

## SUMMATION FORMULAS ARISING FROM THE HADAMARD PRODUCT OF HORADAM NUMBERS OF SECOND AND THIRD ORDER

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**ABSTRACT.** In this study, a recurrence relation of the sequence generated by the Hadamard product of the generating functions of second and third-order Horadam numbers is constructed. Furthermore, we show that the finite sum of the generating function of the mentioned sequence leads to various summation formulas.

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### 1. INTRODUCTION

Horadam numbers  $\{W_n(a, b; p, q)\}$ , or shortly  $\{W_n\}$ , are defined by the recurrence relation

$$W_0 = a, \quad W_1 = b, \quad W_n = pW_{n-1} + qW_{n-2} \quad \text{for } n \geq 2, \quad (1)$$

where  $a, b, p, q \in \mathbb{Z}$ . Equation (1) is the general form of linear recurrence relations of second order with integer coefficients and integer initial conditions. Horadam numbers are reputed to be a generalization of Fibonacci numbers  $\{F_n\} = \{W_n(0, 1; 1, 1)\}$ , Lucas numbers  $\{L_n\} = \{W_n(2, 1; 1, 1)\}$ , Pell numbers  $\{P_n\} = \{W_n(0, 1; 2, 1)\}$ , Jacobsthal numbers  $\{J_n\} = \{W_n(0, 1; 1, 2)\}$ , and many other well-known sequences.

Fundamental properties of Horadam Numbers were first studied by Horadam [1–3], hence the numbers are named after him. There are various ways of discovering new identities for these numbers, some of which can be found in [4–7]. Articles involving element-wise multiplication between two Horadam numbers or between Horadam numbers and other sequences include but are not limited to [8–10].

The recurrence relation (1) can be extended to higher order. Third-order Horadam numbers  $\{H_n\} = \{H_n(c, d, e; r, s, t)\}$  are similarly defined as

$$H_0 = c, \quad H_1 = d, \quad H_2 = e, \quad H_n = rH_{n-1} + sH_{n-2} + tH_{n-3} \quad \text{for } n \geq 3, \quad (2)$$

where  $c, d, e, r, s, t \in \mathbb{Z}$ . Some special case of third order Horadam numbers are tribonacci numbers  $\{T_n\} = \{H_n(0, 0, 1; 1, 1, 1)\}$ , Padovan numbers  $\{\mathcal{P}_n\} = \{H_n(1, 1, 1; 0, 1, 1)\}$ , Perrin numbers  $\{Q_n\} = \{H_n(3, 0, 2; 0, 1, 1)\}$ , etc. Basic properties and some identities of third order Horadam numbers were investigated in [11–14].

This article focuses on the Hadamard product of generating functions of second- and third-order Horadam numbers. We use the connection between eigenvalues of the companion matrix and roots of the characteristic equation to construct the recurrence relation of the coefficients of the Hadamard product. Considered as a generating function, this product can be rearranged in closed form, leading to a wealth of summation formulas.

## 2. MATERIALS AND METHODS

Consider Horadam numbers satisfying (1). We can write the current vector  $[W_n \ W_{n+1}]^T$  as a linear transformation of the previous one, i.e.,

$$\begin{bmatrix} W_n \\ W_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q & p \end{bmatrix} \begin{bmatrix} W_{n-1} \\ W_n \end{bmatrix}. \quad (3)$$

In this context, we call the matrix

$$W = \begin{bmatrix} 0 & 1 \\ q & p \end{bmatrix} \quad (4)$$

as the companion matrix of Horadam numbers. We can iterate the transformation to trace back to the initial values:

$$\begin{bmatrix} W_n \\ W_{n+1} \end{bmatrix} = W^n \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = W^n \begin{bmatrix} a \\ b \end{bmatrix}. \quad (5)$$

Similarly, the companion matrix of third-order Horadam number (2) is simply constructed as

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & s & r \end{bmatrix}. \quad (6)$$

In general, the characteristic polynomial corresponding to a linear homogeneous recurrence relation can be computed by  $\det(A - xI)$ , where  $A$  is the companion matrix of the relation and  $I$  is the identity matrix of the same size. Hence, the roots of the characteristic polynomial are, in fact, the eigenvalues of the companion matrix. If the roots are all distinct, they form the basis of the solutions of the recurrence relation.

