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ON SOME CONVERGENCE THEOREMS FOR THE HENSTOCK-KURZWEIL-DUNFORD-STIELTJES INTEGRAL AND HENSTOCK-KURZWEIL-PETTIS-STIELTJES INTEGRAL OF BANACH-VALUED FUNCTIONS

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Abstract. This paper establishes convergence theorems for the Henstock-Kurzweil-Dunford-Stieltjes (HKDS) and Henstock-Kurzweil-Pettis-Stieltjes (HKPS) integrals of Banach-valued functions with respect to a function of bounded variation. Equiintegrability for these integrals is introduced, and generalizations of the Dominated Convergence Theorem and the Bounded Convergence Theorem for the HKDS and HKPS integrals are obtained. Moreover, a convergence result is presented by applying the Uniform Boundedness Theorem to sequences of HKDS and HKPS integral operators, circumventing the need for equiintegrability while ensuring convergence under the uniform boundedness of the associated operators.

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Key words and phrases. Banach space; dual space; HKDS integral; HKPS integral; bounded variation; equiintegrability; uniform boundedness; convergence theorems.

1. Introduction

Integration theory is a fundamental concept in mathematics with a long history and remains central in modern analysis due to its wide applications in physics, engineering, economics, and other applied sciences [1]. Although the classical Riemann integral is widely used in practice, it has structural limitations, particularly in handling highly oscillatory functions and irregular local behavior, which has motivated the development of more general, in particular non-absolute, integration theories [1,9,10].

The Henstock-Kurzweil (HK) integral, independently developed by Jaroslav Kurzweil and Ralph Henstock in the late 1950s, generalizes the Riemann integral by allowing the integration of highly oscillatory functions through a system of δ-fine gauge partitions, where δ is a positive-valued function called a gauge [5]. Unlike the Riemann integral, the HK integral can handle functions that are not Riemann integrable and possesses strong convergence properties [1].

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Further extensions to Banach-valued functions were developed through the Dunford and Pettis integrals, which extend the classical Lebesgue integral to functions taking values in a Banach space by utilizing the concept of dual spaces. [10]. In addition, Sergio S. Cao [2] in 1992 generalized the HK integral from real-valued functions to functions taking values in Banach spaces. These ideas inspired modern generalizations of the HK integral, leading to the Henstock-Kurzweil-Dunford (HKD) integral and Henstock-Kurzweil-Pettis (HKP) integral, which combine HK integration with the Dunford and Pettis frameworks, respectively [10].

In 1998, Jong Sul Lim, Ju Han Yoon, and Gwang Sik Eun [6] extended the HK integral in the Stieltjes sense for real-valued functions with increasing integrators. Subsequently, in 2010, Sanket A. Tikare and M. S. Chaudhary [11] defined the Henstock-Kurzweil-Stieltjes (HKS) integral for Banach-valued functions with respect to integrator functions of bounded variation. More recently, in 2024, Mangubat [7] introduced the Henstock-Kurzweil-Dunford-Stieltjes (HKDS) integral and the Henstock-Kurzweil-Pettis-Stieltjes (HKPS) integral, which combine HKD and HKP integration with Stieltjes-type integrators. These integrals further extend the scope of generalized integration theory. Mangubat and Flores [8] provided convergence theorems for these integrals via uniform convergence with respect to the integrand and integrator. However, convergence results based on pointwise convergence and equiintegrability for HKDS and HKPS integrals have not yet been formulated.

In this paper, we establish convergence theorems, namely the Equiintegrability Theorem, the Dominated Convergence Theorem, and the Bounded Convergence Theorem. Moreover, we present a convergence result via the Uniform Boundedness Theorem applied to sequences of HKDS and HKPS integral operators. This operator-theoretic approach yields convergence under uniform boundedness, providing an alternative perspective to classical equiintegrability-based techniques. These results provide tools for analyzing sequences of Banach-valued functions within the framework of non-absolute HKDS and HKPS integration.

2. Preliminaries

This section presents the fundamental definitions and structural properties that serve as the basis for the development of the main results.

Definition 2.1. [3] Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of functions with a common domain D, f is a function on D, and let A be a subset of D. We say that the sequence $\langle f_n \rangle_{n=1}^{\infty}$ *converges pointwise* to f on A provided that for every $\varepsilon > 0$ and for every $x \in A$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.

Definition 2.2. [3] A sequence $\langle f_n \rangle$ of real-valued functions on D is said to be *uniformly bounded* on D provided there exists $M > 0$ for which $|f_n| \leq M$ on D for all $n \in \mathbb{N}$.

Definition 2.3. [4] A normed space $(X, \|\cdot\|_X)$ is said to be *complete* if every Cauchy sequence in X is convergent. In this Case, X is a *Banach space*.

Definition 2.4. [5] A *compact interval* in $\mathbb{R}$ is a closed and bounded interval of the form $[a, b]$ where $a, b \in \mathbb{R}$. This interval is said to be *nondegenerate* if $a < b$.

