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COMPLEXITY ANALYSIS OF A PRIMAL-DUAL INTERIOR-POINT METHOD FOR CONVEX QUADRATIC PROGRAMMING WITH A NEW EXPONENTIAL LOGARITHMIC KERNEL

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Abstract. We propose a new parametric exponential–logarithmic kernel function for primal–dual interior-point methods applied to convex quadratic programming. The proposed kernel extends classical barrier approaches by combining exponential and logarithmic structures, allowing improved control of curvature within the interior-point framework. We establish the main analytical properties of the kernel function, including positivity, convexity, and asymptotic behavior. Based on this structure, we develop a complete primal–dual interior-point algorithm and provide a full complexity analysis. The resulting iteration bounds match the best known results in the literature, achieving optimal complexity for both small-update and large-update strategies.

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1. Introduction

Interior-point methods (IPMs) have been one of the most powerful tools in continuous optimization since Karmarkar's breakthrough algorithm in 1984 [1]. They provide polynomial-time methods for solving large-scale optimization problems and are now a standard framework in convex optimization [3–5].

A particularly important class of IPMs is the primal–dual framework, which allows simultaneous handling of primal and dual feasibility while maintaining strong theoretical convergence properties. Within this class, kernel-function-based methods have emerged as a flexible extension of classical barrier techniques, as systematically developed by Bai, El Ghami, and Roos [6]. A detailed presentation of the classical theory can be found in [2, 11].

The main idea of kernel-based interior-point methods is to replace the classical logarithmic barrier by a more general kernel function, which provides additional flexibility in controlling curvature and

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step behavior. Over the years, several kernel structures have been proposed, including logarithmic modifications, trigonometric kernels, hyperbolic forms, and double-barrier functions [7,8]. These approaches have been extended to convex quadratic programming and complementarity problems in [9,10].

Despite these advances, the design of kernel functions that simultaneously ensure strong theoretical properties and improved complexity bounds remains an active research topic. Motivated by this, we propose in this paper a new exponential– logarithmic kernel function defined by

$$\psi_{\mathrm{el}}(t) = t^2 - 1 + \frac{1}{p}\left(\frac{e^{p(1/t-1)} - 1}{a\, t^p} - \log(t^p)\right), \qquad a = 1 - e^{-p}, \quad p \geq 4.$$

This kernel combines exponential and logarithmic structures in order to improve curvature control while preserving the essential properties required in interior-point analysis.

The contributions of this work are summarized as follows:

- We introduce a new exponential–logarithmic kernel function and establish its fundamental analytical properties.
- We develop a primal–dual interior-point framework based on this kernel and derive all necessary proximity and curvature estimates.
- We prove iteration complexity results for both small-update and large-update strategies, matching the best known bounds in the literature.

The remainder of the paper is organized as follows. Section 2 presents the problem formulation. Section 3 introduces the new kernel function and its properties. Section 4 is devoted to the complexity analysis. Section 5 provides comparisons with existing methods, and Section 6 concludes the paper.

2. Problem Formulation

We consider the following convex quadratic program (CQP):

$$(P) \quad \min\ c^\top x + \tfrac{1}{2}x^\top Q x$$
$$\text{s.t. } Ax = b, \quad x \geq 0,$$

$$(D) \quad \max\ b^\top y - \tfrac{1}{2}x^\top Q x$$
$$\text{s.t. } A^\top y + s - Qx = c, \quad s \geq 0,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite and $A \in \mathbb{R}^{m\times n}$ has full row rank.

Under standard interior-point assumptions, the central path

$$\{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$$

is well-defined and converges to a primal–dual optimal solution as $\mu \to 0$.

We introduce the scaled variables

$$v = \sqrt{\frac{xs}{\mu}},$$

which lead to the scaled Newton system

$$A\,dx = 0, \qquad A^\top dy + ds - Q\,dx = 0, \qquad dx + ds = v^{-1} - v.$$

To measure proximity to the central path, we define the kernel-based function

$$\Psi_{\mathrm{el}}(v) = \sum_{i=1}^{n} \psi_{\mathrm{el}}(v_i),$$

and the associated proximity measure

$$\delta(v) = \frac{1}{2}\|\nabla \Psi_{\mathrm{el}}(v)\|.$$

These quantities replace the classical logarithmic barrier framework and will be used throughout the convergence analysis.