Let  $C_W(x)$  and  $C_H(x)$  be the characteristic polynomials of Horadam numbers and third-order Horadam numbers, respectively. Then

$$C_W(x) = x^2 - px - q \quad (7)$$

and

$$C_H(x) = x^3 - rx^2 - sx - t. \quad (8)$$

The generating functions of Horadam numbers and third-order Horadam numbers denoted by  $G_W(x)$  and  $G_H(x)$  are defined as follows:

$$G_W(x) = \sum_{j=0}^{\infty} W_j x^j, \quad G_H(x) = \sum_{j=0}^{\infty} H_j x^j. \quad (9)$$

By a traditional approach, we easily obtain the closed forms of the power series as

$$G_W(x) = \frac{a + (b - pa)x}{1 - px - qx^2} \quad (10)$$

and

$$G_H(x) = \frac{c + (d - rc)x + (e - rd - sc)x^2}{1 - rx - sx^2 - tx^3}, \quad (11)$$

for any  $x$  in the intervals of convergence. Note here that the radius of convergence is the reciprocal of the absolute maximum root of the characteristic equation in each case. Now the partial sum of each generating function can be calculated by the difference of the generating function and its remainder. The whole procedure leads to the results expressed in the following theorems.

**Theorem 1.** For  $n \in \mathbb{N}_0$ , the sum of the first  $n + 1$  terms of  $G_W(x)$  is

$$\sum_{j=0}^n W_j x^j = \frac{a + (b - pa)x - W_{n+1}x^{n+1} - qW_n x^{n+2}}{1 - px - qx^2}. \quad (12)$$

**Theorem 2.** For  $n \in \mathbb{N}_0$ , the sum of the first  $n + 1$  terms of  $G_H(x)$  is

$$\sum_{j=0}^n H_j x^j = \frac{\Delta_1(n, x)}{1 - rx - sx^2 - tx^3}, \quad (13)$$

where

$$\begin{aligned} \Delta_1(n, x) &= c + (d - rc)x + (e - rd - sc)x^2 \\ &\quad - H_{n+1}x^{n+1} - (H_{n+2} - rH_{n+1})x^{n+2} - tH_n x^{n+3}. \end{aligned}$$

Since the sums are finite, the formulas (12) and (13) are true for any complex number  $x$ , as long as the denominators are not zero, without having to worry about convergence. The following Corollaries are then immediate consequences.

**Corollary 1.** For  $n \in \mathbb{N}_0$ , the following formulas hold:

(1) For  $p + q \neq 1$ ,

$$\sum_{j=0}^n W_j = \frac{a + (b - pa) - W_{n+1} - qW_n}{1 - p - q}. \quad (14)$$

(2) For  $p - q \neq -1$ ,

$$\sum_{j=0}^n (-1)^j W_j = \frac{a + (pa - b) + (-1)^n W_{n+1} + (-1)^{n+1} qW_n}{1 + p - q}. \quad (15)$$

(3) For  $p + q \neq 1$ ,

$$\sum_{j=0}^n jW_j = \frac{\Delta_2(n)}{(1 - p - q)^2}, \quad (16)$$

where

$$\begin{aligned} \Delta_2(n) &= (1 - p - q)[(b - pa) - (n + 1)W_{n+1} - (n + 2)qW_n] \\ &\quad - [a + (b - pa) - W_{n+1} - qW_n](-p - 2q) \end{aligned}$$

*Proof.* Substituting  $x = 1$  and  $x = -1$  in (12) simply results in formulas 1 and 2. Differentiate (12) and evaluate at  $x = 1$  to obtain formula 3 for  $j = 1$  to  $n$  which is also true for  $j = 0$ .  $\square$

In Corollary 1, formulas (14), (15), and (16) coincide with the equations (3.5) and (3.7) in [1] and Theorem 2.1 (a) in [5], respectively, but in different approaches.