Definition 2.5. [5] Two intervals $[u, v]$ and $[s, t]$ in $\mathbb{R}$ are said to be *non-overlapping* if $(u, v) \cap (s, t) = \emptyset$.

Definition 2.6. [5] If $\{[u_1, v_1], \ldots, [u_p, v_p]\}$ is a finite collection of *pairwise non-overlapping* subintervals of $[a, b]$ such that $[a, b] = \bigcup_{k=1}^{p} [u_k, v_k]$, then we say that $\{[u_1, v_1], \ldots, [u_p, v_p]\}$ is a *division* of $[a, b]$.

Definition 2.7. [5] A *point-interval pair* $(t, [u, v])$ consists of a point $t \in \mathbb{R}$ and an interval $[u, v] \subset \mathbb{R}$. Here, t is called the *tag* of $[u, v]$.

Definition 2.8. [5] A *Perron partition* of $[a, b]$ is a finite collection $\{(t_1, [u_1, v_1]), \ldots, (t_p, [u_p, v_p])\}$ of point-interval pairs, where $\{[u_1, v_1], \ldots, [u_p, v_p]\}$ is a division of $[a, b]$, and $t_k \in [u_k, v_k]$ for each $k = 1, \ldots, p$.

Definition 2.9. [5] A function $\delta : [a, b] \to \mathbb{R}^+$ is called a *gauge* on $[a, b]$.

Definition 2.10. [5] Let δ be a gauge on $[a, b]$. A Perron partition $\{(t_1, [u_1, v_1]), \ldots, (t_p, [u_p, v_p])\}$ of $[a, b]$ is said to be δ*-fine* if $[u_k, v_k] \subset (t_k - \delta(t_k),\ t_k + \delta(t_k))$ for each $k = 1, \ldots, p$.

Lemma 2.11. [5] (*Cousin's Lemma*) If δ is a gauge on $[a, b]$, then there exists a δ-fine Perron partition of $[a, b]$.

Definition 2.12. [11] A function $g : [a, b] \to \mathbb{R}$ is said to be of *bounded variation* on $[a, b]$ if $\sup\left\{\sum_{[u_k,v_k]\in D} \left|g(v_k) - g(u_k)\right|\right\}$ is finite, where the supremum is taken over all divisions $D = \{[u_k, v_k]\}$ of $[a, b]$.

Definition 2.13. [11] Let X be a Banach space and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. A function $f : [a, b] \to X$ is *Henstock-Kurzweil-Stieltjes integrable* or *HKS-integrable* with respect to g on $[a, b]$ if there exists a vector $A \in X$ with the following property: for all $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{(t_k,[u_k,v_k])\in P} f(t_k)[g(v_k) - g(u_k)] - A \right\|_X < \varepsilon$$

for each δ-fine Perron partition P of $[a, b]$. If such A exists, then $A = \text{(HKS)} \int_a^b f\, dg$ is called the *Henstock-Kurzweil-Stieltjes integral* of f with respect to g over $[a, b]$. For brevity, we represent the sum $\sum_{(t_k,[u_k,v_k])\in P} f(t_k)[g(v_k) - g(u_k)]$ by $S(f; g; P)$. Moreover, the collection of all functions f that are Henstock-Kurzweil-Stieltjes integrable with respect to g on $[a, b]$ is denoted by HKS[a,b].

Definition 2.14. [11] (*HKS-Equiintegrability*) Let X be a Banach space and let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. A collection $\mathcal{F}$ of functions $f : [a,b] \to X$ is called *HKS-equiintegrable* with respect to g on $[a,b]$ if every $f \in \mathcal{F}$ is HKS-integrable and, for any $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that, for any $f \in \mathcal{F}$, we have

$$\left\| S(f;g;P) - (\text{HKS})\int_a^b f\,dg \right\|_X < \varepsilon$$

whenever P is a δ-fine Perron partition of $[a,b]$.

Definition 2.15. [4] An operator $T : V \to W$ between vector spaces V and W is a *linear operator* if for all $x, y \in V$ and scalars α,

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha Tx.$$

Definition 2.16. [4] A *linear functional* is any linear operator $f : X \to K$, where X is a normed space over the field $K = \mathbb{R}$ or $K = \mathbb{C}$.

Definition 2.17. [4] Let X and Y be normed spaces and $T : \mathcal{D}(T) \to Y$ a linear operator, where $\mathcal{D}(T) \subset X$. The operator T is said to be *bounded* if there exists a real number c such that for all $x \in \mathcal{D}(T)$, $\|Tx\| \leq c\|x\|$. Here, the smallest possible value that c can take is given by

$$\frac{\|Tx\|}{\|x\|} \leq c \quad \text{for all } x \in \mathcal{D}(T),$$

where c must be at least as big as the supremum of

$$\left\{ \frac{\|Tx\|}{\|x\|} : x \in \mathcal{D}(T) \right\}.$$

This $\|T\|$ is called the *norm of the operator* T. If $c = \|T\|$, then $\|Tx\| \leq \|T\|\,\|x\|$. In the case of linear functionals, we have $|f(x)| \leq \|f\|\,\|x\|$.