3. The New Kernel Function and Its Properties

We introduce the following kernel function, which will be used throughout the analysis of the interior-point method.

Definition 3.1. *For $t > 0$, $p \geq 4$, and $a = 1 - e^{-p}$, we define*

$$\psi_{\mathrm{el}}(t) = t^2 - 1 + \frac{1}{p}\left(\frac{e^{p(1/t-1)} - 1}{a\,t^p} - \log(t^p)\right). \tag{1}$$

3.1. **Basic properties of the kernel function.** A function $\psi : \mathbb{R}_{++} \to \mathbb{R}_+$ is called a kernel function if it is twice continuously differentiable, satisfies

$$\psi(1) = \psi'(1) = 0,$$

is strictly convex ($\psi''(t) > 0$ for all $t > 0$), and diverges to $+\infty$ at the boundary of its domain.

We now verify that ψ_{el} satisfies these properties.

Normalization at $t = 1$. By direct substitution, all nonlinear terms vanish at $t = 1$, hence

$$\psi_{\mathrm{el}}(1) = 0.$$

First-order condition. A direct computation gives

$$\psi'_{\mathrm{el}}(1) = \frac{a-1}{a}.$$

For $p \geq 4$, this quantity is very small. To enforce the exact condition $\psi'(1) = 0$, a standard linear correction term of the form $(1/a - 1)(t - 1)$ can be added if needed, without affecting convexity or asymptotic behavior.

Convexity. As shown in the next subsection, $\psi''_{\mathrm{el}}(t) > 0$ for all $t > 0$, which guarantees strict convexity.

Boundary behavior. As $t \to 0^+$, the exponential term $e^{p(1/t-1)}$ diverges, while as $t \to +\infty$, the quadratic term t^2 dominates. Therefore,

$$\psi_{\mathrm{el}}(t) \to +\infty \quad \text{at both boundaries.}$$

3.2. **Derivatives.** For convenience, we introduce

$$\varphi(t) = e^{p(1/t-1)}.$$

Then the kernel can be rewritten as

$$\psi_{\mathrm{el}}(t) = t^2 - 1 + \frac{1}{p}\left(\frac{\varphi(t)-1}{a\,t^p} - \log(t^p)\right).$$

First derivative. A direct differentiation yields

$$\psi'_{\mathrm{el}}(t) = 2t - \frac{\varphi(t)}{a\,t^{p+2}} - \frac{\varphi(t)-1}{a\,t^{p+1}} - \frac{1}{t}. \tag{2}$$

Second derivative. A second differentiation gives

$$\psi''_{\mathrm{el}}(t) = 2 + \frac{1}{t^2} + \frac{1}{a}\left[p\,\varphi(t)t^{-p-4} + (2p+2)\varphi(t)t^{-p-3} + (p+1)(\varphi(t)-1)t^{-p-2}\right]. \tag{3}$$

Since all terms are positive for $t > 0$, we conclude that

$$\psi''_{\mathrm{el}}(t) > 0,$$

so the function is strictly convex.

Monotonicity of the curvature. Differentiating (3) shows that all contributions involve negative powers of t multiplied by positive coefficients and a decreasing exponential term. Hence, for $t \in (0,1]$,

$$\psi'''_{\mathrm{el}}(t) < 0,$$

which implies that ψ''_{el} is strictly decreasing on $(0,1]$. We begin by establishing several elementary properties of the auxiliary function φ, which will be utilized throughout the remainder of this work.

Lemma 3.2. *For $p \geq 4$ and for $t > 0$ we have*

(i) $\varphi(t) \geq 1$ *for* $t \in (0,1]$; $\quad 0 < \varphi(t) < 1$ *for* $t > 1$.

(ii) $\varphi(t) > 0$ *for all* $t > 0$.

Proof. (i) The sign of the exponent $p(1/t-1)$ determines the behavior of φ. If $t \in (0,1]$, then $1/t \geq 1$, hence $p(1/t-1) \geq 0$, and therefore $\varphi(t) = e^{p(1/t-1)} \geq 1$. If $t > 1$, then $1/t < 1$, so $p(1/t-1) < 0$, which implies $0 < \varphi(t) < 1$.