**Corollary 2.** For  $n \in \mathbb{N}_0$ , the following formulas hold:

(1) For  $p + q \neq 1$  and  $p - q \neq -1$ ,

$$\sum_{j=0}^n W_{2j} = \frac{(1 - q)a + p(b - pa) - pW_{2n+1} - q(1 - q)W_{2n}}{(1 - q)^2 - p^2}. \quad (17)$$

(2) For  $p + q \neq 1, p - q \neq -1$  and  $n \geq 1$ ,

$$\sum_{j=1}^n W_{2j-1} = \frac{pa + (1 - q)(b - pa) - (1 - q)W_{2n+1} - pqW_{2n}}{(1 - q)^2 - p^2}. \quad (18)$$

(3) For  $p \neq 0$  and  $q \neq -1$ ,

$$\sum_{j=0}^n (-1)^j W_{2j} = \frac{(1 + q)a - p(b - pa) + p(-1)^n W_{2n+1} + q(1 + q)(-1)^n W_{2n}}{(1 + q)^2 + p^2}. \quad (19)$$

(4) For  $p \neq 0, q \neq -1$  and  $n \geq 1$ ,

$$\sum_{j=1}^n (-1)^j W_{2j-1} = \frac{-pa - (1 + q)(b - pa) + (1 + q)(-1)^n W_{2n+1} - pq(-1)^n W_{2n}}{(1 + q)^2 + p^2}. \quad (20)$$

*Proof.* From formulas (14) and (15) of Corollary 1, perform the summations of the first  $2n + 1$  terms, and then split into sums of even and odd terms:

$$\sum_{j=0}^n W_{2j} + \sum_{j=1}^n W_{2j-1} = \frac{a + (b - pa) - W_{2n+1} - qW_{2n}}{1 - p - q}, \quad (21)$$

$$\sum_{j=0}^n W_{2j} - \sum_{j=1}^n W_{2j-1} = \frac{a - (b - pa) + W_{2n+1} - qW_{2n}}{1 + p - q}. \quad (22)$$

The addition and subtraction of (21) and (22) conclude (17) and (18). For the later 2 formulas, we return to Theorem 1. Perform the summation of the first  $2n + 1$  terms of (1) and evaluate at  $x = i$ :

$$\sum_{j=0}^{2n} W_j = \frac{a + (b - pa)i - (-1)^n W_{2n+1}i + q(-1)^n W_{2n}}{(1 + q) - pi}. \quad (23)$$

Split the left hand side into sums of even and odd terms, and then equate the real part and imaginary part to obtain (19) and (20).  $\square$

It is worth noting that all formulas in Corollary 1 and 2 can be derived almost directly from Theorem 1. The proofs require neither a repetition of the relation nor a mathematical induction argument. Using the same approach, we also prove the following consequences of Theorem 2.

**Corollary 3.** For  $n \in \mathbb{N}_0$ , the following formulas hold:

(1) For  $r + s + t \neq 1$ ,

$$\sum_{j=0}^n H_j = \frac{c + (d - rc) + (e - rd - sc) - H_{n+2} + (r - 1)H_{n+1} - tH_n}{1 - r - s - t} \quad (24)$$

(2) For  $r - s + t \neq -1$ ,

$$\sum_{j=0}^n (-1)^j H_j = \frac{c - (d - rc) + (e - rd - sc) + (-1)^n [-H_{n+2} + (r + 1)H_{n+1} + tH_n]}{1 + r - s + t} \quad (25)$$

(3) For  $r + s + t \neq 1$ ,

$$\sum_{j=0}^n jH_j = \frac{\Delta_3(n)}{(1 - r - s - t)^2} \quad (26)$$

where

$$\begin{aligned} \Delta_3(n) &= (1 - r - s - t) \frac{\partial \Delta_1(n, 1)}{\partial x} - \Delta_1(n, 1)(r + 2s + 3t), \\ \frac{\partial \Delta_1(n, 1)}{\partial x} &= (d - rc) + 2(e - rd - sc) - [(n + 1) - r(n + 2)]H_{n+1} \\ &\quad - (n + 2)H_{n+2} - (n + 3)tH_n, \\ \Delta_1(n, 1) &= c + (d - rc) + (e - rd - sc) - H_{n+2} + (r - 1)H_{n+1} - tH_n. \end{aligned}$$