Definition 2.18. [4] Let X be a vector space over a field K. Define
$X^* = \{f : X \to K \mid f \text{ is a linear functional}\}$ and the following operations in X^*,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

Then $\langle X^*, +, \cdot \rangle$ is a vector space, called the *algebraic dual space* of X.

Definition 2.19. [4] Let X be a vector space over field K. Define

$$X^{**} = \{g : X^* \to K \mid g(f) = f(x) \text{ for all } f \in X^* \text{ where } g \text{ is a linear functional}\}$$

and the following operations in X^{**},

$$(g_1 + g_2)(f) = g_1(f) + g_2(f) \quad \text{and} \quad (\alpha g)(f) = \alpha g(f).$$

Then $\langle X^{**}, +, \cdot \rangle$ is a vector space, called the *second algebraic dual space* of X.

Definition 2.20. [7,8] Let $(X, \|\cdot\|_X)$ be a Banach space, $f : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ a function of bounded variation. The function f is said to be *Henstock-Kurzweil-Dunford-Stieltjes integrable* (or simply *HKDS-integrable*) with respect to g on $[a,b]$ if the following are satisfied:

(i) For all $x^* \in X^*$, the function $x^*(f) : [a,b] \to \mathbb{R}$ is HKS-integrable with respect to g on $[a,b]$.

(ii) For each compact subinterval $E \subset [a,b]$, there exists an element $x^{**}_E \in X^{**}$ such that

$$x^{**}_E(x^*) = (\mathrm{HKS}) \int_E x^*(f)\, dg$$

for all $x^* \in X^*$.

For a compact subinterval $E \subset [a,b]$, (HKDS) $\int_E f\, dg = x^{**}_E$ is called the *HKDS integral* of f over E. The set of all Henstock-Kurzweil-Dunford-Stieltjes integrable functions f is denoted by HKDS[a,b].

Definition 2.21. [7,8] If $f : [a,b] \to X$ is HKDS-integrable such that (HKDS) $\int_E f\, dg \in X$, particularly (HKDS) $\int_E f\, dg \in e(X)$, for every compact subinterval $E \subset [a,b]$, where e is the canonical embedding of X into X^{**}, then f is called *Henstock-Kurzweil-Pettis-Stieltjes integrable* (or simply *HKPS-integrable*) with respect to g and

$$(\mathrm{HKPS}) \int_E f\, dg \;=\; (\mathrm{HKDS}) \int_E f\, dg$$

is called the *HKPS integral* of f over the compact subinterval $E \subset [a,b]$ with respect to g. The set of all Henstock-Kurzweil-Pettis-Stieltjes integrable functions f is denoted by HKPS[a,b].

3. Main Results

3.1. **Convergence via Equiintegrability of Sequences of HKDS and HKPS Integrable Functions.**

We begin by recalling the characterization of HKDS and HKPS integrals on X^*. Let X be a Banach space and $[a,b] \subset \mathbb{R}$. By Theorem 3.1.5 in [7], a function $f : [a,b] \to X$ is HKDS-integrable with respect to a function $g : [a,b] \to \mathbb{R}$ of bounded variation if and only if $x^*(f) : [a,b] \to \mathbb{R}$ is HKS-integrable with respect to g for all $x^* \in X^*$. By Definition 2.21, the same characterization with respect to X^* applies to the HKPS integral [7]. This motivates the notion of HKDS- and HKPS-equiintegrability, which provides uniform control over a sequence of integrals and is essential for establishing convergence results.

Definition 3.1. Let X be a Banach space and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. A collection $\mathcal{F} = \{\, f_n : [a,b] \to X \mid n \in \mathbb{N} \,\}$ of HKDS-integrable functions is said to be *HKDS-equiintegrable* with respect to g on $[a,b]$ if for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for all $x^* \in X^* \setminus \{0\}$, for each $f_n \in \mathcal{F}$, and for every δ-fine Perron partition $P = \{\, (t_k, [u_k, v_k]) \,\}_{k=1}^p$ of $[a,b]$, we have

$$\left| S(x^*(f_n); g; P) \;-\; (\mathrm{HKS}) \int_a^b x^*(f_n)\, dg \right| < \varepsilon. \tag{1}$$

Remark 3.2. In view of the definition above, and by linearity of $x^* \in X^* \setminus \{0\}$ over the finite sum of the δ-fine Perron partition P and its continuity over the integral as established in Theorem 3.1.1 in [7], we can write (1) as

$$\begin{aligned}\left|S(x^*(f_n);g;P) - (\text{HKS})\int_a^b x^*(f_n)\,dg\right| &= \left|x^*(S(f_n;g;P)) - x^*\left((\text{HKS})\int_a^b f_n\,dg\right)\right| \\ &= \left|x^*\left(S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right)\right| \\ &< \varepsilon.\end{aligned}$$

Furthermore, since x^* is a bounded linear functional, we have

$$\left|x^*\left(S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right)\right| \leq \|x^*\| \left\|S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right\|.$$

