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FIXED POINT THEOREMS FOR ENRICHED P-CYCLIC AND REICH-TYPE CYCLIC CONTRACTIONS WITH APPLICATIONS TO IFS

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Abstract. In this paper, we introduce two new classes of enriched cyclic contractions in the setting of Banach spaces: enriched P-cyclic contractions and enriched Reich-type cyclic contractions. The enriched P-cyclic condition employs the absolute difference of the displacement norms, which is a strictly weaker requirement than the sum used in earlier works. The enriched Reich-type cyclic contraction allows independent coefficients for $\|x - Tx\|$ and $\|y - Ty\|$, thereby generalizing both the classical Reich contraction and known enriched cyclic contractions. For each class, we prove the existence and uniqueness of a fixed point in the intersection of the cyclic subsets. Moreover, we show that the Krasnosel'skii iteration converges to this fixed point for any initial guess, and we establish a priori and a posteriori error estimates together with the rate of convergence. As an application, we consider a generalized iterated function system formed by enriched Reich-type cyclic contractions with $\beta = \gamma = 0$. We prove that the associated Hutchinson operator on the hyperspace of compact sets possesses a unique attractor, and that the iteration of this operator converges to the attractor in the Hausdorff metric. Numerical examples in one and two dimensions illustrate the theoretical results.

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1. Introduction

Let (X, d) be a metric space and consider a mapping $T : X \to X$. The notion of fixed points, where there exists $z \in X$ such that $Tz = z$, is central in the study of nonlinear mappings. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$. When $\mathrm{Fix}(T)$ reduces to a single point and the Picard sequence $\{T^n x_0\}$ converges to that point for any initial $x_0 \in X$, we call T a Picard operator.

A landmark result in fixed point theory is the Banach contraction principle [3], which asserts that every contraction self-mapping on a complete metric space is a Picard operator. Recall that T is a

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contraction if there exists $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \leq \alpha\, d(x,y) \qquad \forall x,y \in X.$$

Over the years, many generalizations of the Banach contraction principle have been obtained, including the Kannan [12], Chatterjea [10], Ćirić–Reich–Rus [11,17,18] and other contractive conditions. In a different direction, Popescu [16] introduced the notion of P-contraction: a self-mapping T on a metric space is called a P-contraction if there exists $\beta \in [0,1)$ such that

$$d(Tx,Ty) \leq \beta\big[d(x,y) + |d(x,Tx) - d(y,Ty)|\big] \qquad \forall x,y \in X.$$

Every contraction is a P-contraction, but the converse is false. Popescu showed that P-contractions on complete metric spaces are Picard operators.

Another recent direction, initiated by Berinde and Păcurar [4], is the study of enriched contractions in normed spaces. A mapping $T : C \to C$ on a convex subset C of a normed space $(X, \|\cdot\|)$ is called an enriched contraction if there exist $b \geq 0$ and $\theta \in [0, b+1)$ such that

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x-y\| \qquad \forall x,y \in C.$$

Setting $b = 0$ recovers a classical contraction. However, there exist enriched contractions that are not contractions (and not even P-contractions), as shown in [4]. For such mappings, the Picard iteration may not converge, but the Krasnosel'skii iteration

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad \lambda \in (0,1],$$

does converge to the unique fixed point. Enriched versions of Kannan, Chatterjea, Ćirić–Reich–Rus and other contractive inequalities have been studied in [5–8]. The notion of enriched mappings has been fruitfully extended in multiple directions, encompassing various enriched contractive and nonexpansive-type operators across Banach spaces, convex metric spaces, and other nonlinear settings [14,15,19,20].

Independently, Kirk et al [13] extended the Banach principle to mappings that need not be continuous by using a cyclic representation of the space. A finite family $\{B_i\}_{i=1}^k$ of nonempty subsets of X is a cyclic representation with respect to T if $X = \bigcup_{i=1}^k B_i$ and $T(B_i) \subseteq B_{i+1}$ for $i = 1, \ldots, k-1$, $T(B_k) \subseteq B_1$. Under a cyclic contractive condition, they proved the existence of a unique fixed point in the intersection $\bigcap_{i=1}^k B_i$.

The purpose of this paper is to merge the above ideas by introducing and studying *enriched cyclic contractions* of two new types: *enriched P-cyclic contractions* and *enriched Reich-type cyclic contractions*. Our work is strongly motivated by the recent paper of Abbas et al [1], where the authors initiated a class of generalized enriched cyclic contractions and applied them to iterated function systems. However, the contractive condition in [1] involves the sum $\|x - Tx\| + \|y - Ty\|$ (see their inequality (2.0.1)). In contrast, our enriched P-cyclic contraction uses the absolute difference $|\|x-Tx\| - \|y-Ty\||$, which is a strictly weaker condition (see [2] for more details). Moreover, we introduce an enriched Reich-type cyclic contraction where the coefficients on $\|x - Tx\|$ and $\|y - Ty\|$ are independent, thereby generalizing both the classical Reich contraction and the enriched cyclic contractions studied in the literature.

More precisely, let $\{B_i\}_{i=1}^k$ be nonempty closed convex subsets of a Banach space X and let $T : \bigcup_{i=1}^k B_i \to X$ be a mapping. We define:

- T is an **enriched P-cyclic contraction** if there exist $b \geq 0$ and $\theta \in [0, b+1)$ such that for all $x \in B_i$, $y \in B_{i+1}$,

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| + \frac{\theta}{b+1}\big|\|x-Tx\| - \|y-Ty\|\big|.$$

- T is an **enriched Reich-type cyclic contraction** if there exist $b \geq 0$ and constants $\alpha, \beta, \gamma \geq 0$ with $\frac{\alpha}{b+1} + \beta + \gamma < 1$ such that for all $x \in B_i$, $y \in B_{i+1}$,

$$\|b(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\|x-Tx\| + \gamma\|y-Ty\|.$$

For both classes we prove:

(1) Existence and uniqueness of a fixed point $u \in \bigcap_{i=1}^k B_i$;
(2) Convergence of the Krasnosel'skii iteration to u for any initial point;
(3) A priori and a posteriori error estimates;
(4) Rate of convergence estimates.

These results unify and extend several known theorems: the classical cyclic contraction theorem of Kirk et al. [13], the enriched contraction theorem of Berinde and Păcurar [4], the P-contraction theorem of Popescu [16], and the Reich-type cyclic contraction results of Rus [18]. We also provide illustrative examples, including a two-dimensional example where the enriched Reich-type cyclic contraction is verified and the Krasnosel'skii iterates are computed numerically. Furthermore, as an application, we study a generalized iterated function system built from enriched Reich-type cyclic contractions with $\beta = \gamma = 0$, and we prove the existence of a unique fractal attractor in the hyperspace of compact sets.

The paper is organized as follows. In Section 2 we introduce the enriched P-cyclic contraction, prove the main fixed point theorem, and discuss the convergence of the Krasnosel'skiĭ iteration. Section 3 is devoted to enriched Reich-type cyclic contractions; we show that this class is distinct from the P-type class and establish analogous fixed point results. Finally, Section 4 presents applications to iterated function systems and provides numerical illustrations.

2. Enriched P-cyclic contractions

Definition 2.1. [18] Let X be a nonempty set, k a positive integer, and T a self mapping on X. A finite collection $\{B_i \subseteq X : i = 1, 2, 3, \cdots, k\}$ is called its cyclic representation of X with respect to T if

1 . $X = \bigcup_{i=1}^{k} B_i$;
2 . $T(B_1) \subseteq B_2, \cdots T(B_{k-1}) \subseteq B_k$, and $T(B_k) \subseteq B_1$.