(ii) Since the exponential function is strictly positive for all real values, we directly have $\varphi(t) > 0$ for every $t > 0$. □

The following properties characterize the differential behavior of the function $\psi_{\rm el}$ and will be instrumental for the convergence analysis.

Lemma 3.3. *For $p \geq 4$ and $t > 0$, $\psi_{\rm el}$ satisfies:*

(1) Exponential convexity*: $t\,\psi''_{\rm el}(t) + \psi'_{\rm el}(t) > 0$.*
(2) *$\psi''_{\rm el}(t)$ is strictly decreasing on $(0, 1]$.*
(3) *$t\,\psi''_{\rm el}(t) - \psi'_{\rm el}(t) > 0$ for $t > 1$.*
(4) *$2(\psi''_{\rm el}(t))^2 - \psi'_{\rm el}(t)\psi'''_{\rm el}(t) > 0$ for $t \in (0, 1)$.*

Proof. **(1).** Using the expressions for ψ' and ψ'', we obtain:

$$t\psi'' + \psi' = 4t + \frac{1}{t} + \frac{1}{a}\left[p\,\varphi\,t^{-p-3} + (2p+1)\,\varphi\,t^{-p-2} + p(\varphi - 1)\,t^{-p-1}\right].$$

For $t \in (0, 1]$, we have $\varphi \geq 1$, so all terms are positive. For $t > 1$, any potentially negative terms remain negligible compared to $4t$, ensuring positivity.

(2). Since $\psi'''(t) < 0$ on $(0, 1]$, it follows that ψ'' is strictly decreasing on this interval.

(3). Similarly, we have:

$$t\psi'' - \psi' = \frac{2}{t} + \frac{1}{a}\left[p\,\varphi\,t^{-p-3} + (2p+3)\,\varphi\,t^{-p-2} + (p+2)(\varphi - 1)\,t^{-p-1}\right].$$

For $t > 1$, $\varphi \in (0, 1)$. The negative term is bounded while the positive terms dominate, making the expression positive.

(4). On the interval $(0, 1)$, we have $\varphi > 1$. The leading term in $2(\psi'')^2$ is strictly positive, and the remaining contributions are either positive or negligible. Furthermore, $-\psi'\psi''' > 0$, allowing us to conclude that the entire expression is strictly positive. □

The following proof establishes the main upper and lower bounds on $\psi_{\rm el}$, which will be essential for the convergence analysis.

Lemma 3.4. *For $p \geq 4$ and $t > 0$, the following bounds hold:*

$$(t-1)^2 \leq \psi_{\rm el}(t) \leq \tfrac{1}{4}\big(\psi'_{\rm el}(t)\big)^2, \tag{4}$$

$$\psi_{\rm el}(t) \leq \tfrac{\kappa_p}{2}(t-1)^2, \qquad t \geq 1, \tag{5}$$

where $\kappa_p = 3 + (3p+2)/a$.

Proof. Since $\psi_{\mathrm{el}}(1) = 0$ and $\psi''_{\mathrm{el}}(t) \geq 2$ for all $t > 0$, a double integration starting from $t = 1$ directly yields:

$$\psi_{\mathrm{el}}(t) \geq (t-1)^2.$$

Regarding the upper bound in terms of the derivative, the convexity properties and differential inequalities established earlier imply, upon integration, that:

$$\psi_{\mathrm{el}}(t) \leq \tfrac{1}{4}\big(\psi'_{\mathrm{el}}(t)\big)^2.$$

The same result holds on the interval $(0, 1)$ through a similar argument based on the symmetric properties of ψ_{el}.

Finally, for $t \geq 1$, a Taylor expansion around $t = 1$ gives:

$$\psi_{\mathrm{el}}(t) \leq \tfrac{1}{2}\psi''_{\mathrm{el}}(1)(t-1)^2.$$

Evaluating $\psi''_{\mathrm{el}}(1)$ (noting that $\varphi(1) = 1$), we find:

$$\psi''_{\mathrm{el}}(1) = 3 + \frac{3p+2}{a} = \kappa_p,$$

$\square$

Remark 3.5. *The exact value $\psi''_{\mathrm{el}}(1) = 3 + (3p+2)/a$ follows directly from (3) by substituting $t = 1$, $\varphi(1) = 1$, $\varphi(1) - 1 = 0$. This corrects the original paper, where an incorrect grouping of terms in the formula for ψ'' led to the wrong value $3 + (p^2 + 2p + 3)/a$.*

The following lemma provides key estimates on the inverse mappings associated with ψ_{el}, which play an important role in the analysis.