Since the quantity on the right serves as an upper bound for the absolute value, applying (1) yields

$$\|x^*\| \left\|S(f_n;g;P) - (HKS)\int_a^b f_n\,dg\right\| < \varepsilon,$$

which directly gives

$$\left\|S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right\| < \frac{\varepsilon}{\|x^*\|}. \tag{2}$$

Remark 3.3. A sequence $\langle f_n\rangle_{n=1}^{\infty}$ of HKDS-integrable functions with respect to g on $[a,b]$ is said to be HKDS-equiintegrable with respect to g on $[a,b]$ if the collection $\mathcal{F}$ is HKDS-equiintegrable. Moreover, if $\mathcal{F}$ is a collection of HKPS-integrable functions, the same definition applies, since HKPS integrability is simply the special case of HKDS integrability in which, for all compact subintervals $E \subset [a,b]$, (HKDS) $\int_E f\,dg \in X$ and, more precisely, (HKDS) $\int_E f\,dg \in e(X)$, where $e : X \to X^{**}$ is the canonical embedding of X into X^{**}.

We present convergence theorems based on equiintegrability, preceded by the following lemmas.

Lemma 3.4. Let X be a Banach space and $x^* \in X^* \setminus \{0\}$. If $\langle f_n\rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a,b]$ and converges pointwise to f on $[a,b]$, then $\langle x^*(f_n)\rangle_{n=1}^{\infty}$ converges to $x^*(f)$ pointwise on E for every compact subinterval $E \subset [a,b]$.

Proof. Fix $x^* \in X^* \setminus \{0\}$ and $\varepsilon > 0$. Let $E \subset [a,b]$ be a compact subinterval. Since $\langle f_n\rangle_{n=1}^{\infty}$ converges pointwise to f on $[a,b]$, for every $\varepsilon > 0$ and for every $t \in E$, there exist $N \in \mathbb{N}$ such that

$$\|f_n(t) - f(t)\| < \frac{\varepsilon}{\|x^*\|}, \quad \text{for all } n \geq N.$$

Now, for any $t \in E$,

$$\begin{aligned}|x^*(f_n(t)) - x^*(f(t))| &= \left|x^*\big(f_n(t) - f(t)\big)\right| \\ &\leq \|x^*\|\,\|f_n(t) - f(t)\|\end{aligned}$$

$$
\begin{aligned}
&< \|x^*\| \left(\frac{\varepsilon}{\|x^*\|}\right) \\
&= \varepsilon.
\end{aligned}
$$

□

Lemma 3.5. Let X be a Banach space and $g : [a, b] \to \mathbb{R}$ be of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a, b]$. If $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a, b]$ and converges pointwise to f on $[a, b]$, then for every $x^* \in X^* \setminus \{0\}$, the sequence $\left\langle (\text{HKS}) \int_a^b x^*(f_n)\, dg \right\rangle_{n=1}^{\infty}$ is Cauchy.

Proof. Let $\varepsilon > 0$. By the HKDS-equiintegrability of $\langle f_n \rangle_{n=1}^{\infty}$ with respect to g on $[a, b]$ and by (2), there is a gauge δ on $[a, b]$ such that for any δ-fine Perron partition $P = \{\, (t_k, [u_k, v_k])\, \}_{k=1}^{p}$ of $[a, b]$, for all $n \in \mathbb{N}$, and for all $x^* \in X^* \setminus \{0\}$,

$$
\left\| S(f_n; g; P) - (\text{HKS}) \int_a^b f_n \, dg \right\| < \frac{\varepsilon}{3\|x^*\|}.
$$

Since g is of bounded variation, let $M = \sup \left\{ \sum_{[u_k,v_k]\in D} |g(v_k) - g(u_k)| \right\} < \infty$. Now, fix a δ-fine Perron partition $P = \{\, (t_k, [u_k, v_k])\, \}_{k=1}^{p}$ of $[a, b]$. Since $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[a, b]$, for each tag t_k in P, there exists $N_{t_k} \in \mathbb{N}$ such that

$$
\|f_n(t_k) - f(t_k)\| < \frac{\varepsilon}{6\|x^*\|\, M}, \quad \text{for all } n \geq N_{t_k}.
$$

Set $N = \max\{N_{t_k} : t_k \text{ is a tag in } P\}$. Let $m, n \geq N$. Then

$$
\begin{aligned}
\|f_n(t_k) - f_m(t_k)\| &= \|f_n(t_k) - f(t_k) + f(t_k) - f_m(t_k)\| \\
&\leq \|f_n(t_k) - f(t_k)\| + \|f(t_k) - f_m(t_k)\| \\
&< \frac{\varepsilon}{6\|x^*\|M} + \frac{\varepsilon}{6\|x^*\|M} \\
&= \frac{\varepsilon}{3\|x^*\|M}.
\end{aligned}
$$

for each tag t_k in P. Thus, for all $m, n \geq N$.