**Corollary 4.** For  $n \in \mathbb{N}_0$ , the following formulas hold:

(1) For  $r + s + t \neq 1$  and  $r - s + t \neq -1$ ,

$$\sum_{j=0}^n H_{2j} = \frac{\Delta_4(n)}{(1 - s)^2 - (r + t)^2}, \quad (27)$$

where

$$\begin{aligned}\Delta_4(n) &= (r+t)(d-rc) + (1-s)(c+e-rd-sc) \\ &\quad - (1-s)H_{2n+2} - (rs+t)H_{2n+1} - t(r+t)H_{2n}.\end{aligned}$$

(2) For  $r+s+t \neq 1, r-s+t \neq -1$  and  $n \geq 1$ ,

$$\sum_{j=1}^n H_{2j-1} = \frac{\Delta_5(n)}{(1-s)^2 - (r+t)^2}, \quad (28)$$

where

$$\begin{aligned}\Delta_5(n) &= (1-s)(d-rc) + (r+t)(c+e-rd-sc) \\ &\quad - (r+t)H_{2n+2} + (r^2+rt+s-1)H_{2n+1} - t(1-s)H_{2n}.\end{aligned}$$

(3) For  $r+s+t \neq 1$  and  $r-s+t \neq -1$ ,

$$\sum_{j=0}^n (-1)^j H_{2j} = \frac{\Delta_6(n)}{(1+s)^2 + (r-t)^2}, \quad (29)$$

where

$$\begin{aligned}\Delta_6(n) &= -(r-t)(d-rc) + (1+s)(c-e+rd+sc) \\ &\quad + (-1)^n[(1+s)H_{2n+2} - (rs+t)H_{2n+1} - t(r-t)H_{2n}].\end{aligned}$$

(4) For  $r+s+t \neq 1, r-s+t \neq -1$  and  $n \geq 1$ ,

$$\sum_{j=1}^n (-1)^j H_{2j-1} = \frac{\Delta_7(n)}{(1+s)^2 + (r-t)^2}, \quad (30)$$

where

$$\begin{aligned}\Delta_7(n) &= (1+s)(d-rc) + (r-t)(c-e+rd+sc) \\ &\quad + (-1)^n[(r-t)H_{2n+2} - (r^2-rt+s+1)H_{2n+1} + t(1+s)H_{2n}].\end{aligned}$$

### 3. RESULTS

**3.1. Hadamard Product of  $G_W(x)$  and  $G_H(x)$ .** Let  $F(x) := \sum_{i=0}^{\infty} f_i x^i$  and  $G(x) := \sum_{i=0}^{\infty} g_i x^i$  be two power series with integer coefficients. Hadamard product of  $F(x)$  and  $G(x)$  is defined by

$$F(x) * G(x) = \sum_{i=0}^{\infty} f_i g_i x^i \quad (31)$$

Some applications of the Hadamard product in combinatorial aspects can be observed in [15–17].

Consider Hadamard product of the generating functions  $G_W(x)$  and  $G_H(x)$ :

$$G_W(x) * G_H(x) = \sum_{i=0}^{\infty} W_i H_i x^i. \quad (32)$$

At this point, we set  $\{\mathcal{H}_n\} = \{W_n H_n\}$  as a sequence of element-wise multiplication of second and third-order Horadam numbers. Obviously, (32) is the generating function of the sequence. In this section, we use the same approach as in [17] to construct a recurrence relation for this sequence.

We first calculate the Kronecker product of the companion matrix of second and third-order Horadam numbers:

$$\mathcal{H} = W \otimes H = \begin{bmatrix} 0 & 1 \\ q & p \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & s & r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & t & s & r \\ 0 & q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & 0 & p \\ qt & qs & qr & pt & ps & pr \end{bmatrix}. \quad (33)$$

Let  $\lambda_1, \lambda_2$  be the eigenvalues of  $W$  and  $\mu_1, \mu_2, \mu_3$  be the eigenvalues of  $H$ . Since any eigenvalue of  $W \otimes H$  is a product of eigenvalues of  $W$  and  $H$  [18], we immediately obtain a monic polynomial such that  $\lambda_1\mu_1, \lambda_1\mu_2, \lambda_1\mu_3, \lambda_2\mu_1, \lambda_2\mu_2, \lambda_2\mu_3$  are its roots. Let

$$\begin{aligned} C_{\mathcal{H}}(x) &= \det(W \otimes H - xI) \\ &= x^6 - prx^5 - (p^2s + qr^2 + 2qs)x^4 - (3pqt + pqr s + p^3t)x^3 \end{aligned} \quad (34)$$

$$- (2q^2rt + p^2qrt - q^2s^2)x^2 - (-pq^2st)x - q^3t^2 \quad (35)$$

This characteristic polynomial corresponds to our desired recurrence relation which needed to be confirmed in the next theorem.

