f} \in AC^n(\mathfrak{a}, \mathfrak{b})$. This means that $\mathfrak{f}^{(n)} \in \mathfrak{L}^1(\mathfrak{a}, \mathfrak{b})$, since $\mathfrak{f}^{(n-1)}$ is absolutely continuous.By fulfilling the conditions $\gamma > 0$ and $t > \mathfrak{a}$, the kernel $\exp_{n-\gamma}\left(-\frac{n-\gamma}{\gamma-n+1}t\right)$ is continuous on ($0, \infty$) and belongs to Locally Integrable Functions $\mathfrak{L}^1_{\text{loc}}(0, \infty)$. For every $t \in (\mathfrak{a}, \mathfrak{b}]$, the function $\exp_{n-\gamma}\left(-\frac{n-\gamma}{\gamma-n+1}t\right)$ is integrable on $[\mathfrak{a}, t]$, and since $\mathfrak{f}^{(n)} \in \mathfrak{L}^1(\mathfrak{a}, t)$, the two integrals $^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a}+,t}\mathfrak{f}(t)$ and $^{*}\mathfrak{D}^{\gamma}_{t,\mathfrak{b}-}\mathfrak{f}(t)$ is finite. Hence the left and right sides proposed fractional derivative exists.

Necessity. Let $^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a}+,t}\mathfrak{f}(t)$ be finite and exists for all $t \in (\mathfrak{a}, \mathfrak{b}]$. Using (2.5) and (2.6), we have $\lim_{\gamma\to n}\exp_{n-\gamma}\left(-\frac{n-\gamma}{\gamma-n+1}t\right) = \delta(t)$, Consequently, $\lim_{\gamma\to n} {}^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a}+,t}\mathfrak{f}(t) = \mathfrak{f}^{(n)}(t)$. Since each $^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a}+,t}\mathfrak{f}(t)$ is locally integrable (because the defining integral is finite by assumption); hence $\mathfrak{f}^{(n)} \in \mathfrak{L}^1_{\text{loc}}(\mathfrak{a}, \mathfrak{b})$. Integrating $\mathfrak{f}^{(n)}$ shows that $\mathfrak{f}^{(n-1)}$ is absolutely continuous on every compact subinterval, and iterating this argument yields $\mathfrak{f} \in AC^n[\mathfrak{a}, \mathfrak{b}]$.

It should be noted that this study will only focus on the left-hand fractional derivative of order $0 < \gamma \leq 1$, and researchers can address the other cases of the new fractional derivative.

Using (2.6) and Laplace transform of nth -derivative, the Laplace transform of the proposed fractional derivative with order $0 < \gamma \leq 1$ can be easily found.

$$\mathcal{L}\left\{^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{f}(t)\right\} = \frac{1}{\gamma s^{-\gamma} + (1-\gamma)s^{-1}}\hat{\mathfrak{f}}(s). \tag{3.3}$$

It is very important to define the fractional integral corresponding to the proposed derivative, and for that purpose we will begin with a simple assumption,

$$^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{f}(t) = \mathfrak{g}(t), \tag{3.4}$$

By (3.3), we get the following result

$$\frac{1}{\gamma s^{-\gamma} + (1-\gamma)s^{-1}}\hat{\mathfrak{f}}(s) = \hat{\mathfrak{g}}(s). \tag{3.5}$$

By performing simple algebraic operations,

$$\hat{\mathfrak{f}}(s) = \left(\frac{\gamma}{s^{\gamma}} + \frac{1-\gamma}{s}\right)\hat{\mathfrak{g}}(s). \tag{3.6}$$

Taking the inverse of the Laplace transform to both sides,

$$\mathfrak{f}(t) = \gamma \mathfrak{T}^{\gamma}_{0,t}\mathfrak{g}(t) + (1-\gamma)\mathfrak{T}_t\mathfrak{g}(t). \tag{3.7}$$

Therefore, the fractional integral of order γ corresponding to the proposed fractional derivative with order $0 < \gamma \leq 1$ can be written as follows:

$$^{*}\mathfrak{T}^{\gamma}_{0,t}\mathfrak{f}(t) = \left(\gamma\mathfrak{T}^{\gamma}_{0,t} + (1-\gamma)\mathfrak{T}_t\right)\mathfrak{f}(t). \tag{3.8}$$

The Laplace transform of the fractional integral of order γ corresponding to the proposed fractional derivative is given by

$$\mathcal{L}\left\{^{*}\mathfrak{T}^{\gamma}_{0,t}\mathfrak{f}(t)\right\} = \left(\frac{\gamma}{s^{\gamma}} + \frac{1-\gamma}{s}\right)\hat{\mathfrak{f}}(s). \tag{3.9}$$