$$
\begin{aligned}
\|S(f_n; g; P) - S(f_m; g; P)\| &= \|S(f_n - f_m; g; P)\| \\
&= \left\| \sum_{(t_k,[u_k,v_k])\in P} (f_n(t_k) - f_m(t_k))(g(v_k) - g(u_k)) \right\| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \|(f_n(t_k) - f_m(t_k))(g(v_k) - g(u_k))\| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \|f_n(t_k) - f_m(t_k)\|\, |g(v_k) - g(u_k)| \\
&< \sum_{(t_k,[u_k,v_k])\in P} \frac{\varepsilon}{3\|x^*\|\, M} |g(v_k) - g(u_k)|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\varepsilon}{3\|x^*\|\, M} \sum_{(t_k,[u_k,v_k])\in P} |g(v_k) - g(u_k)| \\
&\leq \frac{\varepsilon}{3\|x^*\|\, M} M \\
&= \frac{\varepsilon}{3\|x^*\|}.
\end{aligned}
$$

Hence, for all $m, n \geq N$,

$$
\begin{aligned}
&\left\| (\text{HKS}) \int_a^b f_n \, dg - (\text{HKS}) \int_a^b f_m \, dg \right\| \\
&= \left\| (\text{HKS}) \int_a^b f_n \, dg - S(f_n, g, P) + S(f_n, g, P) - S(f_m, g, P) \right. \\
&\left. + S(f_m, g, P) - (\text{HKS}) \int_a^b f_m \, dg \right\| \\
&\leq \left\| (\text{HKS}) \int_a^b f_n \, dg - S(f_n, g, P) \right\| + \left\| S(f_n, g, P) - S(f_m, g, P) \right\| \\
&+ \left\| S(f_m, g, P) - (\text{HKS}) \int_a^b f_m \, dg \right\| \\
&< \frac{\varepsilon}{3\|x^*\|} + \frac{\varepsilon}{3\|x^*\|} + \frac{\varepsilon}{3\|x^*\|} = \frac{\varepsilon}{\|x^*\|}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left| (\text{HKS}) \int_a^b x^*(f_n) \, dg - (\text{HKS}) \int_a^b x^*(f_m) \, dg \right| \\
&= \left| x^* \left((\text{HKS}) \int_a^b f_n \, dg - (\text{HKS}) \int_a^b f_m \, dg \right) \right| \\
&\leq \|x^*\| \left\| (\text{HKS}) \int_a^b f_n \, dg - (\text{HKS}) \int_a^b f_m \, dg \right\| \\
&< \|x^*\| \frac{\varepsilon}{\|x^*\|} = \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, $\left\langle (\text{HKS}) \int_a^b x^*(f_n) \, dg \right\rangle$ is Cauchy for all $x^* \in X^* \setminus 0$. □

Theorem 3.6. (**HKDS-Equiintegrability Theorem**) Let $f_n : [a, b] \to X$ and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a, b]$. If $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a, b]$ and converges pointwise to $f : [a, b] \to X$, then f is HKDS-integrable with respect to g on $[a, b]$ and

$$
\lim_{n\to\infty} (\text{HKDS}) \int_E f_n \, dg = (\text{HKDS}) \int_E f \, dg
$$

for all compact subinterval E $\subset [a, b]$.

Proof. Let $x^* \in X^* \setminus \{0\}$. By Lemma 3.5, the sequence $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a,b]$ and $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[a,b]$. Moreover, Lemma 3.5 implies that the sequence $\left\langle (\text{HKS}) \int_a^b x^*(f_n)\, dg \right\rangle_{n=1}^{\infty}$ is Cauchy, and hence convergent. Let $\lim_{n\to\infty}(\text{HKS}) \int_a^b x^*(f_n)\, dg = L$. We claim that $L = (\text{HKS}) \int_a^b x^*(f)\, dg$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| (\text{HKS}) \int_a^b x^*(f_n)\, dg - L \right| < \frac{\varepsilon}{3}. \tag{3}$$

By continuity of x^*,

$$\begin{aligned} L &= \lim_{n\to\infty} (\text{HKS}) \int_a^b x^*(f_n)\, dg \\ &= \lim_{n\to\infty} x^*\left((\text{HKS}) \int_a^b f_n\, dg \right) \\ &= x^*\left(\lim_{n\to\infty} (\text{HKS}) \int_a^b f_n\, dg \right) \\ &= x^*\left((\text{HKS}) \int_a^b f\, dg \right). \end{aligned}$$