**Theorem 3.** *The sequence  $\{\mathcal{H}_n\} = \{W_n H_n\}$  satisfies the following recurrence relation for all natural number  $n \geq 6$ :*

$$\mathcal{H}_n = \alpha_1 \mathcal{H}_{n-1} + \alpha_2 \mathcal{H}_{n-2} + \alpha_3 \mathcal{H}_{n-3} + \alpha_4 \mathcal{H}_{n-4} + \alpha_5 \mathcal{H}_{n-5} + \alpha_6 \mathcal{H}_{n-6}, \quad (36)$$

where

$$\begin{aligned} \alpha_1 &= pr, \\ \alpha_2 &= p^2s + qr^2 + 2qs, \\ \alpha_3 &= 3pqt + pqr s + p^3t, \\ \alpha_4 &= 2q^2rt + p^2qrt - q^2s^2, \\ \alpha_5 &= -pq^2st, \\ \alpha_6 &= q^3t^2, \end{aligned}$$

with the initial conditions  $\mathcal{H}_i = W_i H_i$  for  $i = 0, 1, 2, 3, 4, 5$ .

*Proof.* Let's start with  $\mathcal{H}_n = W_n H_n$ . Use the recurrence relations of  $W_n$  and  $H_n$  to expand the product repeatedly, and collect terms step by step from the newest to the oldest term:

$$\begin{aligned}
 \mathcal{H}_n &= (pW_{n-1} + qW_{n-2})(rH_{n-1} + sH_{n-2} + tH_{n-3}) \\
 &= prW_{n-1}H_{n-1} + psW_{n-1}H_{n-2} + ptW_{n-1}H_{n-3} \\
 &\quad + qrW_{n-2}H_{n-1} + qsW_{n-2}H_{n-2} + qtW_{n-2}H_{n-3} \\
 &= [\alpha_1\mathcal{H}_{n-1} + 2qsW_{n-2}H_{n-2}] + psW_{n-1}H_{n-2} + ptW_{n-1}H_{n-3} \\
 &\quad + qrW_{n-2}H_{n-1} - qsW_{n-2}H_{n-2} + qtW_{n-2}H_{n-3} \\
 &= [\alpha_1\mathcal{H}_{n-1} + \alpha_2\mathcal{H}_{n-2} + 3pqtW_{n-3}H_{n-3}] \\
 &\quad - qsW_{n-2}H_{n-2} + (p^2t + qrs + qt)W_{n-2}H_{n-3} + qrtW_{n-2}H_{n-4} \\
 &\quad + qrsW_{n-3}H_{n-2} - 2pqtW_{n-3}H_{n-3} \\
 &= [\alpha_1\mathcal{H}_{n-1} + \alpha_2\mathcal{H}_{n-2} + \alpha_3\mathcal{H}_{n-3} - q^2s^2W_{n-4}H_{n-4}] \\
 &\quad - pqtW_{n-3}H_{n-3} + pqrtW_{n-3}H_{n-4} \\
 &\quad + (p^2qt + q^2t)W_{n-4}H_{n-3} - q^2rtW_{n-4}H_{n-4} - q^2stW_{n-4}H_{n-5} \\
 &= [\alpha_1\mathcal{H}_{n-1} + \alpha_2\mathcal{H}_{n-2} + \alpha_3\mathcal{H}_{n-3} + \alpha_4\mathcal{H}_{n-4} + \alpha_5\mathcal{H}_{n-5}] \\
 &\quad + q^2t^2W_{n-4}H_{n-6} - pq^2t^2W_{n-5}H_{n-6} \\
 &= \alpha_1\mathcal{H}_{n-1} + \alpha_2\mathcal{H}_{n-2} + \alpha_3\mathcal{H}_{n-3} + \alpha_4\mathcal{H}_{n-4} + \alpha_5\mathcal{H}_{n-5} + \alpha_6\mathcal{H}_{n-6}.
 \end{aligned}$$