Theorem 2. *Suppose that $\mathfrak{f}, \mathfrak{g} \in AC[\mathfrak{a}, \mathfrak{b}]$, $\mathfrak{a} < \mathfrak{b}$, $0 < \gamma \leq 1$, m is a non-negative integer number and ϑ is a constant, then*

i.

$$^{*}\mathfrak{D}^{\gamma}_{0,t}(\vartheta\mathfrak{f}(t) + \mathfrak{g}(t)) = \vartheta\, {}^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{f}(t) + {}^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{g}(t), \tag{3.10}$$

ii.

$$^{*}\mathfrak{T}^{\gamma}_{0,t}(\vartheta\mathfrak{f}(t) + \mathfrak{g}(t)) = \vartheta\, {}^{*}\mathfrak{T}^{\gamma}_{0,t}\mathfrak{f}(t) + {}^{*}\mathfrak{T}^{\gamma}_{0,t}\mathfrak{g}(t), \tag{3.11}$$

iii.

$$^{*}\mathfrak{D}^{\gamma}_{0,t}\, {}^{*}\mathfrak{T}^{\gamma}_{0,t}\mathfrak{f}(t) = \mathfrak{f}(t), \tag{3.12}$$

iv.

$$^{*}\mathfrak{T}^{\gamma}_{0,t}\, {}^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{f}(t) = \mathfrak{f}(t), \tag{3.13}$$

v.

$$\frac{d^m}{dt^m}\, {}^{*}\mathfrak{D}^{\gamma}_{0,t}\mathfrak{f}(t) = {}^{*}\mathfrak{D}^{m+\gamma}_{0,t}\mathfrak{f}(t). \tag{3.14}$$

4. The existence and uniqueness

In this section, we will use Banach fixed-point theorem to prove the existence and uniqueness of the solution to Cauchy's problems involving the new derivative.

Consider the following Cauchy-type problem of order $0 < \gamma \leq 1$.

$$^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon(t) = \mathfrak{f}(t, \Upsilon(t)), \mathfrak{a} \leq t \leq \mathfrak{b}, \tag{4.1}$$

with initial conditions

$$\Upsilon(\mathfrak{a}) = \mathfrak{b}; \mathfrak{b} \in \mathbb{C}. \tag{4.2}$$

We will study conditions for a unique solution to (4.1)-(4.2) in $\mathfrak{L}[\mathfrak{a},\mathfrak{b}]$ defined for $0 < \gamma \leq 1$, where $\mathfrak{L}(\mathfrak{a},\mathfrak{b})$ is the set of measurable functions $\mathfrak{f}$ on $[\mathfrak{a},\mathfrak{b}]$ for which $\|\mathfrak{f}\| < \infty$, where

$$\|\mathfrak{f}\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} = \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathfrak{f}(t)| dt, \tag{4.3}$$

and

$$\mathfrak{L}^{\gamma}(\mathfrak{a},\mathfrak{b}) = \left\{ \mathfrak{f} \in \mathfrak{L}(\mathfrak{a},\mathfrak{b}) : {}^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\mathfrak{f} \in \mathfrak{L}(\mathfrak{a},\mathfrak{b}) \right\}. \tag{4.4}$$

We prove that the following Volterra integral equation and the Cauchy problem (4.1)-(4.2)

$$\Upsilon(t) = \int_{\mathfrak{a}}^{t} \mathfrak{f}(t,\Upsilon(t)) \left(\frac{\gamma}{\Gamma(\gamma)}(t-\tau)^{\gamma-1} + 1 - \gamma \right) d\tau, \tag{4.5}$$

are equivalent.

Lemma 2. [22] . *The fractional integral of Riemann-Liouville $\mathfrak{T}^{\gamma}_{\mathfrak{a},t}$ with $0 < \gamma \leq 1$ is bounded in $\mathfrak{L}[\mathfrak{a},\mathfrak{b}]$,*

$$\left\|\mathfrak{T}^{\gamma}_{\mathfrak{a},t}\mathfrak{f}\right\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} \leq K\|\mathfrak{f}\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})}, \tag{4.6}$$

where $K = \frac{(\mathfrak{b}-\mathfrak{a})^{\gamma}}{\Gamma(\gamma+1)}$.

Theorem 3. *The new fractional integral ${}^{*}\mathfrak{T}^{\gamma}_{\mathfrak{a},t}$ with $0 < \gamma \leq 1$ is bounded in $\mathfrak{L}[\mathfrak{a},\mathfrak{b}]$,*

$$\left\|{}^{*}\mathfrak{T}^{\gamma}_{\mathfrak{a},t}\mathfrak{f}\right\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} \leq C\|\mathfrak{f}\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})}, \tag{4.7}$$

where $C = \gamma\frac{(\mathfrak{b}-\mathfrak{a})^{\gamma}}{\Gamma(\gamma+1)} + (1-\gamma)(\mathfrak{b}-\mathfrak{a})$.

Proof. This theorem can be easily proven through the Lemma 2.