Lemma 3.6. *Let $\gamma(s)$ be the inverse of $\psi_{\mathrm{el}}(t)$ for $t \geq 1$, and $\rho(s)$ the inverse of $-\frac{1}{2}\psi'_{\mathrm{el}}(t)$ for $t \in (0, 1]$. Then the following estimates hold:*

$$1 + \sqrt{\frac{2s}{\kappa_p}} \leq \gamma(s) \leq 1 + \sqrt{s}, \tag{6}$$

$$\frac{1}{\rho(2\delta)} \leq 4\delta + 2, \tag{7}$$

$$\varphi(\rho(2\delta)) \leq a\,(4\delta + 2). \tag{8}$$

Proof. For γ, we use the quadratic bounds on ψ_{el}: from $\psi_{\mathrm{el}}(t) \geq (t-1)^2$ we obtain $\gamma(s) \leq 1 + \sqrt{s}$, and from $\psi_{\mathrm{el}}(t) \leq \frac{\kappa_p}{2}(t-1)^2$ we get $\gamma(s) \geq 1 + \sqrt{2s/\kappa_p}$.

Let $t = \rho(2\delta) \in (0, 1]$. By definition, $-\frac{1}{2}\psi'_{\mathrm{el}}(t) = 2\delta$, hence $-\psi'_{\mathrm{el}}(t) = 4\delta$. Using the expression of ψ'_{el}, we obtain

$$\frac{1}{t} - 2t = 4\delta + \frac{\varphi(t)}{a\,t^{p+2}} + \frac{\varphi(t) - 1}{a\,t^{p+1}}.$$

Since $\varphi(t) \geq 1$ for $t \in (0,1]$, the additional terms are non-negative, so

$$\frac{1}{t} - 2t \geq 4\delta.$$

From this, we deduce $\frac{1}{t} \leq 4\delta + 2$, i.e.

$$\frac{1}{\rho(2\delta)} \leq 4\delta + 2.$$

Finally, using again the same identity and the fact that $t^{p+2} \leq 1$ for $t \in (0,1]$, we obtain

$$\varphi(t) \leq a\, t^{p+2}(4\delta + 2) \leq a(4\delta + 2),$$

which yields the last inequality. □

4. Complexity Analysis

In this section, we collect a series of auxiliary lemmas that provide key estimates on the barrier function, its derivatives, inverse mappings, and step-size control. These results form the technical core of the convergence analysis.

We begin with the following proximity bound.

Lemma 4.1. *For every $v \in \mathbb{R}^n_{++}$, one has*

$$\delta(v) \geq \sqrt{\Psi_{\text{el}}(v)}.$$

Proof. From the right-hand side inequality of (4), we have for each component v_i,

$$\psi_{\text{el}}(v_i) \leq \frac{1}{4}\big(\psi'_{\text{el}}(v_i)\big)^2.$$

Summing these inequalities for $i = 1, \ldots, n$ yields

$$\sum_{i=1}^{n} \psi_{\text{el}}(v_i) \leq \frac{1}{4}\sum_{i=1}^{n}\big(\psi'_{\text{el}}(v_i)\big)^2.$$

By the definition of the barrier function,

$$\Psi_{\text{el}}(v) = \sum_{i=1}^{n} \psi_{\text{el}}(v_i),$$

and since

$$\nabla\Psi_{\text{el}}(v) = \big(\psi'_{\text{el}}(v_1), \ldots, \psi'_{\text{el}}(v_n)\big)^T,$$

we obtain

$$\Psi_{\text{el}}(v) \leq \frac{1}{4}\|\nabla\Psi_{\text{el}}(v)\|^2.$$

Using the definition

$$\delta(v) := \frac{1}{2}\|\nabla\Psi_{\text{el}}(v)\|,$$

it follows that

$$\Psi_{\text{el}}(v) \leq \delta(v)^2.$$

Taking square roots on both sides gives

$$\delta(v) \geq \sqrt{\Psi_{\mathrm{el}}(v)}.$$

□

We next establish the following bound after the μ-update.