Definition 2.2. A mapping $T : \cup_{i=1}^{k} B_i \to X$ is called an enriched P-cyclic contraction if it satisfies the following conditions:

(i) $\{B_i : i = 1, 2, 3, \cdots, p\}$ is cyclic representation of $\cup_{i=1}^{k} B_i$ with respect to T_λ,
(2) there exists $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that for all $x \in B_i$ and $y \in B_{i+1}$ for $1 \leq i \leq k$,

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| + \frac{\theta}{b+1}|\|x-Tx\| - \|y-Ty\||. \tag{2.1}$$

Recall that, a mapping $T_\lambda : C \to C$ given by $T_\lambda(x) = (1-\lambda)x + \lambda Tx$ is called an averaged mapping, where $\lambda \in (0, 1]$ and T is a self mapping defined on a convex subset C of a linear space X. Note that $Fix(T_\lambda) = Fix(T)$.

Theorem 2.3. *Let $T : \cup_{i=1}^{k} B_i \to X$ be an enriched P-cyclic contraction. Then*

1. *Fix (T)=$\{u\}$ for some $u \in \cap_{i=1}^{k} B_i$;*
2. *There exists $\lambda \in (0, 1]$ such that an iterative method $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \; n \geq 0, \tag{2.2}$$

converges to u for any $x_0 \in \cup_{i=1}^{k} B_i$;

3. *The following priori and posteriori estimate hold;*

$$\|x_n - u\| \leq \frac{\epsilon^n}{1-\epsilon}\|x_0 - x_1\|, \; n \geq 0;$$

$$\|x_{n+1} - u\| \leq \frac{\epsilon}{1-\epsilon}\|x_n - x_{n+1}\|, \; n > 0.$$

4. *Moreover, the inequality*

$$\|x_n - u\| \leq \epsilon\|x_{n-1} - u\|, \; n = 1, 2, \cdots$$

determine the rate of convergence of the Krasnosel'skii iteration where $\epsilon = \dfrac{2\theta\lambda}{1+\theta\lambda}$.

Proof. We will consider two cases:

Let $b > 0$ and let $\lambda = \dfrac{1}{b+1}$. Obviously, $0 < \lambda < 1$ and the enriched P-cyclic contraction condition (2.1) becomes

$$\|\frac{1}{\lambda}(x-y) + Tx - Ty\| \leq \theta\|x-y\| + \lambda\theta|\|x-Tx\| - \|y-Ty\||$$

for all $x \in B_i$ and $y \in B_{i+1}$. The above condition can be written in an equivalent form as

$$\|T_\lambda x - T_\lambda y\| \leq \delta[\|x-y\| + |\|x - T_\lambda x\| - \|y - T_\lambda y\||] \tag{2.3}$$

where $\delta = \theta\lambda$. We note that $\delta \in [0,1)$ since $\theta \in [0, b+1)$. This implies that T_λ is a P-enriched mapping. Assume that $\|x_{n+1} - x_n\| > 0$ for all $n \in \mathbb{N}$. Otherwise, x_n will be a fixed point of T_λ and hence of T. Consider an arbitrary point $x_0 \in \cup_{i=1}^k B_i$ and define the sequence of Krasnosel'skii iterates $x_{n+1} = (1-\lambda)x_n + \lambda T x_n, n \geq 0$ which is equivalent to the Picard iterates $x_{n+1} = T_\lambda x_n$ for all $n = 0, 1, 2, \cdots$. Notice that, there exists $i \in \{1, 2, \cdots k\}$ such that $x_0 \in B_i$. As $\{B_i; i = 1, 2, \cdots k\}$ is a cyclic representation of $\cup_{i=1}^k B_i$ with T_λ, we have $x_1 = T_\lambda x_0 \in B_{i+1}$. Therefore,

$$\begin{aligned} \|x_2 - x_1\| &= \|T_\lambda x_1 - T_\lambda x_0\| \\ &\leq \delta[\|x_1 - x_0\| + |\|x_1 - T_\lambda x_1\| - \|x_0 - T_\lambda x_0\||] \\ &= \delta[\|x_1 - x_0\| + |\|x_1 - x_2\| - \|x_0 - x_1\||]. \end{aligned} \tag{2.4}$$

If $\|x_1 - x_2\| \geq \|x_0 - x_1\|$, then, by (2.4), we obtain $\|x_2 - x_1\| \leq \delta \|x_2 - x_1\|$, which is a contradiction. Therefore,

$$\|x_1 - x_2\| < \|x_0 - x_1\|. \tag{2.5}$$

Then, from (2.4), we have

$$\|x_2 - x_1\| \leq \delta[\|x_1 - x_0\| + \|x_0 - x_1\| - \|x_1 - x_2\|],$$

which implies,

$$\|x_2 - x_1\| \leq \frac{2\delta}{1+\delta}\|x_1 - x_0\|.$$

Since $\delta \in [0,1)$, we have $\frac{2\delta}{1+\delta} < 1$. Let $\epsilon = \frac{2\delta}{1+\delta}$; then $\epsilon \in [0,1)$.

By induction, we obtain

$$\|x_{n+1} - x_n\| \leq \epsilon^n \|x_1 - x_0\|. \tag{2.6}$$

Now, for any numbers $n, m \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (\epsilon^{n+m-1} + \epsilon^{n+m-2} + \cdots + \epsilon^n)\|x_1 - x_0\| \\ &= \epsilon^n(\epsilon^{m-1} + \epsilon^{m-2} + \cdots + 1)\|x_1 - x_0\| \\ &= \epsilon^n \frac{(1-\epsilon^m)}{1-\epsilon}\|x_1 - x_0\|. \end{aligned} \tag{2.7}$$

Taking the limit in (2.7) as $n \to \infty$, we get $\|x_{n+m} - x_n\| \to 0$, which shows that the sequence $\{x_n\}$ is a Cauchy sequence in $\cup_{i=1}^k B_i$. Since $\cup_{i=1}^k B_i$ is a complete subspace, $\{x_n\}$ converges to a point in $\cup_{i=1}^k B_i$. Let us denote

$$u = \lim_{n \to \infty} x_n. \tag{2.8}$$

Moreover, since $\{B_i : i = 1, 2, \cdots, k\}$ forms a cyclic representation of $\cup_{i=1}^{k} B_i$ with respect to T_λ, the sequence $\{x_n\}$ has infinitely many terms in each B_i for all $i \in \{1, 2, \cdots, k\}$. Consequently, $u \in \cap_{i=1}^{k} B_i$. We need to prove that u is the fixed point of T_λ. From (2.3), it follows that

$$\begin{aligned} \|x_{n+1} - T_\lambda u\| &= \|T_\lambda x_n - T_\lambda u\| \\ &\leq \delta[\|x_n - u\| + \|x_n - T_\lambda x_n\| - \|u - T_\lambda u\|] \\ &= \delta[\|x_n - u\| + \|x_n - x_{n+1}\| - \|u - T_\lambda u\|]. \end{aligned} \tag{2.9}$$

By letting $n \to \infty$ in (2.9), we obtain

$$\|u - T_\lambda u\| \leq \delta \|u - T_\lambda u\|.$$

Hence we get $\|u - T_\lambda u\| = 0$. ie; $u \in Fix(T_\lambda)$. To prove the uniqueness of u, let $p \in \cap_{i=1}^{k} B_i$ such that $T_\lambda p = p$.

$$\begin{aligned} 0 < \|u - p\| &= \|T_\lambda u - T_\lambda p\| \\ &\leq \delta[\|u - p\| + \|u - T_\lambda u\| - \|p - T_\lambda p\|] \\ &= \delta \|u - p\|, \end{aligned}$$

a contradiction. Therefore, $Fix(T_\lambda) = u = Fix(T)$ and hence (i) holds. Condition (ii) follows from (2.8).