Theorem 4. *Let $0 < \gamma \leq 1$, $\mathfrak{f}(t,\Upsilon) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$ for any $\Upsilon \in G$ and G be an open set in $\mathbb{R}$ where $\mathfrak{f} : (\mathfrak{a},\mathfrak{b}] \times G \to \mathbb{R}$. If $\Upsilon(t) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$, then $\Upsilon(t)$ satisfies (4.1) and (4.2) if and only if $\Upsilon(t)$ satisfies (4.5).*

Proof. Let $\Upsilon(t) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$ be a solution to the Cauchy problem (4.1)-(4.2). Applying the operator ${}^{*}\mathfrak{T}^{\gamma}_{\mathfrak{a},t}$ on (4.1), and talking (3.13) and (4.2), we get (4.5).

If $\Upsilon(t) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$ satisfies (4.5), by taking ${}^{*}\mathfrak{D}^{\gamma}_{\mathfrak{a},t}$ to both sides of (4.5) and by using (3.12), we get the Cauchy problem (4.1)-(4.2), the proof is complete.

The Lipschitzian condition on $\mathfrak{f}(x,\Upsilon)$ with respect to Υ for all $t \in (\mathfrak{a},\mathfrak{b}]$ and for all $\Upsilon_1, \Upsilon_2 \in G \subset \mathbb{R}$,

$$\left|\mathfrak{f}\left(t,\Upsilon_1\right) - \mathfrak{f}\left(t,\Upsilon_2\right)\right| \leqq A\left|\Upsilon_1 - \Upsilon_2\right|, \tag{4.8}$$

where $A > 0$.

Theorem 5. [22] *Let (X,d) be a complete metric space, and let $T : X \to X$ be the map such that, for every $x_1, x_2 \in X$, the relation $d\left(Tx_1, Tx_2\right) \leq \omega d\left(x_1, x_2\right)$ holds, where $0 < \omega < 1$. Then the operator T has a unique fixed point $x \in X$ and the sequence $\left\{T^k x_0\right\}_{k=1}^{\infty}$ converges to the above fixed point x for any $x_0 \in X$.*

Theorem 6. *Suppose that the conditions of Theorem 4 and the Lipschitz condition (4.8) hold. Then there exists a unique solution* $\Upsilon(t)$ *to (4.1)-(4.2) in* $\mathfrak{L}^{\gamma}[\mathfrak{a},\mathfrak{b}]$.

Proof. By Theorem 4, it suffices to prove the existence of a unique solution $\Upsilon(t) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$ to (4.5). In any interval $[\mathfrak{a}, t_1] \subset [\mathfrak{a},\mathfrak{b}]$ where $t_1 \in (\mathfrak{a},\mathfrak{b})$, (4.5) makes sense, choose t_1 such that the inequality

$$A\left(\frac{\gamma}{\Gamma(\gamma)}(t_1-\mathfrak{a})^{\gamma-1}+1-\gamma\right)<1, \tag{4.9}$$

holds, and establish that $\Upsilon(x) \in \mathfrak{L}[\mathfrak{a}, t_1]$ is exist to (4.5) on the closed interval $[\mathfrak{a}, t_1]$, so we use the Banach fixed point for $C^{m-1}[a,b]$, which is the complete metric space, and the distance between points is given by

$$d(\Upsilon_1,\Upsilon_2)=\|\Upsilon_1-\Upsilon_2\|=\int_{\mathfrak{a}}^{t_1}|\Upsilon_1(t)-\Upsilon_2(t)|\,dt. \tag{4.10}$$

We rewrite (4.5) by using the operator T as follows

$$(T\Upsilon)(t)=\int_{\mathfrak{a}}^{t}\mathfrak{f}(t,\Upsilon(t))\left(\frac{\gamma}{\Gamma(\gamma)}(t-\tau)^{\gamma-1}+1-\gamma\right)d\tau. \tag{4.11}$$

To apply Theorem 5, we must prove that (1) if $\Upsilon(t) \in \mathfrak{L}[\mathfrak{a}, t_1]$, then $(T\Upsilon)(t) \in \mathfrak{L}[\mathfrak{a}, t_1]$; and (2) for any $\Upsilon_1, \Upsilon_2 \in \mathfrak{L}[\mathfrak{a}, t_1]$,

$$\|T\Upsilon_1-T\Upsilon_2\|_{\mathfrak{L}(\mathfrak{a},t_1)}\leqq\omega\,\|\Upsilon_1-\Upsilon_2\|_{\mathfrak{L}(\mathfrak{a},t_1)} \tag{4.12}$$

with

$$\omega=A\left(\frac{\gamma}{\Gamma(\gamma)}(t_1-\mathfrak{a})^{\gamma-1}+1-\gamma\right) \tag{4.13}$$