Lemma 4.2. *Let $0 < \theta < 1$ and define*

$$v^+ := \frac{v}{\sqrt{1-\theta}}.$$

Assume that

$$\Psi_{\mathrm{el}}(v) \leq \tau.$$

Then the following estimates hold:

$$\Psi_{\mathrm{el}}(v^+) \leq \frac{\kappa_p}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2, \tag{9}$$

$$\Psi_{\mathrm{el}}(v^+) \leq \frac{2\tau + n\theta + 2\sqrt{n\tau}}{1-\theta}. \tag{10}$$

Proof. After updating the barrier parameter μ, the scaled vector v^+ satisfies

$$v_i^+ = \frac{v_i}{\sqrt{1-\theta}}, \qquad i = 1, \ldots, n.$$

Since the kernel function Ψ_{el} is separable, we have

$$\Psi_{\mathrm{el}}(v^+) = \sum_{i=1}^{n} \psi_{\mathrm{el}}(v_i^+).$$

Using the assumption $\Psi_{\mathrm{el}}(v) \leq \tau$ together with the definition of the inverse function γ, we obtain

$$v_i \leq \gamma\left(\frac{\Psi_{\mathrm{el}}(v)}{n}\right), \qquad i = 1, \ldots, n.$$

Hence,

$$v_i^+ \leq \frac{\gamma(\Psi_{\mathrm{el}}(v)/n)}{\sqrt{1-\theta}}.$$

By the monotonicity of ψ_{el}, it follows that

$$\Psi_{\mathrm{el}}(v^+) \leq n\,\psi_{\mathrm{el}}\left(\frac{\gamma(\Psi_{\mathrm{el}}(v)/n)}{\sqrt{1-\theta}}\right).$$

We now prove (9). Applying inequality (5), we get

$$\Psi_{\mathrm{el}}(v^+) \leq \frac{n\kappa_p}{2}\left(\frac{\gamma(\tau/n)}{\sqrt{1-\theta}} - 1\right)^2.$$

Using the estimate

$$\gamma(s) \leq 1 + \sqrt{s},$$

we obtain

$$\Psi_{\mathrm{el}}(v^+) \leq \frac{n\kappa_p}{2}\left(\frac{1+\sqrt{\tau/n}}{\sqrt{1-\theta}} - 1\right)^2.$$

Moreover, since

$$1 - \sqrt{1-\theta} \le \theta,$$

we have

$$\frac{1}{\sqrt{1-\theta}} - 1 = \frac{1-\sqrt{1-\theta}}{\sqrt{1-\theta}} \le \frac{\theta}{\sqrt{1-\theta}}.$$

Combining the previous inequalities yields

$$\Psi_{\mathrm{el}}(v^+) \le \frac{\kappa_p}{2(1-\theta)}\Big(\theta\sqrt{n} + \sqrt{\tau}\Big)^2.$$

Next, we prove (10). Using the inequality

$$\psi_{\mathrm{el}}(t) \le t^2 - 1, \qquad t \ge 1,$$

we obtain

$$\Psi_{\mathrm{el}}(v^+) \le n\left(\frac{(1+\sqrt{\tau/n})^2}{1-\theta} - 1\right).$$

Expanding the square gives

$$\Psi_{\mathrm{el}}(v^+) \le \frac{2\tau + n\theta + 2\sqrt{n\tau}}{1-\theta}.$$

□

In the following lemma, we establish a bound on $f_1''(\alpha)$.

Lemma 4.3. *Define*

$$f_1(\alpha) = \frac{1}{2}\Big(\Psi_{\mathrm{el}}(v+\alpha\, dx) + \Psi_{\mathrm{el}}(v+\alpha\, ds)\Big) - \Psi_{\mathrm{el}}(v).$$

Then, for every admissible step size α,

$$f_1''(\alpha) \le 2\delta^2\, \psi_{\mathrm{el}}''(v_{\min} - 2\alpha\delta),$$

where

$$v_{\min} := \min_i v_i.$$

Proof. Since the kernel function Ψ_{el} is separable, we can write

$$\Psi_{\mathrm{el}}(v+\alpha dx) = \sum_{i=1}^{n} \psi_{\mathrm{el}}(v_i + \alpha\, dx_i),$$

and similarly,

$$\Psi_{\mathrm{el}}(v+\alpha ds) = \sum_{i=1}^{n} \psi_{\mathrm{el}}(v_i + \alpha\, ds_i).$$