On taking the limit as $m \to \infty$ in (2.7), we obtain a priori estimate given by

$$\|x_n - u\| \leq \frac{\epsilon^n}{1 - \epsilon} \|x_0 - x_1\|, \; n \geq 0.$$

To obtain the posteriori estimate, we consider $x = x_{n-1}$ and $y = x_n$ in (2.3), then

$$\|x_n - x_{n+1}\| \leq \delta[\|x_{n-1} - x_n\| + |\|x_{n-1} - x_n\| - \|x_n - x_{n+1}\||]. \tag{2.10}$$

Using the same arguement as in (2.5), we obtain $\|x_n - x_{n+1}\| < \|x_{n-1} - x_n\|$. It then follows from (2.10), that

$$\|x_n - x_{n+1}\| \leq \epsilon \|x_{n-1} - x_n\|.$$

By induction we therefore have

$$\|x_{n+m} - x_{n+m+1}\| \leq \epsilon^{m+1} \|x_{n-1} - x_n\|$$

which implies that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (\epsilon^m + \epsilon^{m-1} + \cdots + \epsilon) \|x_{n-1} - x_n\| \\ &= \epsilon(\epsilon^{m-1} + \epsilon^{m-2} + \cdots + 1) \|x_{n-1} - x_n\| \end{aligned}$$

$$= \quad \epsilon\frac{1-\epsilon^m}{1-\epsilon}\|x_{n-1}-x_n\|. \tag{2.11}$$

Letting $m \to \infty$, we obtain the posteriori estimate

$$\|x_{n+1}-u\| \leq \frac{\epsilon}{1-\epsilon}\|x_n - x_{n+1}\|,\ n > 0.$$

Consider $x \in B_i$ and $y \in B_{i+1}$ for $i \in \{1, 2, \cdots, k\}$. From (2.3), we have

$$\begin{aligned}\|T_\lambda x - T_\lambda y\| &\leq \delta[\|x-y\| + |\|x-y\| + \|y - T_\lambda y\| + \|T_\lambda y - T_\lambda x\| - \|y - T_\lambda y\||] \\ &= \delta[\|x-y\| + |\|x-y\| + \|T_\lambda y - T_\lambda x\||].\end{aligned}$$

This implies that

$$\|T_\lambda x - T_\lambda y\| \leq \epsilon\|x-y\|.$$

By taking $x = x_{n-1}$ and $y = u$, we obtain the rate of convergence of the Krasnosel'skii iteration. If $b = 0$, then $\lambda = 1$ and $\delta = \theta$. In this case, the Krasnosel'skii iteration reduces to the Picard iteration associated with T. □

3. Enriched Reich-type cyclic contractions

The enriched P-cyclic contraction (Definition 2.2) employs a single parameter θ. In this section, we introduce a complementary class of enriched cyclic contractions based on the classical Reich contraction [17], where the coefficients on $\|x-Tx\|$ and $\|y-Ty\|$ are independent and decoupled from the coefficient on $\|x-y\|$. This yields a genuinely distinct class of mappings: the enriched P-cyclic contraction class and the enriched Reich-type cyclic contraction class are distinct. In this section, we enrich the Reich-type contractions introduced by Bojor [9] and obtain existence and convergence results for such mappings.

Definition 3.1. A mapping $T : \bigcup_{i=1}^k B_i \to X$ is called an *enriched Reich-type cyclic contraction* if the following conditions are satisfied:

(1) $\{B_i : i = 1, 2, \ldots, k\}$ is a cyclic representation of $\bigcup_{i=1}^k B_i$ with respect to T_λ;

(2) There exist $b \in [0, \infty)$ and constants $\alpha, \beta, \gamma \geq 0$ with

$$\frac{\alpha}{b+1} + \beta + \gamma < 1$$

such that for all $x \in B_i$ and $y \in B_{i+1}$, $1 \leq i \leq k$,

$$\|b(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\|x - Tx\| + \gamma\|y - Ty\|. \tag{3.1}$$

Remark 3.2. Setting $b = 0$ in (3.1) reduces to the classical Reich-type cyclic contraction: $\|Tx - Ty\| \leq \alpha\|x-y\| + \beta\|x-Tx\| + \gamma\|y-Ty\|$ with $\alpha + \beta + \gamma < 1$.

Remark 3.3. The enriched P-cyclic contraction class (Definition 2.2) and the enriched Reich-type cyclic contraction class are distinct. Indeed, since $|\|x-Tx\|-\|y-Ty\|| \le \|x-Tx\|+\|y-Ty\|$, every mapping satisfying (3.1) with $\beta=\gamma=\frac{\theta}{b+1}$ and $\alpha=\theta$ also satisfies the enriched P-cyclic condition (2.1) with the same θ. However, the converse does not hold: the P-contraction condition, through the use of $|\|x-Tx\|-\|y-Ty\||$, is strictly weaker than the sum $\|x-Tx\|+\|y-Ty\|$. On the other hand, the Reich-type condition permits $\beta \ne \gamma$ and allows the three coefficients to be independent, which the P-cyclic condition (with its single parameter θ) does not. Thus, neither class contains the other.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and let $B_1, B_2, \dots, B_k$ be nonempty closed convex subsets of X. Let $T : \bigcup_{i=1}^k B_i \to X$ be an enriched Reich-type cyclic contraction. Then:*

(1) $\mathrm{Fix}(T) = \{u\}$ *for some* $u \in \bigcap_{i=1}^k B_i$;

(2) *There exists $\lambda \in (0,1]$ such that the Krasnosel'skii iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \ge 0,$$

converges to u for any $x_0 \in \bigcup_{i=1}^k B_i$;

(3) *The following a priori estimate holds:*

$$\|x_n - u\| \le \frac{\varepsilon^n}{1-\varepsilon}\|x_0 - x_1\|, \quad n \ge 0,$$

where $\varepsilon = \dfrac{a+p}{1-q}$, with $a = \dfrac{\alpha}{b+1}$, $p=\beta$, and $q=\gamma$;

(4) *The following a posteriori estimate holds:*

$$\|x_n - u\| \le \frac{\varepsilon}{1-\varepsilon}\|x_{n-1} - x_n\|, \quad n \ge 1.$$

Proof. **Case 1:** $b>0$. Let $\lambda = \frac{1}{b+1}$. Then $0<\lambda<1$. We set $a = \frac{\alpha}{b+1}$, $p=\beta$, $q=\gamma$, and note that $a+p+q<1$ by hypothesis.

Since $T_\lambda x = (1-\lambda)x + \lambda T x$, we have $Tx = \frac{1}{\lambda}(T_\lambda x - (1-\lambda)x)$, and consequently

$$\|x - Tx\| = \frac{1}{\lambda}\|x - T_\lambda x\|$$

for all $x \in \bigcup_{i=1}^k B_i$. Moreover, $\|b(x-y)+Tx-Ty\| = \frac{1}{\lambda}\|T_\lambda x - T_\lambda y\|$.