since $f(t,\Upsilon(t)) \in \mathfrak{L}(\mathfrak{a},\mathfrak{b})$ and by Lemma 2, $(T\Upsilon)(t) \in \mathfrak{L}(\mathfrak{a}, t_1)$. By (4.11), $\mathfrak{L}^{\gamma}(\mathfrak{a},\mathfrak{b})$ and (4.8) and applying (4.6),

$$\begin{aligned}\|T\Upsilon_1-T\Upsilon_2\|_{\mathfrak{L}(\mathfrak{a},t_1)} &\leq \left\|{}^{*}\mathfrak{T}^{\gamma}_{\mathfrak{a},t}\left[\left|\mathfrak{f}(t,\Upsilon_1(t))-\mathfrak{f}(t,\Upsilon_2(t))\right|\right]\right\|_{\mathfrak{L}(\mathfrak{a},t_1)}\\ &\leq A\left\|{}^{*}\mathfrak{T}^{\gamma}_{\mathfrak{a},t}\left[\left|\Upsilon_1(t)-\Upsilon_2(t)\right|\right]\right\|_{\mathfrak{L}(\mathfrak{a},t_1)}\\ &\leq A\left(\frac{\gamma}{\Gamma(\gamma)}(t_1-\mathfrak{a})^{\gamma-1}+1-\gamma\right)\|\Upsilon_1(t)-\Upsilon_2(t)\|_{\mathfrak{L}(\mathfrak{a},t_1)}\end{aligned} \tag{4.14}$$

which yields (4.12). According to Theorem 5 and (4.13), $0<\omega<1$, there exists a unique solution $\Upsilon(t)=\Upsilon^*(t) \in \mathfrak{L}(\mathfrak{a}, t_1)$ to (4.5) on $[\mathfrak{a}, t_1]$

By Theorem 5, $\Upsilon^*(t)$ is a limit of the convergent sequence $\Upsilon_m(t)=(T^m\Upsilon^*)(t)$:

$$\lim_{m\to\infty}\|T^m\Upsilon_0^*-\Upsilon^*\|_{\mathfrak{L}(\mathfrak{a},t_1)}=0 \tag{4.15}$$

where $\Upsilon_0^*(t)$ is any function in $\mathscr{L}[a,b]$. If $b\neq 0$ in (4.2), we can take $\Upsilon_0^*(t)=\Upsilon_0(t)$ with $\Upsilon_0(t)$ defined in (4.11). Using the same method as above, we conclude that there is a unique solution $\Upsilon^*(t) \in \mathfrak{L}(t_1, t_2)$ to (4.5) on $[t_1, t_2]$, where $t_2 = t_1 + h_1$ and $h_1 > 0$ are such that $t_2 < \mathfrak{b}$, and repeating this process, we conclude that there exists a unique solution $\Upsilon^*(t) \in \mathfrak{L}(\mathfrak{a},\mathfrak{b})$ for (4.5) on $[a,b]$.

Thus $\Upsilon_m(t)=(T^m\Upsilon^*)(t)$

$$\lim_{m\to\infty} \|\Upsilon_m - \Upsilon\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} = 0$$

Using this formula and taking (4.1) and (4.8) into account, we have

$$\left\|{}^*\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon_m - {}^*\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon\right\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} \leqq \|\mathfrak{f}[\tau, \Upsilon_m(\tau)] - \mathfrak{f}[\tau, \Upsilon(\tau)]\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})}$$

$$\leq A\|\Upsilon_m - \Upsilon\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} \leqq A(\mathfrak{b}-\mathfrak{a})^{\gamma}\|\Upsilon_m - \Upsilon\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})} \leqq A\|\Upsilon_m - \Upsilon\|_{\mathfrak{L}(\mathfrak{a},\mathfrak{b})}$$

From this and (4.15), we obtain

$$\lim_{m\to\infty} \left\|{}^*\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon_m - {}^*\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon\right\|_{C_\gamma[\mathfrak{a},\mathfrak{b}]} = 0$$

Thus ${}^*\mathfrak{D}^{\gamma}_{\mathfrak{a},t}\Upsilon(t) \in \mathfrak{L}[\mathfrak{a},\mathfrak{b}]$, and hence, in accordance with (4.3), $\Upsilon \in \mathfrak{L}^{\gamma}[\mathfrak{a},\mathfrak{b}]$.

5. Approximation Solutions of Fractional Equations

This section will include approximate solutions to Cauchy-type problems with RL-fractional derivative, AB -fractional derivative, CF -fractional derivative, and the proposed fractional derivative using the Adomian decomposition method, and then discuss the efficiency of the solutions through graphs and value tables. This section will start by presenting the algorithm for the Adomian decomposition method applied to Cauchy-type problems; the algorithms for other fractional derivatives can be found in the referenced studies [23], [24], [25], [26].