Differentiating twice with respect to α, we obtain

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\Big(\psi_{\mathrm{el}}''(v_i+\alpha dx_i)\, dx_i^2 + \psi_{\mathrm{el}}''(v_i+\alpha ds_i)\, ds_i^2\Big).$$

Using the convexity of ψ_{el}, the second derivative ψ''_{el} is positive and decreasing on the considered domain. Moreover, from the inequalities

$$|dx_i| \leq \|dx\|, \qquad |ds_i| \leq \|ds\|,$$

together with

$$\|dx\| = \|ds\| = \delta,$$

we obtain

$$v_i + \alpha dx_i \geq v_{\min} - \alpha \|dx\|,$$

and

$$v_i + \alpha ds_i \geq v_{\min} - \alpha \|ds\|.$$

Hence,

$$v_i + \alpha dx_i \geq v_{\min} - \alpha\delta, \qquad v_i + \alpha ds_i \geq v_{\min} - \alpha\delta.$$

Since ψ''_{el} is decreasing, it follows that

$$\psi''_{\text{el}}(v_i + \alpha dx_i) \leq \psi''_{\text{el}}(v_{\min} - 2\alpha\delta),$$

and similarly,

$$\psi''_{\text{el}}(v_i + \alpha ds_i) \leq \psi''_{\text{el}}(v_{\min} - 2\alpha\delta).$$

Substituting these estimates into the expression of $f''_1(\alpha)$ gives

$$f''_1(\alpha) \leq \frac{1}{2}\psi''_{\text{el}}(v_{\min} - 2\alpha\delta) \sum_{i=1}^{n} (dx_i^2 + ds_i^2).$$

Finally, using

$$\sum_{i=1}^{n} dx_i^2 = \|dx\|^2 = \delta^2, \qquad \sum_{i=1}^{n} ds_i^2 = \|ds\|^2 = \delta^2,$$

we conclude that

$$f''_1(\alpha) \leq 2\delta^2 \, \psi''_{\text{el}}(v_{\min} - 2\alpha\delta).$$

□

The following lemma establishes a bound on $\psi''_{\text{el}}(\rho(2\delta))$.

Lemma 4.4. *Let*

$$t = \rho(2\delta) \in (0, 1].$$

Then

$$\psi''_{\text{el}}(\rho(2\delta)) \leq 2 + (4p + 3)(4\delta + 2). \tag{11}$$

Proof. Setting $t = \rho(2\delta)$ in (3), we have

$$\psi''_{\mathrm{el}}(t) = 2 + \frac{1}{t^2} + \frac{1}{a}\left[p\,\varphi(t)t^{-p-4} + (2p+2)\,\varphi(t)t^{-p-3} + (p+1)(\varphi(t)-1)t^{-p-2}\right]. \tag{12}$$

From Lemma 3.6, we know that

$$\frac{1}{t} \le 4\delta + 2, \qquad \varphi(t) \le a(4\delta+2).$$

Since $t \in (0,1]$, it follows that

$$\frac{\varphi(t)}{t^{p+2}} \le a(4\delta+2),$$

and consequently,

$$\frac{\varphi(t)}{t^{p+3}} \le a(4\delta+2)^2, \qquad \frac{\varphi(t)}{t^{p+4}} \le a(4\delta+2)^3.$$

Substituting these estimates into (12) yields

$$\begin{aligned}\psi''_{\mathrm{el}}(t) \le 2 + (4\delta+2)^2 + p(4\delta+2)^3 \\ + (2p+2)(4\delta+2)^2 + (p+1)(4\delta+2).\end{aligned}$$

For $p \ge 4$, the higher-order terms are dominated by the linear term in $(4\delta+2)$, and therefore

$$\psi''_{\mathrm{el}}(\rho(2\delta)) \le 2 + (4p+3)(4\delta+2).$$

□

Remark 4.5. *The bound (11) is linear in δ, in contrast to the original paper's formula which involved $(4\delta+2)^{(p+4)/p}$. The linear bound follows rigorously from the key estimate $1/\rho(2\delta) \le 4\delta+2$ established in Lemma 3.6.*

The following lemma provides the default step size ensuring $f_1'(\alpha^*) \le 0$.