Substituting into (3.1) and multiplying both sides by λ, the enriched condition reduces to

$$\|T_\lambda x - T_\lambda y\| \le a\|x-y\| + p\|x - T_\lambda x\| + q\|y - T_\lambda y\| \tag{3.2}$$

for all $x \in B_i$, $y \in B_{i+1}$, $1 \le i \le k$.

Let $x_0 \in \bigcup_{i=1}^k B_i$ be arbitrary, and define $x_{n+1} = T_\lambda x_n$ for all $n \ge 0$. We assume $\|x_{n+1} - x_n\| > 0$ for all n; otherwise x_n is a fixed point of T_λ and hence of T.

Since $\{B_i\}$ is a cyclic representation with respect to T_λ, there exists $i \in \{1, 2, \ldots, k\}$ such that $x_0 \in B_i$, $x_1 \in B_{i+1}$, $x_2 \in B_{i+2}$, and so on. Applying (3.2) with $x = x_{n-1} \in B_j$ and $y = x_n \in B_{j+1}$, we obtain

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|T_\lambda x_{n-1} - T_\lambda x_n\| \\ &\le a\|x_{n-1} - x_n\| + p\|x_{n-1} - T_\lambda x_{n-1}\| + q\|x_n - T_\lambda x_n\| \\ &= a\|x_{n-1} - x_n\| + p\|x_{n-1} - x_n\| + q\|x_n - x_{n+1}\|.\end{aligned}$$

This implies that

$$(1-q)\|x_{n+1} - x_n\| \le (a+p)\|x_n - x_{n-1}\|.$$

Since $q < 1$ (as $a + p + q < 1$), we have

$$\|x_{n+1} - x_n\| \le \varepsilon\|x_n - x_{n-1}\|, \quad \text{where } \varepsilon = \frac{a+p}{1-q} < 1. \tag{3.3}$$

By induction, (3.3) yields

$$\|x_{n+1} - x_n\| \le \varepsilon^n\|x_1 - x_0\|, \quad n \ge 0. \tag{3.4}$$

For $m > n$, using (3.3):

$$\begin{aligned}\|x_m - x_n\| &\le \sum_{j=n}^{m-1}\|x_{j+1} - x_j\| \\ &\le \sum_{j=n}^{m-1}\varepsilon^j\|x_1 - x_0\| \\ &= \varepsilon^n \cdot \frac{1-\varepsilon^{m-n}}{1-\varepsilon}\|x_1 - x_0\|. \end{aligned}\tag{3.5}$$

Since $\varepsilon \in [0, 1)$, $\|x_m - x_n\| \to 0$ as $n \to \infty$. So $\{x_n\}$ is Cauchy in the complete subspace $\bigcup_{i=1}^k B_i$. Hence $\{x_n\}$ converges to some $u = \lim_{n\to\infty} x_n$.

Since $\{x_n\}$ has infinitely many terms in each B_i (by the cyclic structure) and each B_i is closed, $u \in \bigcap_{i=1}^k B_i$.

Since $u \in \bigcap_{i=1}^k B_i$, for any $x_{n-1} \in B_j$ we have $u \in B_{j+1}$. Applying (3.2) with $x = x_{n-1}$ and $y = u$:

$$\begin{aligned}\|x_n - T_\lambda u\| &= \|T_\lambda x_{n-1} - T_\lambda u\| \\ &\le a\|x_{n-1} - u\| + p\|x_{n-1} - x_n\| + q\|u - T_\lambda u\|.\end{aligned}$$

Letting $n \to \infty$:

$$\|u - T_\lambda u\| \le q\|u - T_\lambda u\|.$$

Since $q < 1$, it follows that $\|u - T_\lambda u\| = 0$, i.e., $T_\lambda u = u$. Since $\mathrm{Fix}(T_\lambda) = \mathrm{Fix}(T)$, we conclude $Tu = u$.

Suppose $v \in \bigcap_{i=1}^k B_i$ with $T_\lambda v = v$ and $v \ne u$. Taking $x = u \in B_j$ and $y = v \in B_{j+1}$ in (3.2):

$$\|u - v\| = \|T_\lambda u - T_\lambda v\| \le a\|u - v\| + p\|u - T_\lambda u\| + q\|v - T_\lambda v\| = a\|u - v\|.$$

Since $a < 1$, this gives $\|u - v\| = 0$, a contradiction. Thus $\mathrm{Fix}(T) = \{u\}$, establishing (1).

Condition (2) follows from $\lim_{n\to\infty} x_n = u$.

A priori estimate.

Letting $m \to \infty$ in the Cauchy estimate (3.5):

$$\|x_n - u\| \le \frac{\varepsilon^n}{1-\varepsilon}\|x_0 - x_1\|, \quad n \ge 0, \tag{3.6}$$

establishing (3).

A posteriori estimate.

From (3.4), by induction: $\|x_{n+j+1} - x_{n+j}\| \le \varepsilon^{j+1}\|x_{n-1} - x_n\|$ for all $j \ge 0$. Therefore, for $m > n$:

$$\begin{aligned}\|x_m - x_n\| &\le \sum_{j=0}^{m-n-1} \|x_{n+j+1} - x_{n+j}\| \le \sum_{j=0}^{m-n-1} \varepsilon^{j+1}\|x_{n-1} - x_n\| \\ &= \varepsilon \cdot \frac{1-\varepsilon^{m-n}}{1-\varepsilon}\|x_{n-1} - x_n\|.\end{aligned}$$

Letting $m \to \infty$:

$$\|x_n - u\| \le \frac{\varepsilon}{1-\varepsilon}\|x_{n-1} - x_n\|, \quad n \ge 1,$$

establishing (4).

Case 2: $b = 0$. Then $\lambda = 1$ and $a = \alpha$, so the Krasnosel'skii iteration reduces to the Picard iteration $x_{n+1} = Tx_n$, and (3.2) becomes the classical Reich-type cyclic contraction condition. □

If we consider $b = 0$ and $\beta = \gamma = 0$ in Theorem 3.4, we obtain Theorem 1.3 of [13] in the setting of a Banach space.

Corollary 3.5. [13] *Let $(X, \|.\|)$ be a Banach space and $T : \cup_{i=1}^k B_i \to \cup_{i=1}^k B_i$ satisfying the following conditions (where $B_{i+1} = B_1$) :*

(1) $T(B_i) \subseteq B_{i+1}$

(2) $\exists\, \alpha \in (0, 1)$ *such that* $\|Tx - Ty\| \le \alpha\|x - y\|$ *for all* $x \in B_i$ *and* $y \in B_{i+1}$ *for* $1 \le i \le k$.

Then T has a unique fixed point.

Remark 3.6. Theorem 3.4 unifies several known results:

(1) **Enriched Banach-type cyclic contraction:** Setting $\beta = \gamma = 0$ yields $\|b(x - y) + Tx - Ty\| \le \alpha\|x - y\|$, and the contraction factor becomes $\varepsilon = \frac{\alpha}{b+1}$.

(2) **Enriched Kannan-type cyclic contraction:** Setting $\alpha = 0$ and $\beta = \gamma = \delta$ yields $\|b(x - y) + Tx - Ty\| \le \delta(\|x - Tx\| + \|y - Ty\|)$ with $2\delta < 1$, and $\varepsilon = \frac{\delta}{1-\delta}$.