We consider the differential equation of fractional order $0 < \gamma \leq 1$.

$$ {}^*\mathfrak{D}^{\gamma}_{0+,t}\Upsilon(t) + R\Upsilon(t) + N\Upsilon(t) = \mathfrak{g}(t) \tag{5.1}$$

and

$$\Upsilon(0) = \mathfrak{b} \tag{5.2}$$

where R, N and g are linear operator, nonlinear operator and known function respectively.
Applying (3.8) on (5.1) and by using (5.2), (3.13), we obtain

$$\Upsilon(t) = {}^*\mathfrak{I}^{\gamma}_{0+,t}(\mathfrak{g}(t) - R\Upsilon(t) - N\Upsilon(t)) \tag{5.3}$$

Now, we write

$$\Upsilon(t) = \sum_{m=0}^{\infty}\Upsilon_m(t) \text{ and } N\Upsilon(t) = \sum_{m=0}^{\infty} A_m(t) \tag{5.4}$$

where $A_m(t)$ are the Adomian polynomials (see [27]).
Putting (5.4) in (5.3)

$$\sum_{m=0}^{\infty}\Upsilon_m(t) = {}^*\mathfrak{I}^{\gamma}_{0+,t}\left(\mathfrak{g}(t) - \sum_{m=0}^{\infty} R\Upsilon_m(t) - \sum_{m=0}^{\infty} A_m(t)\right) \tag{5.5}$$

Assuming $\mathfrak{S}(t) = \mathfrak{b} + {}^*\mathfrak{I}^{\gamma}_{0+,t}(\mathfrak{g}(t))$, we have

$$\sum_{m=0}^{\infty}\Upsilon_m(t) = \mathfrak{S}(t) - {}^*\mathfrak{I}^{\gamma}_{0+,t}\left(\sum_{m=0}^{\infty} R\Upsilon_m(t) + \sum_{m=0}^{\infty} A_m(t)\right) \tag{5.6}$$

Therefore, we have

$$\Upsilon_0(t) = \mathfrak{S}(t) \tag{5.7}$$

$$\Upsilon_{m+1}(t) = -{}^{*}\mathfrak{T}^{\gamma}_{0+,t}\left(R\Upsilon_m(t) + A_m(t)\right), m = 0, 1, 2, \ldots \tag{5.8}$$

It is clear that when some of the components $\Upsilon_m(t)$ are known, the solution $\Upsilon(t)$ can be approximated as a series.

$$\Upsilon(t) = \lim_{\eta\to\infty}\sum_{m=0}^{\eta}\Upsilon_m(t) \tag{5.9}$$

Example 1. *Consider the below differential equations with order* $0 < \gamma \leq 1$

$$ {}^{*}D^{\gamma}_{0,t}\Upsilon(t) + te^{\Upsilon(t)} = 0 \tag{5.10}$$

$$\mathfrak{D}^{\gamma}_{0,t}\Upsilon(t) + te^{\Upsilon(t)} = 0 \tag{5.11}$$

$$ {}^{AB}\mathfrak{D}^{\gamma}_{0,t}\Upsilon(t) + te^{\Upsilon(t)} = 0 \tag{5.12}$$

$$ {}^{CF}\mathfrak{D}^{\gamma}_{0,t}\Upsilon(t) + te^{\Upsilon(t)} = 0 \tag{5.13}$$

with initial conditions equal to 1 .

Using the algorithm of the Adomian decomposition method, the approximation solutions of equations (5.10), (5.11), (5.12), and (5.13) respectively are given by