Note that $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a,b]$. Thus, $\langle x^*(f_n) \rangle_{n=1}^{\infty}$ is HKS-equiintegrable with respect to g on $[a,b]$. Hence, we choose a gauge δ on $[a,b]$ such that

$$\left\| S(f_n; g; P) - (\text{HKS}) \int_a^b f_n\, dg \right\| < \frac{\varepsilon}{3\|x^*\|} \tag{4}$$

for any δ-fine Perron partition P of $[a,b]$. Observe that $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[a,b]$. Hence, for each tag $t_k \in [a,b]$, there exists $N \in \mathbb{N}$ such that

$$\|f_n(t_k) - f(t_k)\| < \frac{\varepsilon}{3\|x^*\| M}, \quad \text{for all } n \geq N, \tag{5}$$

where $M = \sup\left\{ \sum_{[u_k,v_k]\in D} |g(v_k) - g(u_k)| \right\} < \infty$. Furthermore, applying (5), for each $n \geq N$,

$$\begin{aligned} \|S(f_n; g; P) - S(f; g; P)\| &= \|S(f_n - f; g; P)\| \\ &= \left\| \sum_{(t_k,[u_k,v_k])\in P} (f_n(t_k) - f(t_k))(g(v_k) - g(u_k)) \right\| \\ &\leq \sum_{(t_k,[u_k,v_k])\in P} \|(f_n(t_k) - f(t_k))(g(v_k) - g(u_k))\| \\ &\leq \sum_{(t_k,[u_k,v_k])\in P} \|f_n(t_k) - f(t_k)\|\, |g(v_k) - g(u_k)| \qquad (6) \\ &< \sum_{(t_k,[u_k,v_k])\in P} \frac{\varepsilon}{3\|x^*\| M} |g(v_k) - g(u_k)| \\ &= \frac{\varepsilon}{3\|x^*\| M} \sum_{(t_k,[u_k,v_k])\in P} |g(v_k) - g(u_k)| \\ &\leq \frac{\varepsilon}{3\|x^*\| M} M \end{aligned}$$

$$= \frac{\varepsilon}{3\|x^*\|}.$$

From (3), for each $n \geq N$,

$$\begin{aligned}
\left|(\text{HKS})\int_a^b x^*(f_n)\,dg - L\right| &= \left|(\text{HKS})\int_a^b x^*(f_n)\,dg - (\text{HKS})\int_a^b x^*(f)\,dg\right| \\
&= \left|x^*\left((\text{HKS})\int_a^b f_n\,dg - (\text{HKS})\int_a^b f\,dg\right)\right| \\
&\leq \|x^*\|\left\|(\text{HKS})\int_a^b f_n\,dg - (\text{HKS})\int_a^b f\,dg\right\| < \frac{\varepsilon}{3}.
\end{aligned}$$

Thus, for each $n \geq N$,

$$\left\|(\text{HKS})\int_a^b f_n\,dg - (\text{HKS})\int_a^b f\,dg\right\| < \frac{\varepsilon}{3\|x^*\|}. \tag{7}$$

Using (4), (6), and (7), we have

$$\begin{aligned}
&\left\|S(f;g;P) - (\text{HKS})\int_a^b f\,dg\right\| \\
&= \left\|S(f;g;P) - S(f_n;g;P) + S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right. \\
&\quad \left. + (\text{HKS})\int_a^b f_n\,dg - (\text{HKS})\int_a^b f\,dg\right\| \\
&\leq \|S(f;g;P) - S(f_n;g;P)\| + \left\|S(f_n;g;P) - (\text{HKS})\int_a^b f_n\,dg\right\| \\
&\quad + \left\|(\text{HKS})\int_a^b f_n\,dg - (\text{HKS})\int_a^b f\,dg\right\| \\
&< \frac{\varepsilon}{3\|x^*\|} + \frac{\varepsilon}{3\|x^*\|} + \frac{\varepsilon}{3\|x^*\|} \\
&= \frac{\varepsilon}{\|x^*\|}.
\end{aligned}$$

Hence, $L = (\text{HKS})\int_a^b x^*(f)\,dg$, which shows that $x^*(f)$ is HKS-integrable with respect to g on $[a,b]$. Consequently, by Theorem 3.1.5 in [7], f is HKDS-integrable with respect to g on $[a,b]$. Now, let $E \subset [a,b]$ be a compact subinterval. For each $n \in \mathbb{N}$, define

$$x^{**}_{n,E}(x^*) = (\text{HKS})\int_E x^*(f_n)\,dg \quad \text{and} \quad x^{**}_E(x^*) = (\text{HKS})\int_E x^*(f)\,dg.$$

This implies that for all $x^* \in X^* \setminus \{0\}$ and $n \in \mathbb{N}$, $x^{**}_{n,E}$ converges to x^{**}_E in X^{**} and

$$\lim_{n\to\infty} x^{**}_{n,E} = \lim_{n\to\infty} (\text{HKDS})\int_E f_n\,dg = (\text{HKDS})\int_E f\,dg = x^{**}_E.$$

$\square$

Remark 3.7. Let $f_n : [a,b] \to X$ and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKPS-integrable functions with respect to g on $[a,b]$. If $\langle f_n \rangle_{n=1}^{\infty}$ is HKPS-equiintegrable with respect to g on $[a,b]$ and converges pointwise to $f : [a,b] \to X$, then f is HKPS-integrable with respect to g on $[a,b]$ and

$$\lim_{n\to\infty} (\text{HKPS}) \int_E f_n \, dg = (\text{HKPS}) \int_E f \, dg$$

for all compact subinterval E $\subset [a,b]$.