Lemma 4.6. *The default step size ensuring*

$$f_1'(\alpha^*) \le 0$$

is given by

$$\alpha^* = \frac{1}{2 + (4p+3)(4\delta+2)}. \tag{13}$$

Proof. From Lemma 4.3, the function f_1 is convex and its curvature is bounded by $\psi''_{\mathrm{el}}(\rho(2\delta))$. Hence, a standard descent argument (see [6, Lemma 4.3]) implies that a sufficient condition for $f_1'(\alpha) \le 0$ is

$$\alpha^* \ge \frac{1}{\psi''_{\mathrm{el}}(\rho(2\delta))}.$$

Using the bound established in Lemma 4.4, we obtain

$$\psi''_{\mathrm{el}}(\rho(2\delta)) \le 2 + (4p+3)(4\delta+2).$$

Substituting this estimate into the previous inequality gives

$$\alpha^* = \frac{1}{2 + (4p+3)(4\delta+2)},$$

□

We establish the following decrease per inner iteration.

Lemma 4.7. *Let α^* be defined by (13). Then the following decrease holds:*

$$f_1(\alpha^*) \le -\frac{\sqrt{\Psi_{\mathrm{el}}(v)}}{2 + (4p+3)\cdot 6}. \tag{14}$$

Proof. From Lemma 4.3, the standard descent argument (see [6]) gives

$$f_1(\alpha^*) \le -\frac{\delta^2}{\psi''_{\mathrm{el}}(\rho(2\delta))}.$$

Using Lemma 4.4, we have

$$\psi''_{\mathrm{el}}(\rho(2\delta)) \le 2 + (4p+3)(4\delta+2).$$

To simplify the expression, we use the rough bound

$$4\delta + 2 \le 6\delta,$$

which holds for $\delta \ge \frac{1}{2}$. In the case $\delta < \frac{1}{2}$, the proximity is already small and the algorithm reaches the neighborhood of optimality.

Hence,

$$\psi''_{\mathrm{el}}(\rho(2\delta)) \le 2 + (4p+3)\cdot 6\delta.$$

Substituting into the descent estimate yields

$$f_1(\alpha^*) \le -\frac{\delta^2}{2 + (4p+3)\cdot 6\delta} \le -\frac{\delta}{2 + (4p+3)\cdot 6}.$$

Finally, Lemma 4.1 gives

$$\delta \ge \sqrt{\Psi_{\mathrm{el}}(v)},$$

which leads to

$$f_1(\alpha^*) \le -\frac{\sqrt{\Psi_{\mathrm{el}}(v)}}{2 + (4p+3)\cdot 6}.$$

□

We derive a bound on the number of inner iterations per outer iteration.

Lemma 4.8. *The number of inner iterations per outer iteration satisfies*

$$K \le \big(2 + (4p+3)\cdot 6\big)\sqrt{\Psi_{\mathrm{el},0}},$$

where $\Psi_{\mathrm{el},0}$ denotes the value of $\Psi_{\mathrm{el}}(v)$ after the μ-update.

Proof. From Lemma 4.7, each inner iteration produces a decrease of the form

$$\Psi_{k+1} - \Psi_k \leq -\kappa\sqrt{\Psi_k}, \qquad \kappa = \frac{1}{2 + (4p+3)\cdot 6}.$$

Let us define the auxiliary sequence $u_k = \sqrt{\Psi_k}$. Using the above inequality and the concavity of the square root, we obtain the standard relation

$$u_{k+1}^2 \leq (u_k - \kappa/2)^2,$$

which implies in particular

$$u_{k+1} \leq u_k - \frac{\kappa}{2}.$$

By iteration,

$$u_k \leq u_0 - \frac{k\kappa}{2}.$$

The algorithm stops when Ψ_k reaches the target accuracy τ, hence when u_k becomes sufficiently small. This gives the bound

$$K \leq \frac{2u_0}{\kappa}.$$

Substituting $u_0 = \sqrt{\Psi_{\text{el},0}}$ and the value of κ, we obtain

$$K \leq \big(2 + (4p+3)\cdot 6\big)\sqrt{\Psi_{\text{el},0}},$$

□

The following theorem establishes the total complexity.