$$\begin{aligned}
\Upsilon_*(t) = {}& 1 - \frac{\gamma t^{1+\gamma}e}{\Gamma(2+\gamma)} + \frac{1}{2}t^2e(-1+\gamma) + \frac{1}{8}t^4e^2(-1+\gamma)^2 - \frac{t^{3+\gamma}e^2\gamma(\gamma+5)(-1+\gamma)}{\Gamma(4+\gamma)} + \frac{t^{2\gamma+2}\gamma^2e^2(2+\gamma)}{\Gamma(3+2\gamma)} \\
& + \frac{1}{4}\frac{t^{2\gamma+4}\gamma^2(-1+\gamma)e^3}{(2+\gamma)\Gamma(2+\gamma)^2} + \frac{1}{24}t^6(-1+\gamma)^3e^3 + \frac{1}{2}\frac{t^{2\gamma+4}\gamma^2\left(\gamma^3+15\gamma^2+58\gamma+76\right)(-1+\gamma)e^3}{\Gamma(5+2\gamma)} \\
& - \frac{1}{2}\frac{t^{\gamma+5}\gamma(-1+\gamma)^2\left(\gamma^3+11\gamma^2+44\gamma+124\right)e^3}{\Gamma(6+\gamma)} \\
& - \frac{1}{2}\frac{\gamma^3t^{3\gamma+3}\left(\Gamma(2\gamma+4)+2\Gamma(2+\gamma)^2(2+\gamma)(3+2\gamma)\right)e^3}{\Gamma(2+\gamma)^2\Gamma(3\gamma+4)}. \\
\Upsilon_{RL}(t) = {}& 1 - \frac{et^{1+\gamma}}{\Gamma(2+\gamma)} + \frac{e^2(2+\gamma)t^{2\gamma+2}}{\Gamma(3+2\gamma)}. \\
\Upsilon_{AB}(t) = {}& 1 - \frac{\gamma t^{1+\gamma}e}{\Gamma(2+\gamma)} + te(-1+\gamma) + t^2e^2(-1+\gamma)^2 - \frac{t^{2+\gamma}e^2\gamma(4+\gamma)(-1+\gamma)}{\Gamma(3+\gamma)} \\
& + \frac{t^{2\gamma+2}\gamma^2e^2(2+\gamma)}{\Gamma(3+2\gamma)} + \frac{1}{2}\frac{t^{3+2\gamma}e^3\gamma^2(-1+\gamma)}{\Gamma(2+\gamma)^2} + \frac{t^{3+2\gamma}\gamma^2e^3(-1+\gamma)\left(4\gamma^2+19\gamma+24\right)}{\Gamma(2\gamma+4)} \\
& + \frac{3}{2}t^3e^3(-1+\gamma)^3 - \frac{1}{2}\frac{\gamma^3t^{3\gamma+3}\left(\Gamma(2\gamma+4)+2\Gamma(2+\gamma)^2(2+\gamma)(3+2\gamma)\right)e^3}{\Gamma(2+\gamma)^2\Gamma(3\gamma+4)} \\
& - \frac{\gamma t^{3+\gamma}e^3\left(2\gamma^2+12\gamma+27\right)(-1+\gamma)^2}{\Gamma(4+\gamma)}. \\
\Upsilon_{CF}(t) = {}& 1 - \frac{1}{2}et(\gamma t-2\gamma+2) + \frac{1}{24}e^2t^2\left(3\gamma^2t^2-20\gamma^2t+24\gamma^2+20\gamma t-48\gamma+24\right) \\
& - \frac{1}{120}e^3t^3\left(5\gamma^3t^3-62\gamma^3t^2+205\gamma^3t+62\gamma^2t^2-180\gamma^3-410\gamma^2t+540\gamma^2+205\gamma t-540\gamma+180\right)\cdot
\end{aligned}$$

Thus, the approximate solution of (5.10), (5.11), (5.12), and (5.13) at $\gamma = 1$ is given by

$$\Upsilon_{\gamma=1}(t) = 1 - \frac{e}{2}t^2 + \frac{e^2}{8}t^4 - \frac{e^3}{24}t^6 + \cdots \tag{5.18}$$

and the exact solution is

$$\Upsilon_E = e^{-\frac{e}{2}t^2} \tag{5.19}$$

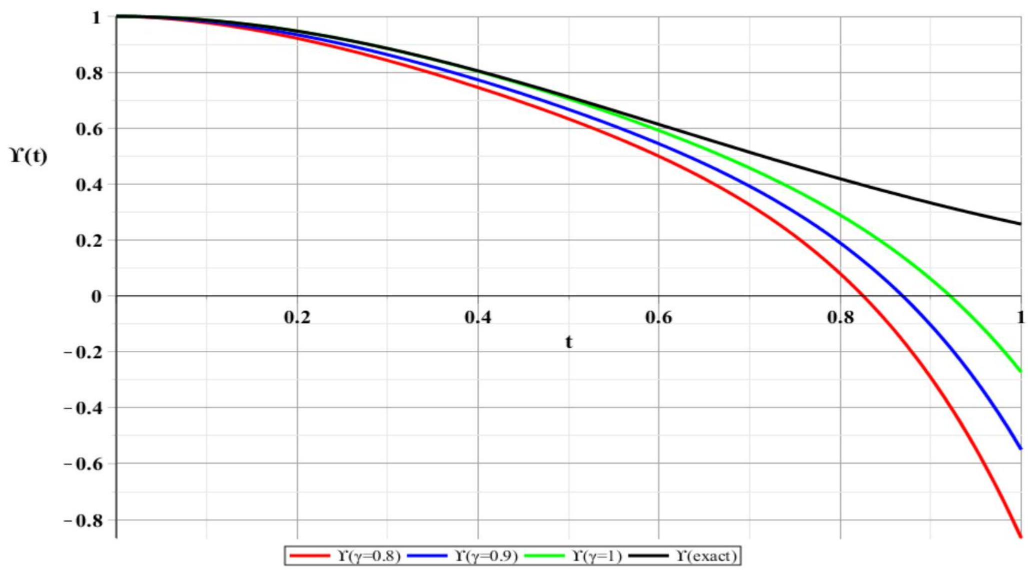

FIGURE 1. The approximate and exact solutions for equation (5.10) at different values of order γ.