Example 3.7. Let $X = \mathbb{R}^2$ with the Euclidean norm, $B_1 = \{(x,0) : 0 \le x \le 1\}$, and $B_2 = \{(0,y) : 0 \le y \le 1\}$. Define $T : B_1 \cup B_2 \to \mathbb{R}^2$ by

$$T(x,0) = \left(0, \frac{x}{4}\right), \quad T(0,y) = \left(\frac{y}{3}, 0\right).$$

Then $T(B_1) \subseteq B_2$ and $T(B_2) \subseteq B_1$, so the cyclic condition holds. We verify (3.1) with $b = 1$ (so $\lambda = 1/2$), $\alpha = 1/2$, $\beta = 1/6$, $\gamma = 1/8$.

$\frac{\alpha}{b+1} + \beta + \gamma = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{13}{24} < 1.$

The contraction factor is $\varepsilon = \frac{a+p}{1-q} = \frac{1/4+1/6}{1-1/8} = \frac{5/12}{7/8} = \frac{10}{21} \approx 0.476.$

The unique fixed point is $u = (0,0) \in B_1 \cap B_2$.

With $b = 1$, the parameter is $\lambda = \dfrac{1}{b+1} = \dfrac{1}{2}$. The Krasnosel'skii iterates are defined by

$$x_{n+1} = T_\lambda x_n = (1-\lambda)x_n + \lambda T x_n = \tfrac{1}{2}x_n + \tfrac{1}{2}Tx_n.$$

The mapping T can be defined on $B_1 \cup B_2$ as follows:

$$T(x,y) = \left(\frac{y}{3}, \frac{x}{4}\right), \quad (x,y) \in \mathbb{R}^2.$$

Indeed, $T(x,0) = (0, \dfrac{x}{4})$ and $T(0,y) = (\dfrac{y}{3}, 0)$, as required. With this extension the Krasnosel'skii operator becomes

$$\begin{aligned} T_\lambda(x,y) &= \frac{1}{2}(x,y) + \tfrac{1}{2}\left(\frac{y}{3}, \frac{x}{4}\right) \\ &= \left(\frac{x}{2} + \frac{y}{6}, \frac{x}{8} + \frac{y}{2}\right). \end{aligned}$$

Numerical illustration.

Take $\mathbf{x}_0 = (1,0) \in B_1$. Then

$$\mathbf{x}_1 = T_\lambda(1,0) = \left(\frac{1}{2}, \frac{1}{8}\right).$$

Then

$$\|\mathbf{x}_1 - \mathbf{x}_0\| = \left\|\left(-\tfrac{1}{2}, \tfrac{1}{8}\right)\right\| = \sqrt{\tfrac{1}{4} + \tfrac{1}{64}} = \frac{\sqrt{17}}{8} \approx 0.515.$$

The a priori estimate (3.6) becomes

$$\|\mathbf{x}_n - u\| \le \frac{21}{11} \cdot \frac{\sqrt{17}}{8} \cdot \left(\frac{10}{21}\right)^n \approx 1.236 \times (0.476)^n.$$

The norm $\|\mathbf{x}_n - u\|$ converges to $u = (0,0)$.

4. Applications involving iterated function systems

As an application of the results presented in Section 3, we study an iterated function system (IFS) consisting of a finite family of enriched Reich-type cyclic contractions with $\beta = \gamma = 0$. First, we introduce some notations that will be used throughout this section.

- Let (X, d) be a complete metric space. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{C}(X)$ the family of all nonempty compact subsets of X. For $A, B \in \mathcal{C}(X)$ define the *gap* between A and B by

$$D(A, B) = \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b),$$

and the *Hausdorff distance* by

$$H(A, B) = \max\{D(A, B),\ D(B, A)\}.$$

It is well known that $(\mathcal{C}(X), H)$ is a complete metric space whenever (X, d) is complete.

- For a real number α we write $\alpha A = \{\alpha a : a \in A\}$ and, more generally, for two sets $A, B \subset X$ the *Minkowski sum* is $A + B = \{a + b : a \in A,\ b \in B\}$.

The class of mappings we employ is a special case of the enriched Reich-type cyclic contraction introduced above, where we set $\beta = \gamma = 0$. Explicitly, a mapping $T : \bigcup_{i=1}^{k} B_i \to X$ satisfies condition

$$\|b(x - y) + Tx - Ty\| \le \alpha\|x - y\| \qquad (\forall x \in B_i,\ \forall y \in B_{i+1},\ 1 \le i \le k),$$

with $b \in [0, \infty)$, $\alpha \ge 0$ and $\frac{\alpha}{b+1} < 1$.

Remark 4.1. If we take $b = 0$ in the above condition, then $\alpha < 1$ and each T reduces to a classical cyclic Banach contraction. In that case the Hutchinson operator that we will associate with our IFS becomes the classical Hutchinson–Barnsley operator.

We need the following elementary lemmas whose proofs are standard.

Lemma 4.2. *Let $\{C_i\}_{i \in \Lambda}$ and $\{D_i\}_{i \in \Lambda}$ be two finite families of sets in $(\mathcal{C}(X), H)$, where $\Lambda = \{1, 2, \ldots, N\}$. Then*

$$H\Big(\bigcup_{i \in \Lambda} C_i,\ \bigcup_{i \in \Lambda} D_i\Big) \le \max_{i \in \Lambda} H(C_i, D_i).$$

Lemma 4.3. *Let (X, d) be a complete metric space. If A is a closed subset of X, then $\mathcal{C}(A)$ is a closed subset of the complete metric space $(\mathcal{C}(X), H)$.*

Before constructing the IFS operator, we prove a theorem that describes the behaviour of the induced mapping on the hyperspace $\mathcal{C}(X)$.

Theorem 4.4. *Let $T : \bigcup_{j=1}^{k} A_j \to X$ be an enriched Reich-type cyclic contraction with $\beta = \gamma = 0$, i.e. there exist $b \in [0, \infty)$ and $\alpha \geq 0$ with $\frac{\alpha}{b+1} < 1$ such that for all $x \in A_j$, $y \in A_{j+1}$ $(1 \leq j \leq k)$,*

$$\|b(x-y) + Tx - Ty\| \leq \alpha \|x - y\|.$$

Define the induced map $T^* : \bigcup_{j=1}^{k} \mathcal{C}(A_j) \to \mathcal{C}(X)$ by $T^*(A) = \{Tx : x \in A\}$. Then the following hold:

(*i*) *For every $j \in \{1, \ldots, k\}$, every $A \in \mathcal{C}(A_j)$ and $B \in \mathcal{C}(A_{j+1})$,*

$$H\big(bA + T^*(A),\ bB + T^*(B)\big) \leq \alpha\, H(A, B).$$

(*ii*) *The family $\{\mathcal{C}(A_j) : j = 1, \ldots, k\}$ is a cyclic representation of $\bigcup_{j=1}^{k} \mathcal{C}(A_j)$ with respect to the mapping T_λ^*, where $\lambda = \frac{1}{b+1}$ and $T_\lambda^*(A) = (1-\lambda)A + \lambda T^*(A)$, provided that T_λ (the pointwise analogue) is continuous.*

Proof. Set $\lambda = \frac{1}{b+1}$ and define the mapping $T_\lambda : X \to X$ by

$$T_\lambda x = (1-\lambda)x + \lambda Tx.$$

A direct computation shows that

$$b(x-y) + Tx - Ty = (b+1)\big(T_\lambda x - T_\lambda y\big).$$

Therefore, the contraction condition of the hypothesis rewrites as

$$\|T_\lambda x - T_\lambda y\| \leq \frac{\alpha}{b+1}\, \|x - y\| = \alpha\lambda\, \|x - y\| \qquad (\forall x \in A_j,\ \forall y \in A_{j+1}). \tag{4.1}$$

Since $\alpha\lambda = \frac{\alpha}{b+1} < 1$, T_λ is a cyclic contraction.