This completes the proof. □

The simplest special case of the theorem 3 is that of the Hadamard product of Fibonacci numbers ( $\{F_n\} = \{W_n(0, 1; 1, 1)\}$ ) and Tribonacci numbers ( $\{T_n\} = \{H_n(0, 0, 1; 1, 1, 1)\}$ ). The recurrence relation in this case becomes

$$a_n = a_{n-1} + 4a_{n-2} + 5a_{n-3} + 2a_{n-4} - a_{n-5} + a_{n-6},$$

which gives the sequence 0, 0, 1, 2, 6, 20, 56, 169, 504, ...

**3.2. Summation Formulas for  $\mathcal{H}_n$ .** Recall the generating function of the sequence of element-wise multiplication of second and third-order Horadam numbers, and rename as  $G_{\mathcal{H}}(x)$ . Let  $R_{\mathcal{H}}(n, x)$  be the remainder of order  $n$  for this infinite power series. We can write (32) as

$$G_{\mathcal{H}}(x) = \mathcal{H}_0 + \mathcal{H}_1x + \mathcal{H}_2x^2 + \cdots + \mathcal{H}_nx^n + R_{\mathcal{H}}(n, x). \quad (37)$$

Multiply (37) by  $-\alpha_1x, -\alpha_2x^2, -\alpha_3x^3, -\alpha_4x^4, -\alpha_5x^5, -\alpha_6x^6$ , and add them all together. For any  $x$  in the interval of convergence, only the six leftmost terms have nonzero sums. As a result, the closed form

of the power series is acquired:

$$\begin{aligned}
 & (1 - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 - \alpha_4 x^4 - \alpha_5 x^5 - \alpha_6 x^6) G_{\mathcal{H}}(x) \\
 &= \mathcal{H}_0 + (\mathcal{H}_1 - \alpha_1 \mathcal{H}_0)x + (\mathcal{H}_2 - \alpha_1 \mathcal{H}_1 - \alpha_2 \mathcal{H}_0)x^2 \\
 &+ (\mathcal{H}_3 - \alpha_1 \mathcal{H}_2 - \alpha_2 \mathcal{H}_1 - \alpha_3 \mathcal{H}_0)x^3 \\
 &+ (\mathcal{H}_4 - \alpha_1 \mathcal{H}_3 - \alpha_2 \mathcal{H}_2 - \alpha_3 \mathcal{H}_1 - \alpha_4 \mathcal{H}_0)x^4 \\
 &+ (\mathcal{H}_5 - \alpha_1 \mathcal{H}_4 - \alpha_2 \mathcal{H}_3 - \alpha_3 \mathcal{H}_2 - \alpha_4 \mathcal{H}_1 - \alpha_5 \mathcal{H}_0)x^5.
 \end{aligned} \tag{38}$$

Likewise, the remainder terms can be rearranged into an explicit form:

$$\begin{aligned}
 & (1 - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 - \alpha_4 x^4 - \alpha_5 x^5 - \alpha_6 x^6) R_{\mathcal{H}}(n, x) \\
 &= x^{n+1} [ \mathcal{H}_{n+1} + (\mathcal{H}_{n+2} - \alpha_1 \mathcal{H}_{n+1})x + (\mathcal{H}_{n+3} - \alpha_1 \mathcal{H}_{n+2} - \alpha_2 \mathcal{H}_{n+1})x^2 \\
 &+ (\mathcal{H}_{n+4} - \alpha_1 \mathcal{H}_{n+3} - \alpha_2 \mathcal{H}_{n+2} - \alpha_3 \mathcal{H}_{n+1})x^3 \\
 &+ (\mathcal{H}_{n+5} - \alpha_1 \mathcal{H}_{n+4} - \alpha_2 \mathcal{H}_{n+3} - \alpha_3 \mathcal{H}_{n+2} - \alpha_4 \mathcal{H}_{n+1})x^4 \\
 &+ (\mathcal{H}_{n+6} - \alpha_1 \mathcal{H}_{n+5} - \alpha_2 \mathcal{H}_{n+4} - \alpha_3 \mathcal{H}_{n+3} - \alpha_4 \mathcal{H}_{n+2} - \alpha_5 \mathcal{H}_{n+1})x^5 ] \\
 &= x^{n+1} [ \mathcal{H}_{n+1} + (\mathcal{H}_{n+2} - \alpha_1 \mathcal{H}_{n+1})x + (\mathcal{H}_{n+3} - \alpha_1 \mathcal{H}_{n+2} - \alpha_2 \mathcal{H}_{n+1})x^2 \\
 &+ (\mathcal{H}_{n+4} - \alpha_1 \mathcal{H}_{n+3} - \alpha_2 \mathcal{H}_{n+2} - \alpha_3 \mathcal{H}_{n+1})x^3 \\
 &+ (\mathcal{H}_{n+5} - \alpha_1 \mathcal{H}_{n+4} - \alpha_2 \mathcal{H}_{n+3} - \alpha_3 \mathcal{H}_{n+2} - \alpha_4 \mathcal{H}_{n+1})x^4 \\
 &+ \alpha_6 \mathcal{H}_n x^5 ]
 \end{aligned} \tag{39}$$

We then obtain the sum of the first  $n$  terms of our generating function by performing  $G_{\mathcal{H}}(x) - R_{\mathcal{H}}(n, x)$ .

Now we can substitute any complex number  $x$  into the partial sum to derive any number of formulas.

Here we show the results from setting  $x = 1$  and  $x = -1$ .

**Theorem 4.** For  $n \in \mathbb{N}_0$  and  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \neq 1$ , the following formula holds:

$$\begin{aligned}
 & (1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6) \sum_{j=0}^n \mathcal{H}_j \\
 &= (1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \mathcal{H}_0 - \alpha_6 \mathcal{H}_n \\
 &+ (1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) (\mathcal{H}_1 - \mathcal{H}_{n+1}) \\
 &+ (1 - \alpha_1 - \alpha_2 - \alpha_3) (\mathcal{H}_2 - \mathcal{H}_{n+2}) \\
 &+ (1 - \alpha_1 - \alpha_2) (\mathcal{H}_3 - \mathcal{H}_{n+3}) \\
 &+ (1 - \alpha_1) (\mathcal{H}_4 - \mathcal{H}_{n+4}) \\
 &+ (\mathcal{H}_5 - \mathcal{H}_{n+5}).
 \end{aligned} \tag{40}$$

*Proof.* The summation formula is derived by simply putting  $x = 1$  into (38) and (39) manipulating terms in the subtraction  $G_{\mathcal{H}}(1) - R_{\mathcal{H}}(n, 1)$ .  $\square$

**Corollary 5.** For Fibonacci numbers ( $\{F_n\} = \{W_n(0, 1; 1, 1)\}$ ) and tribonacci numbers ( $\{T_n\} = \{H_n(0, 0, 1; 1, 1, 1)\}$ ), the following formula holds for all  $n \in \mathbb{N}_0$ :

$$\sum_{j=0}^n F_j T_j = \frac{F_{n+5} T_{n+5} - 4F_{n+3} T_{n+3} - 9F_{n+2} T_{n+2} - 11F_{n+1} T_{n+1} + F_n T_n - 3}{11} \quad (41)$$

*Proof.* Use  $p = q = r = s = t = 1$  to find the values of all coefficients  $\alpha_i$  in Theorem 3:

$$\alpha_1 = 1, \alpha_2 = 4, \alpha_3 = 5, \alpha_4 = 2, \alpha_5 = -1, \alpha_6 = 1. \quad (42)$$

We calculate the initial terms as the followings:

$$\mathcal{H}_0 = 0, \mathcal{H}_1 = 0, \mathcal{H}_2 = 1, \mathcal{H}_3 = 2, \mathcal{H}_4 = 6, \mathcal{H}_5 = 20. \quad (43)$$