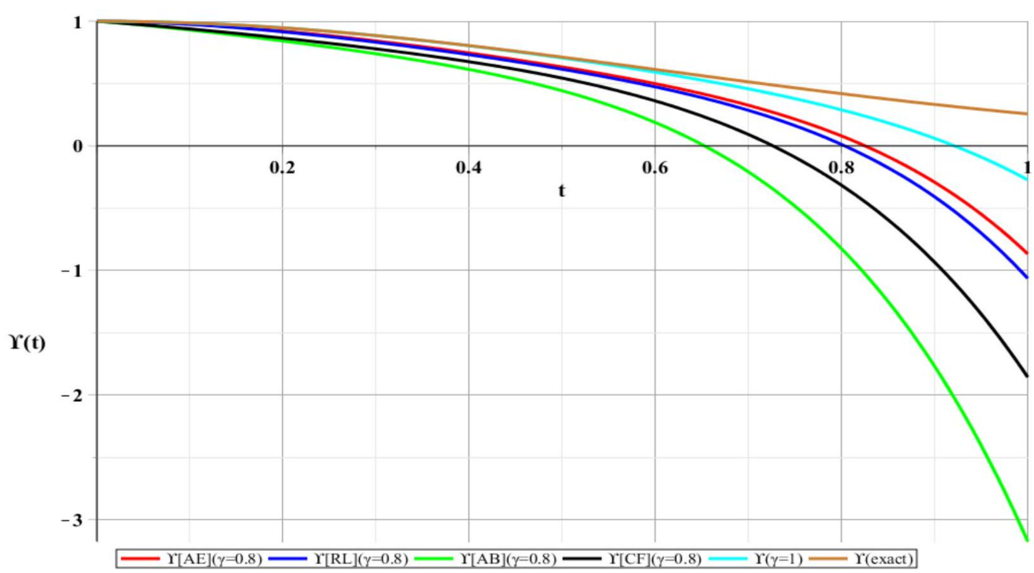

FIGURE 2. The approximate solutions for equations (5.10)–(5.13) at $\gamma = 0.8$ and $\gamma = 1$ and the exact solution.

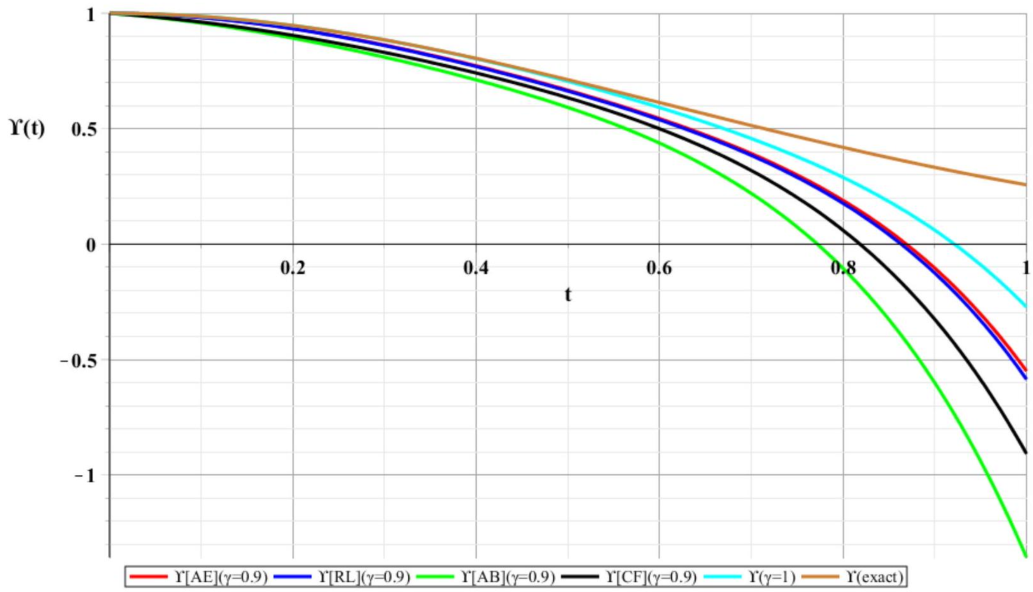

FIGURE 3. The approximate solutions for equations (5.10)-(5.13) at $\gamma = 0.9$ and $\gamma = 1$, together with the exact solution.