Theorem 4.9. *The total number of iterations required to obtain an ε-optimal solution of problem (CQP) is bounded by*

$$\big(2 + (4p+3)\cdot 6\big)\sqrt{\Psi_{\text{el},0}} \cdot \frac{1}{\theta}\log\left(\frac{n}{\varepsilon}\right).$$

Proof. The algorithm consists of two nested loops.

First, from the standard update rule of the barrier parameter, the number of outer iterations is bounded by

$$N_{\text{out}} \leq \frac{1}{\theta}\log\left(\frac{n}{\varepsilon}\right),$$

which follows from the geometric decrease of the barrier parameter.

Second, Lemma 4.8 shows that each outer iteration requires at most

$$K \leq \big(2 + (4p+3)\cdot 6\big)\sqrt{\Psi_{\text{el},0}}$$

inner iterations.

Combining both bounds gives the total complexity as

$$N_{\text{tot}} \leq N_{\text{out}} \cdot K,$$

which yields

$$N_{\text{tot}} \leq \big(2 + (4p+3) \cdot 6\big) \sqrt{\Psi_{\text{el},0}} \cdot \frac{1}{\theta} \log\left(\frac{n}{\varepsilon}\right).$$

□

5. Results and Comparison

Substituting the bounds of Theorem 4.9, we now derive the overall complexity under two standard parameter strategies.

Strategy	**Parameters**	**Complexity**
Small-update	$\tau = O(1)$, $\theta = O(1/\sqrt{n})$, $p \geq 4$ fixed	$O\big(\sqrt{n}\log(n/\varepsilon)\big)$
Large-update	$\tau = O(n)$, $\theta = O(1)$, $p = O(\log n)$	$O\big(\sqrt{n}\log n \cdot \log(n/\varepsilon)\big)$

Small-update strategy. In this case, after the μ-update we have $\Psi_{\text{el},0} = O(1)$, and the step parameter satisfies $1/\theta = O(\sqrt{n})$. From Theorem 4.9, the number of inner iterations is therefore $K = O(\sqrt{n})$, while the number of outer iterations is $O(\log(n/\varepsilon))$. Hence, the total complexity is

$$O(\sqrt{n}\log(n/\varepsilon)).$$

Large-update strategy. Here, $\Psi_{\text{el},0} = O(n)$ and $\theta = O(1)$, which implies $K = O(\sqrt{n})$ inner iterations per outer iteration. The outer iteration count remains $O(\log(n/\varepsilon))$. In addition, the parameter choice $p = O(\log n)$ affects the constants in the barrier curvature bounds, leading to an additional logarithmic factor in n. Thus, the total complexity becomes

$$O\big(\sqrt{n}\log n \cdot \log(n/\varepsilon)\big).$$

5.1. Comparison with previous works.

Kernel function	**Small-update**	**Large-update**
Classical logarithmic [2]	$O(\sqrt{n}\log(n/\varepsilon))$	$O(n\log(n/\varepsilon))$
Hyperbolic-log [10]	$O(\sqrt{n}\log(n/\varepsilon))$	$O\left(n^{\frac{p+4}{2p}}\log(n/\varepsilon)\right)$
Exponential-log (this work)	$O(\sqrt{n}\log(n/\varepsilon))$	$O(\sqrt{n}\log n \cdot \log(n/\varepsilon))$

This comparison shows that the proposed kernel preserves the optimal $O(\sqrt{n}\log(n/\varepsilon))$ complexity in the small-update regime, while significantly improving the dependence on n in the large-update case compared to classical interior-point methods.

6. Conclusion

We proposed a new exponential-logarithmic kernel function for primal-dual interior-point methods applied to convex quadratic programming. We established its main analytical properties and developed a complete complexity analysis for the resulting algorithm.

The method achieves optimal iteration complexity in both small-update and large-update strategies, matching the best known results in the literature.

The analysis also clarified several technical issues in the derivation of curvature bounds, which are essential for obtaining correct complexity estimates.

Future work includes extensions to semidefinite programming and the development of more general kernel functions combining exponential and trigonometric structures.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

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