We substitute all of these values into (40) to complete the proof.  $\square$

**Theorem 5.** For  $n \in \mathbb{N}_0$  and  $\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6 \neq -1$ , the following formula holds:

$$\begin{aligned} & (1 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6) \sum_{j=0}^n (-1)^j \mathcal{H}_j \\ &= (1 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5) \mathcal{H}_0 - \alpha_6 (-1)^n \mathcal{H}_n \\ & \quad - (1 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) (\mathcal{H}_1 - (-1)^n \mathcal{H}_{n+1}) \\ & \quad + (1 + \alpha_1 - \alpha_2 + \alpha_3) (\mathcal{H}_2 - (-1)^n \mathcal{H}_{n+2}) \\ & \quad - (1 + \alpha_1 - \alpha_2) (\mathcal{H}_3 - (-1)^n \mathcal{H}_{n+3}) \\ & \quad + (1 + \alpha_1) (\mathcal{H}_4 - (-1)^n \mathcal{H}_{n+4}) \\ & \quad - (\mathcal{H}_5 - (-1)^n \mathcal{H}_{n+5}). \end{aligned} \quad (44)$$

*Proof.* Similar to the proof of Theorem 4 but put  $x = -1$  instead.  $\square$

**Corollary 6.** For Fibonacci numbers ( $\{F_n\} = \{W_n(0, 1; 1, 1)\}$ ) and tribonacci numbers ( $\{T_n\} = \{H_n(0, 0, 1; 1, 1, 1)\}$ ), the following formula holds for all  $n \in \mathbb{N}_0$ :

$$\begin{aligned} & \sum_{j=0}^n (-1)^j F_j T_j \\ &= (-1)^n [F_n T_n - F_{n+1} T_{n+1} + 3F_{n+2} T_{n+2} + 2F_{n+3} T_{n+3} + 2F_{n+4} T_{n+4} - F_{n+5} T_{n+5}] + 1 \end{aligned}$$

*Proof.* We substitute all values in (42) and (43) into (44) to complete the proof.  $\square$

## 4. DISCUSSION

During our efforts, we have found some beautiful pattern which can be applied to any other sequences from linear recurrence relations of finite order. In particular, Theorem 4 and Theorem 5 also work with other sequences of linear recurrence relations of order six with arbitrary coefficients  $\alpha_i$  that fulfill the assumption of the theorem. In general, we can derive nice formulas for the sequence of order  $k$ :

$$S_n = \alpha_1 S_{n-1} + \alpha_2 S_{n-2} + \alpha_3 S_{n-3} + \cdots + \alpha_k S_{n-k}, \quad (45)$$

in the same manner. So let us end with the formulas below without proof.

Define  $\alpha_0 = 0$ . For  $n \in \mathbb{N}_0$  and  $\sum_{i=1}^k \alpha_i \neq 1$ , the following formula holds:

$$\left(1 - \sum_{i=1}^k \alpha_i\right) \sum_{j=0}^n S_j = \left(1 - \sum_{i=1}^{k-1} \alpha_i\right) S_0 - \alpha_k S_n + \sum_{l=0}^{k-2} \left(1 - \sum_{i=0}^l \alpha_i\right) (S_{k-l-1} - S_{n+k-l-1}). \quad (46)$$

And for  $n \in \mathbb{N}_0$  and  $\sum_{i=1}^k (-1)^{i+1} \alpha_i \neq -1$ , the following formula holds:

$$\begin{aligned} \left(1 - \sum_{i=1}^k (-1)^i \alpha_i\right) \sum_{j=0}^n (-1)^j S_j & \\ &= \left(1 - \sum_{i=1}^{k-1} (-1)^i \alpha_i\right) S_0 - (-1)^{n+k} \alpha_k S_n \\ &+ \sum_{l=0}^{k-2} (-1)^{k-l-1} \left(1 - \sum_{i=0}^l (-1)^i \alpha_i\right) (S_{k-l-1} - (-1)^n S_{n+k-l-1}). \end{aligned} \quad (47)$$

## 5. CONCLUSIONS

This article dealt with the summation formulas of integer sequences from a class of linear recurrence relations of order six, which can be derived from the Hadamard product of Horadam numbers of order two and three. Our approach was primarily concerned with the calculation of generating functions in closed form and the extraction of their finite terms. The main results is comprehensive to the product of any sequences of our types including the square of a sequence. Further investigation may involve the formulas of finite power of a sequence or the formulas with subscripts in arithmetic progression.

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**Conflicts of Interest.** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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