Proof of (ii): By hypothesis $\{A_j\}_{j=1}^{k}$ is a cyclic representation of $\bigcup_{j=1}^{k} A_j$ with respect to T_λ, i.e. $T_\lambda(A_j) \subseteq A_{j+1}$ (with $A_{k+1} = A_1$). As T_λ is continuous and maps compact sets to compact sets, for any $A \in \mathcal{C}(A_j)$ we have $T_\lambda(A) \in \mathcal{C}(A_{j+1})$. Because $T_\lambda^*(A) = T_\lambda(A)$, this yields

$$T_\lambda^*\big(\mathcal{C}(A_j)\big) \subseteq \mathcal{C}(A_{j+1}),$$

which is precisely the required cyclic property.

Proof of (i): Let $A \in \mathcal{C}(A_j)$ and $B \in \mathcal{C}(A_{j+1})$. Observe the relation

$$bA + T^*(A) = \{ba + Ta : a \in A\} = (b+1)\Big\{\Big(1 - \frac{1}{b+1}\Big)a + \frac{1}{b+1}Ta : a \in A\Big\} = (b+1)\, T_\lambda^*(A).$$

Using the Hausdorff distance, we obtain

$$H\big(bA + T^*(A),\, bB + T^*(B)\big) = (b+1)\, H\big(T_\lambda^*(A),\, T_\lambda^*(B)\big).$$

Now, for any compact sets A, B we have

$$D\big(T_\lambda^*(A), T_\lambda^*(B)\big) = \sup_{x \in A} \inf_{y \in B} \|T_\lambda x - T_\lambda y\|$$

$$\leq \alpha\lambda \sup_{x\in A} \inf_{y\in B} \|x-y\|$$
$$= \alpha\lambda\, D(A,B).$$

Symmetrically, $D(T_\lambda^*(B), T_\lambda^*(A)) \leq \alpha\lambda\, D(B,A)$. Consequently,

$$H\big(T_\lambda^*(A), T_\lambda^*(B)\big) \leq \alpha\lambda\, H(A,B).$$

Multiplying by $b+1 = \frac{1}{\lambda}$ we finally obtain

$$H\big(bA + T^*(A),\, bB + T^*(B)\big) \leq \alpha\, H(A,B),$$

which completes the proof. □

Corollary 4.5. *For $n = 1, 2, \ldots, N$, let $T^{(n)} : \bigcup_{j=1}^k A_j \to X$ be a finite family of mappings. Assume that:*

(1) *each $T^{(n)}$ is an enriched Reich-type cyclic contraction with $\beta = \gamma = 0$; i.e. there exist $b_n \in [0,\infty)$ and $\alpha_n \geq 0$ with $\frac{\alpha_n}{b_n+1} < 1$ such that for every $x \in A_j$, $y \in A_{j+1}$ $(1 \leq j \leq k)$,*

$$\|b_n(x-y) + T^{(n)}x - T^{(n)}y\| \leq \alpha_n \|x-y\|;$$

(2) *$\{A_j : j = 1, \ldots, k\}$ is a cyclic representation of $\bigcup_{j=1}^k A_j$ with respect to each averaged mapping $T_{\lambda_n}^{(n)}$, where $\lambda_n = \frac{1}{b_n+1}$ and $T_{\lambda_n}^{(n)}x = (1-\lambda_n)x + \lambda_n T^{(n)}x$.*

Define the operator $F : \bigcup_{j=1}^k \mathcal{C}(A_j) \to \mathcal{C}(X)$ by

$$F(A) = \bigcup_{n=1}^N T_{\lambda_n}^{(n)*}(A),$$

where $T_{\lambda_n}^{(n)*}(A) = \{T_{\lambda_n}^{(n)}x : x \in A\}$. Then the following hold:

(i) *$\{\mathcal{C}(A_j) : j = 1, \ldots, k\}$ is a cyclic representation of $\bigcup_{j=1}^k \mathcal{C}(A_j)$ with respect to F;*

(ii) *for all $1 \leq j \leq k$, $A \in \mathcal{C}(A_j)$ and $B \in \mathcal{C}(A_{j+1})$,*

$$H\big(F(A), F(B)\big) \leq \theta\, H(A,B),$$

where $\theta = \max\limits_{1\leq n\leq N} \dfrac{\alpha_n}{b_n+1}$, provided that all the averaged mappings $T_{\lambda_n}^{(n)}$ are continuous.

Proof. Let $C \in \mathcal{C}(A_j)$ for some j. By Theorem 4.4(ii), each mapping $T_{\lambda_n}^{(n)*}$ maps $\mathcal{C}(A_j)$ into $\mathcal{C}(A_{j+1})$; therefore the finite union $F(C) = \bigcup_{n=1}^N T_{\lambda_n}^{(n)*}(C)$ is again a compact subset of A_{j+1} and hence belongs to $\mathcal{C}(A_{j+1})$. Consequently,

$$F\big(\mathcal{C}(A_j)\big) \subseteq \mathcal{C}(A_{j+1}) \qquad (1 \leq j \leq k),$$

which proves the cyclic property (i).

For the contractive estimate (ii), take $A \in \mathcal{C}(A_j)$ and $B \in \mathcal{C}(A_{j+1})$. From the proof of Theorem 4.4(i) we have, for each n,

$$H\big(T_{\lambda_n}^{(n)*}(A),\, T_{\lambda_n}^{(n)*}(B)\big) \leq \frac{\alpha_n}{b_n+1}\, H(A,B). \tag{3.0.35}$$

Using Lemma 4.2 and the estimate above,

$$\begin{aligned} H\big(F(A), F(B)\big) &= H\Big(\bigcup_{n=1}^{N} T_{\lambda_n}^{(n)*}(A),\ \bigcup_{n=1}^{N} T_{\lambda_n}^{(n)*}(B)\Big) \\ &\le \max_{1\le n\le N} H\big(T_{\lambda_n}^{(n)*}(A),\, T_{\lambda_n}^{(n)*}(B)\big) \\ &\le \max_{1\le n\le N} \frac{\alpha_n}{b_n+1}\, H(A,B) \\ &= \theta\, H(A,B), \end{aligned}$$

where $\theta = \max\limits_{1\le n\le N} \dfrac{\alpha_n}{b_n+1}$. This completes the proof. □

Theorem 4.6. *Let $(X, \|\cdot\|)$ be a Banach space, let $\{A_j\}_{j=1}^k$ be a finite family of nonempty closed subsets of X, and let $T^{(n)} : \bigcup_{j=1}^k A_j \to X$, $n \in \{1, 2, \dots, N\}$, be a family of mappings. Assume that:*