TABLE 1. Numerical values of the solution $\Upsilon(t)$ in equation (5.10) for different fractional orders γ.

t	$\Upsilon_{\gamma=0.8}$	$\Upsilon_{\gamma=0.9}$	$\Upsilon_{\gamma=1}$	Υ_E	$\lvert\Upsilon_E - \Upsilon_{0.8}\rvert$	$\lvert\Upsilon_E - \Upsilon_{0.9}\rvert$	$\lvert Y_E - Y_1\rvert$
0.1	0.977031	0.981963	0.986500	0.986501	0.009469	0.004537	4.2E-07
0.2	0.921210	0.934056	0.947059	0.947086	0.025876	0.013030	2.71E-05
0.3	0.841932	0.862430	0.884549	0.884863	0.042931	0.022432	0.000314
0.4	0.745354	0.772403	0.802755	0.804558	0.059204	0.032155	0.001803
0.5	0.633041	0.666879	0.704865	0.711923	0.078883	0.045044	0.007058
0.6	0.498879	0.543661	0.591366	0.613061	0.114182	0.069400	0.021695
0.7	0.325384	0.392117	0.457325	0.513770	0.188386	0.121653	0.056445
0.8	0.079559	0.189323	0.289082	0.419014	0.339456	0.229691	0.129932
0.9	-0.291620	-0.104300	0.060329	0.332570	0.624195	0.436874	0.272241
1.0	-0.866090	-0.549850	-0.272410	0.256881	1.122969	0.806729	0.529288

TABLE 2. Comparison of numerical solutions of equations (5.10)–(5.13) at $\gamma = 0.8$.

t	$\lvert\Upsilon_E - \Upsilon_*\rvert$	$\lvert\Upsilon_E - \Upsilon_C\rvert$	$\lvert\Upsilon_E - \Upsilon_{AB}\rvert$	$\lvert\Upsilon_E - \Upsilon_{CF}\rvert$	$\lvert\Upsilon_E - \Upsilon_{\gamma=1}\rvert$
0.1	0.009469	0.011818	0.056821	0.048104	4.2E-07
0.2	0.025876	0.032195	0.105678	0.082797	2.71E-05
0.3	0.042931	0.053232	0.146145	0.10677	0.000314
0.4	0.059204	0.073163	0.191677	0.129406	0.001803
0.5	0.078883	0.097111	0.270174	0.168552	0.007058
0.6	0.114182	0.139733	0.42601	0.252516	0.021695
0.7	0.188386	0.228468	0.72242	0.422399	0.056445
0.8	0.339456	0.407288	1.244184	0.734851	0.129932
0.9	0.624195	0.740926	2.100593	1.265318	0.272241
1	1.122969	1.319593	3.428739	2.111807	0.529288

Table 3. Comparison of numerical solutions of equations (5.10)–(5.13) at $\gamma = 0.9$.

t	$\lvert\Upsilon_E - \Upsilon_*\rvert$	$\lvert\Upsilon_E - \Upsilon_C\rvert$	$\lvert\Upsilon_E - \Upsilon_{AB}\rvert$	$\lvert\Upsilon_E - \Upsilon_{CF}\rvert$	$\lvert\Upsilon_E - \Upsilon_{\gamma=1}\rvert$
0.1	0.004537	0.005040	0.028963	0.024618	4.2E-07
0.2	0.013030	0.014461	0.054939	0.042838	2.71E-05
0.3	0.022432	0.024851	0.075227	0.054631	0.000314
0.4	0.032155	0.035472	0.093246	0.063503	0.001803
0.5	0.045044	0.049250	0.119919	0.078008	0.007058
0.6	0.069400	0.074867	0.175917	0.113616	0.021695
0.7	0.121653	0.129495	0.294382	0.195047	0.056445
0.8	0.229691	0.242185	0.524126	0.359164	0.129932
0.9	0.436874	0.457924	0.933341	0.658488	0.272241
1.0	0.806729	0.842352	1.613840	1.165346	0.529288

Figures 1-3 and Tables 1–3 provide a comprehensive assessment of the performance of the proposed fractional derivative in comparison with the Riemann–Liouville, Atangana–Baleanu, and Caputo–Fabrizio derivatives. Figure 2 illustrates the behavior of the approximate solution of Equation (5.10) for different fractional orders, showing that the numerical solution approaches the exact solution as the fractional order tends to one. Figures 1 and 2 further compare the numerical solutions obtained by the four fractional operators for $(\omega = 0.8)$ and $(\omega = 0.9)$, respectively. The results indicate that the proposed derivative consistently produces solutions closer to the exact solution over the entire computational interval.

Moreover, the numerical data reported in Tables 1–3 support the graphical observations. Table 1 demonstrates the convergence of the proposed model toward the classical solution as the fractional order increases. Tables 2 and 3 present the absolute errors associated with the different fractional derivatives for $(\omega = 0.8)$ and $(\omega = 0.9)$. It can be observed that the proposed derivative yields the smallest absolute errors among the considered fractional operators, particularly when compared with the Atangana–Baleanu and Caputo–Fabrizio derivatives. These results suggest that the proposed operator provides a more accurate approximation for the considered test problem while preserving the expected convergence toward the classical integer-order solution.

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