- *each $T^{(n)}$ is an enriched Reich-type cyclic contraction with $\beta = \gamma = 0$; i.e. there exist $b_n \in [0, \infty)$ and $\alpha_n \ge 0$ with $\frac{\alpha_n}{b_n+1} < 1$ such that for all $x \in A_j$, $y \in A_{j+1}$ $(1 \le j \le k)$,*

$$\|b_n(x-y) + T^{(n)}x - T^{(n)}y\| \le \alpha_n \|x-y\|;$$

- *$\{A_j : j = 1, 2, \dots, k\}$ is a cyclic representation of $\bigcup_{j=1}^k A_j$ with respect to $T_{\lambda_n}^{(n)}$ for every $n = 1, \dots, N$, where $\lambda_n = \frac{1}{b_n+1}$ and $T_{\lambda_n}^{(n)}x = (1-\lambda_n)x + \lambda_n T^{(n)}x$;*
- *$\theta = \max\limits_{1\le n\le N} \dfrac{\alpha_n}{b_n+1} < 1$.*

Then the map F defined in Corollary 4.5 possesses a unique attractor $\mathbf{A}$; i.e. there exists a unique $\mathbf{A} \in \bigcup_{j=1}^k \mathcal{C}(A_j)$ such that $F(\mathbf{A}) = \mathbf{A}$. Moreover,

$$\mathbf{A} = \lim_{m\to\infty} F^m(B)$$

for every $B \in \bigcup_{j=1}^k \mathcal{C}(A_j)$, provided the averaged mappings $T_{\lambda_n}^{(n)}$ are continuous for all n.

Proof. Since $(X, \|\cdot\|)$ is a Banach space, the hyperspace $(\mathcal{C}(X), H)$ is a complete metric space. By Lemma 4.3 each $\mathcal{C}(A_j)$ is a nonempty closed subset of $\mathcal{C}(X)$. Corollary 4.5 shows that the map F satisfies all the requirements of the cyclic contraction IFS theorem (Theorem 4.4 with the parameters $\beta = \gamma = 0$). More precisely, F is a cyclic θ-contraction with respect to the cyclic family $\{\mathcal{C}(A_j)\}_{j=1}^k$. Hence, by Theorem 4.4, F has a unique attractor $\mathbf{A}$, and for any starting set B in the same union the sequence of iterates converges to $\mathbf{A}$ in the Hausdorff metric. □

Example 4.7. Let $X = \mathbb{R}$ be equipped with the usual norm. Define the two closed sets

$$A_1 = [0,2], \qquad A_2 = [1,3].$$

Set $b = 1$; hence $\lambda = \frac{1}{b+1} = \frac{1}{2}$. Consider the mappings $T^{(1)}, T^{(2)} \colon A_1 \cup A_2 \to \mathbb{R}$ given by

$$T^{(1)}x = -0.8\,x + 3.2,$$

$$T^{(2)}x = -0.8\,x + 2.6.$$

The corresponding averaged maps are

$$T^{(1)}_{1/2}x = (1-\lambda)x + \lambda T^{(1)}x = 0.1\,x + 1.6,$$
$$T^{(2)}_{1/2}x = 0.1\,x + 1.3.$$

Cyclic representation. We verify that $\{A_1, A_2\}$ is a cyclic representation of $A_1 \cup A_2$ with respect to both averaged maps:

$$T^{(1)}_{1/2}(A_1) = [1.6,\ 1.8] \subset A_2, \quad T^{(1)}_{1/2}(A_2) = [1.7,\ 1.9] \subset A_1,$$
$$T^{(2)}_{1/2}(A_1) = [1.3,\ 1.5] \subset A_2, \quad T^{(2)}_{1/2}(A_2) = [1.4,\ 1.6] \subset A_1.$$

Enriched contraction condition. For every $x \in A_1,\ y \in A_2$ and $n = 1, 2$ we have

$$\begin{aligned}\left|\, b(x-y) + T^{(n)}x - T^{(n)}y \,\right| &= \left|\, (x-y) + (-0.8)(x-y) \,\right| \\ &= 0.2\,|x-y|.\end{aligned}$$

Thus each $T^{(n)}$ is an enriched Reich-type cyclic contraction with parameters $b = 1$, $\alpha_n = 0.2$ (and $\beta = \gamma = 0$). Since

$$\frac{\alpha_1}{b+1} = \frac{0.2}{2} = 0.1 < 1, \qquad \frac{\alpha_2}{b+1} = 0.1 < 1,$$

the condition $\theta = \max_n \frac{\alpha_n}{b+1} = 0.1 < 1$ holds.

IFS operator and attractor. Define the operator $F\colon \mathcal{C}\big(A_1 \cup A_2\big) \to \mathcal{C}\big(A_1 \cup A_2\big)$ by

$$F(B) = T^{(1)}_{1/2}(B)\ \cup\ T^{(2)}_{1/2}(B).$$

By Corollary 4.5, F is a cyclic θ-contraction and possesses a unique attractor $\mathbf{A} \subset A_1 \cap A_2$. The attractor is a Cantor-type set generated by two affine maps with contraction ratio 0.1; after the first iteration the entire set lies in $[1.3,\ 1.9]$, and successive iterations produce an infinite collection of disjoint intervals whose lengths tend to zero.

Example 4.8. Let $X = \mathbb{R}^2$ with the norm $\|(u,v)\|_1 = |u| + |v|$. Let

$$A_1 = [-1, 1] \times \mathbb{R}, \qquad A_2 = \left[-\tfrac{1}{2}, 1\right] \times \mathbb{R}.$$

Define three mappings $T^{(1)}, T^{(2)}, T^{(3)} : A_1 \cup A_2 \to \mathbb{R}^2$ by

$$T^{(1)}(u,v) = \left(0,\ \tfrac{u}{2} + 0.1v\right),$$
$$T^{(2)}(u,v) = \left(\tfrac{1}{2},\ -\tfrac{3u}{4} - \tfrac{v}{2} + 1\right),$$
$$T^{(3)}(u,v) = \left(0,\ \tfrac{u}{4} + \tfrac{v}{4}\right).$$

Take $b_1 = b_2 = b_3 = 1$, so $\lambda_n = \frac{1}{2}$ for all n. Then

$$T^{(1)}_{\frac{1}{2}}(u,v) = \left(\tfrac{u}{2},\ \tfrac{u}{4} + 0.55v\right),$$

$$T_{\frac{1}{2}}^{(2)}(u,v) = \left(\tfrac{u}{2} + \tfrac{1}{4},\ -\tfrac{3u}{8} + \tfrac{v}{4} + \tfrac{1}{2}\right),$$

$$T_{\frac{1}{2}}^{(3)}(u,v) = \left(\tfrac{u}{2},\ \tfrac{u}{8} + \tfrac{5v}{8}\right).$$

Cyclic representation. For $T_{\frac{1}{2}}^{(1)}$: if $(u,v) \in A_1$ then the first component $\frac{u}{2} \in [-\frac{1}{2}, \frac{1}{2}] \subseteq A_2$; if $(u,v) \in A_2$ then $\frac{u}{2} \in [-\frac{1}{4}, \frac{1}{2}] \subseteq A_1$. For $T_{\frac{1}{2}}^{(2)}$: $(u,v) \in A_1$ gives first component $\frac{u}{2} + \frac{1}{4} \in [-\frac{1}{4}, \frac{3}{4}] \subseteq A_2$; $(u,v) \in A_2$ gives $\frac{u}{2} + \frac{1}{4} \in [0, \frac{3}{4}] \subseteq A_1$. For $T_{\frac{1}{2}}^{(3)}$: the first component is $\frac{u}{2}$, which behaves exactly as for $T_{\frac{1}{2}}^{(1)}$. Hence $\{A_1, A_2\}$ is a cyclic representation of $A_1 \cup A_2$ with respect to each $T_{\frac{1}{2}}^{(n)}$.

Enriched contraction condition. For $(u,v) \in A_1$ and $(\tilde{u},\tilde{v}) \in A_2$ we have

$$b_n\big((u,v) - (\tilde{u},\tilde{v})\big) + T^{(n)}(u,v) - T^{(n)}(\tilde{u},\tilde{v}) = 2\Big(T_{\frac{1}{2}}^{(n)}(u,v) - T_{\frac{1}{2}}^{(n)}(\tilde{u},\tilde{v})\Big).$$

Compute the $\|\cdot\|_1$–contraction factors:

For $T_{\frac{1}{2}}^{(1)}$:

$$\begin{aligned}
&\big\|T_{\frac{1}{2}}^{(1)}(u,v) - T_{\frac{1}{2}}^{(1)}(\tilde{u},\tilde{v})\big\|_1 \\
&= \big|\tfrac{u-\tilde{u}}{2}\big| + \big|\tfrac{u-\tilde{u}}{4} + 0.55(v-\tilde{v})\big| \\
&\le \tfrac{1}{2}|u-\tilde{u}| + \tfrac{1}{4}|u-\tilde{u}| + 0.55|v-\tilde{v}| \\
&= \tfrac{3}{4}|u-\tilde{u}| + 0.55|v-\tilde{v}| \\
&\le \tfrac{3}{4}\big(|u-\tilde{u}| + |v-\tilde{v}|\big) = \tfrac{3}{4}\,\|(u,v) - (\tilde{u},\tilde{v})\|_1,
\end{aligned}$$

hence $\alpha_1 = 2 \times \frac{3}{4} = \frac{3}{2}$.

For $T_{\frac{1}{2}}^{(2)}$:

$$\begin{aligned}
&\big\|T_{\frac{1}{2}}^{(2)}(u,v) - T_{\frac{1}{2}}^{(2)}(\tilde{u},\tilde{v})\big\|_1 \\
&= \big|\tfrac{u-\tilde{u}}{2}\big| + \big|-\tfrac{3(u-\tilde{u})}{8} + \tfrac{v-\tilde{v}}{4}\big| \\
&\le \tfrac{1}{2}|u-\tilde{u}| + \tfrac{3}{8}|u-\tilde{u}| + \tfrac{1}{4}|v-\tilde{v}| \\
&= \tfrac{7}{8}|u-\tilde{u}| + \tfrac{1}{4}|v-\tilde{v}| \\
&\le \tfrac{7}{8}\big(|u-\tilde{u}| + |v-\tilde{v}|\big) = \tfrac{7}{8}\,\|(u,v) - (\tilde{u},\tilde{v})\|_1,
\end{aligned}$$

so $\alpha_2 = 2 \times \frac{7}{8} = \frac{7}{4}$.

For $T_{\frac{1}{2}}^{(3)}$:

$$\begin{aligned}
&\big\|T_{\frac{1}{2}}^{(3)}(u,v) - T_{\frac{1}{2}}^{(3)}(\tilde{u},\tilde{v})\big\|_1 \\
&= \big|\tfrac{u-\tilde{u}}{2}\big| + \big|\tfrac{u-\tilde{u}}{8} + \tfrac{5(v-\tilde{v})}{8}\big| \\
&\le \tfrac{1}{2}|u-\tilde{u}| + \tfrac{1}{8}|u-\tilde{u}| + \tfrac{5}{8}|v-\tilde{v}|
\end{aligned}$$

$$= \tfrac{5}{8}|u - \tilde{u}| + \tfrac{5}{8}|v - \tilde{v}|$$
$$= \tfrac{5}{8}\big(|u - \tilde{u}| + |v - \tilde{v}|\big) = \tfrac{5}{8}\,\|(u,v) - (\tilde{u},\tilde{v})\|_1,$$

hence $\alpha_3 = 2 \times \frac{5}{8} = \frac{5}{4}$.

The global contraction constant of the Hutchinson operator F is

$$\theta = \max_{n=1,2,3} \frac{\alpha_n}{b_n + 1} = \max\left\{\frac{3/2}{2}, \frac{7/4}{2}, \frac{5/4}{2}\right\} = \max\left\{\frac{3}{4}, \frac{7}{8}, \frac{5}{8}\right\} = \frac{7}{8} < 1.$$

Hutchinson operator and attractor. The Hutchinson operator $F : \mathcal{C}(A_1) \cup \mathcal{C}(A_2) \to \mathcal{C}(A_1) \cup \mathcal{C}(A_2)$ defined by

$$F(B) = \bigcup_{n=1}^{3} T_{\frac{1}{2}}^{(n)*}(B)$$

is a cyclic $\frac{7}{8}$–contraction on the hyperspace $(\mathcal{C}(A_1) \cup \mathcal{C}(A_2), H)$. By the cyclic attractor theorem, F has a unique attractor $\mathbf{A}$, and for any $B \in \mathcal{C}(A_1) \cup \mathcal{C}(A_2)$,

$$\mathbf{A} = \lim_{m\to\infty} F^m(B), \qquad H\big(F^m(B),\, \mathbf{A}\big) \le \frac{(7/8)^m}{1/8}\, H\big(B, F(B)\big) \to 0.$$

The attractor $\mathbf{A}$ is a self-similar fractal contained in $[-\frac{1}{2}, 1] \times \mathbb{R}$, generated by the three affine contractions $T_{\frac{1}{2}}^{(1)}, T_{\frac{1}{2}}^{(2)}, T_{\frac{1}{2}}^{(3)}$. Unlike the one-dimensional Cantor-set attractor, this attractor lives genuinely in the plane and exhibits a richer self-similar geometry determined by the three maps.

5. Conclusion

In this paper, we introduced and analysed two new classes of enriched cyclic contractions: enriched P-cyclic contractions and enriched Reich-type cyclic contractions. For both classes, we established the existence and uniqueness of a fixed point in the intersection of the cyclic subsets, and we proved that the Krasnosel'skiĭ iteration converges to that fixed point for any initial guess. Moreover, we derived a priori and a posteriori error estimates as well as the rate of convergence.

The enriched P-cyclic condition uses the absolute difference of the displacement norms, which is a strictly weaker requirement than the sum used in earlier works. The enriched Reich-type cyclic contraction allows independent coefficients for $\|x - Tx\|$ and $\|y - Ty\|$, thereby generalising both the classical Reich contraction and the enriched cyclic contractions studied in the literature. These results unify and extend several fundamental theorems, including the cyclic contraction theorem of Kirk, Srinivasan and Veeramani, the enriched contraction theorem of Berinde and Păcurar, and the P-contraction theorem of Popescu.

As an application, we considered a generalized iterated function system consisting of enriched Reich-type cyclic contractions with $\beta = \gamma = 0$. We proved that the associated Hutchinson operator on the hyperspace of compact sets possesses a unique attractor (a self-similar fractal set) and that the iteration of the operator converges to this attractor in the Hausdorff metric. Numerical examples in one and two dimensions were provided to illustrate the theoretical findings.

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