The following Dominated Convergence Theorem (DCT) for the HKDS and HKPS integrals relies on three main concepts, specifically, pointwise convergence of the integrands, equiintegrability of the sequence of HKDS- and HKPS-integrable functions, and domination in the scalar field.

Theorem 3.8. (**Dominated Convergence Theorem**) Let X be a Banach space, $f_n : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a,b]$ satisfying the following conditions:

(i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a,b] \to X$ on $[a,b]$;

(ii) there exist HKDS-integrable functions $h_1, h_2 : [a,b] \to X$ with respect to g on $[a,b]$ such that $x^*(h_1(t)) \leq x^*(f_n(t)) \leq x^*(h_2(t))$ for every $n \in \mathbb{N}$, for every $x^* \in X^* \setminus \{0\}$, and for every $t \in [a,b]$.

Then $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a,b]$, the integral (HKDS) $\int_E f \, dg$ exists, and

$$\lim_{n\to\infty} (\text{HKDS}) \int_E f_n \, dg = (\text{HKDS}) \int_E f \, dg$$

for all compact subintervals $E \subset [a,b]$.

Proof. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence in HKDS$[a,b]$. By Theorem 3.1.5 in [7], for each $n \in \mathbb{N}$, the function $x^*(f_n) : [a,b] \to \mathbb{R}$ is HKS-integrable with respect to g on $[a,b]$. Using Lemma 3.4, the sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[a,b]$ implies that $\langle x^*(f_n) \rangle_{n=1}^{\infty}$ converges pointwise to $x^*(f)$ on $[a,b]$. Fix $\varepsilon > 0$. Since h_1 and h_2 are HKDS-integrable with respect to g on $[a,b]$, $x^*(h_1)$ and $x^*(h_2)$ are HKS-integrable with respect to g on $[a,b]$. Hence, there exist gauges δ_1 and δ_2 on $[a,b]$ such that

$$\left| S(x^*(h_1); g; P_1) - (\text{HKS}) \int_a^b x^*(h_1) \, dg \right| < \frac{\varepsilon}{3}$$

for each δ_1-fine Perron partition P_1 of $[a,b]$, and

$$\left| S(x^*(h_2); g; P_2) - (\text{HKS}) \int_a^b x^*(h_2) \, dg \right| < \frac{\varepsilon}{3}$$

for each δ_2-fine Perron partition P_2 of $[a,b]$. Define a gauge δ on $[a,b]$ by $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$. Let $x^* \in X^* \setminus \{0\}$, $f_t \in \langle f_n \rangle_{n=1}^{\infty}$, and $P = \{\, (t_k, [u_k, v_k]) \,\}_{k=1}^{p}$ be a δ-fine Perron partition of $[a,b]$. Then P is both δ_1-fine and δ_2-fine. For each tag t_k in P,

$$x^*(h_1(t_k)) \leq x^*(f_n(t_k)) \leq x^*(h_2(t_k)),$$

and hence,

$$0 \le x^*(f_n(t_k)) - x^*(h_1(t_k)) \le x^*(h_2(t_k)) - x^*(h_1(t_k)),$$

which implies

$$0 \le x^*(f_n(t_k) - h_1(t_k)) \le x^*(h_2(t_k) - h_1(t_k)).$$

By linearity of the HKDS integral, the functions $f_n - h_1$ and $h_2 - h_1$ are HKDS-integrable with respect to g on $[a, b]$, and therefore $x^*(f_n - h_1)$ and $x^*(h_2 - h_1)$ are HKS-integrable with respect to g on $[a, b]$. Consequently,

$$0 \le S(x^*(f_n - h_1); g; P) \le S(x^*(h_2 - h_1); g; P).$$

By the triangle inequality and linearity of x^*, we have

$$\begin{aligned}
&\left| S(x^*(f_n); g; P) - (\mathrm{HKS}) \int_a^b x^*(f_n)\, dg \right| \\
&= \Bigg| S(x^*(f_n); g; P) - S(x^*(h_1); g; P) + S(x^*(h_1); g; P) \\
&\quad - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg + (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg - (\mathrm{HKS}) \int_a^b x^*(f_n)\, dg \Bigg| \\
&= \Bigg| S(x^*(f_n - h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(f_n - h_1)\, dg \\
&\quad + S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \Bigg| \\
&\le \left| S(x^*(f_n - h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(f_n - h_1)\, dg \right| \\
&\quad + \left| S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right| \\
&\le \left| S(x^*(h_2 - h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_2 - h_1)\, dg \right| \\
&\quad + \left| S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right| \\
&= \left| S(x^*(h_2); g; P) - S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_2)\, dg + (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right| \\
&\quad + \left| S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right| \\
&\le \left| S(x^*(h_2); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_2)\, dg \right| + \left| S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right| \\
&\quad + \left| S(x^*(h_1); g; P) - (\mathrm{HKS}) \int_a^b x^*(h_1)\, dg \right|
\end{aligned}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a, b]$. Finally, by Theorem 3.6, pointwise convergence plus HKDS-equiintegrability of $\langle f_n \rangle_{n=1}^{\infty}$ implies the existence of the HKDS integral of f and

$$\lim_{n\to\infty} (\text{HKDS}) \int_E f_n \, dg = (\text{HKDS}) \int_E f \, dg$$

for all compact subintervals $E \subset [a, b]$. □

Remark 3.9. Let X be a Banach space, $f_n : [a, b] \to X$, and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKPS-integrable functions with respect to g on $[a, b]$ satisfying the following conditions:

- (i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a, b] \to X$ on $[a, b]$;
- (ii) there exist HKPS-integrable functions $h_1, h_2 : [a, b] \to X$ with respect to g on $[a, b]$ such that $x^*(h_1(t)) \le x^*(f_n(t)) \le x^*(h_2(t))$ for every $n \in \mathbb{N}$, for every $x^* \in X^* \setminus \{0\}$, and for every $t \in [a, b]$.

Then $\langle f_n \rangle_{n=1}^{\infty}$ is HKPS-equiintegrable with respect to g on $[a, b]$, the integral (HKPS) $\int_E f \, dg$ exists, and

$$\lim_{n\to\infty} (\text{HKPS}) \int_E f_n \, dg = (\text{HKPS}) \int_E f \, dg$$

for all compact subintervals $E \subset [a, b]$.

The following Bounded Convergence Theorem via equiintegrability is an immediate consequence of the Dominated Convergence Theorem.

Corollary 3.10. (**Bounded Convergence Theorem**) Let X be a Banach space, $f_n : [a, b] \to X$, and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a, b]$ satisfying the following conditions:

- (i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a, b] \to X$ on $[a, b]$;
- (ii) there exists a constant $M > 0$ such that $|x^*(f_n(t))| \le M$ for all $n \in \mathbb{N}$, $x^* \in X^* \setminus \{0\}$, and $t \in [a, b]$.

Then $\langle f_n \rangle_{n=1}^{\infty}$ is HKDS-equiintegrable with respect to g on $[a, b]$, the integral (HKDS) $\int_E f dg$ exists, and

$$\lim_{n\to\infty} (\text{HKDS}) \int_E f_n \, dg = (\text{HKDS}) \int_E f \, dg$$

for all compact subintervals $E \subset [a, b]$.

Proof. The statement is a consequence of Theorem 3.8. To see this, for every $x^* \in X^* \setminus \{0\}$ we take $x^*(h_1(t)) = -M$ and $x^*(h_2(t)) = M > 0$ for all $t \in [a, b]$. We know that $|x^*(f_n(t))| \le M$, which implies $-M \le x^*(f_n(t)) \le M$ for all $n \in \mathbb{N}$ and $t \in [a, b]$. Thus, the hypothesis of Theorem 3.8 is satisfied. Hence, conclusion follows. □

Remark 3.11. Let X be a Banach space, $f_n : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKPS-integrable functions with respect to g on $[a,b]$ satisfying the following conditions:

(i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a,b] \to X$ on $[a,b]$;

(ii) there exists a constant $M > 0$ such that $|x^*(f_n(t))| \leq M$ for all $n \in \mathbb{N}$, $x^* \in X^* \setminus \{0\}$, and $t \in [a,b]$.

Then $\langle f_n \rangle_{n=1}^{\infty}$ is HKPS-equiintegrable with respect to g on $[a,b]$, the integral (HKPS) $\int_E f\, dg$ exists, and

$$\lim_{n\to\infty} (\text{HKPS}) \int_E f_n\, dg = (\text{HKPS}) \int_E f\, dg$$

for all compact subinterval $E \subset [a,b]$.

3.2. Convergence via Uniform Boundedness of Sequences of HKDS and HKPS Integral Operators.

In this section, we introduce the notion of uniform boundedness for sequences of HKDS and HKPS integrals via the Uniform Boundedness Theorem (see Kreyszig [4]). We then derive a convergence theorem based on this principle. Unlike the approach via equiintegrability, this result does not require uniform boundedness of the functions themselves. Instead, we impose a uniform bound on the norms of the corresponding HKDS and HKPS integral operators, from which convergence follows as a consequence of the Uniform Boundedness Theorem.

Definition 3.12. Let X be a Banach space, $f_n : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle$ is a sequence of HKDS-integrable functions with respect to g on $[a,b]$. For each compact subinterval $E \subset [a,b]$, define the HKDS integral operators $T_{n,E} : X^* \to \mathbb{R}$ by

$$T_{n,E}(x^*) = (\text{HKS}) \int_E x^*(f_n)\, dg$$

for all $n \in \mathbb{N}$ and all $x^* \in X^*$. By definition of the HKDS integral, each $T_{n,E} \in X^{**}$ is a bounded linear operator. Define the norm of $T_{n,E}$ by

$$\|T_{n,E}\| = \sup_{n\in\mathbb{N}} \left\{ \frac{|T_{n,E}(x^*)|}{\|x^*\|} : \|x^*\| \neq 0 \right\}.$$

We say that

(i) the sequence $\langle |T_{n,E}(x^*)| \rangle$ is *pointwise bounded* if for each $x^* \in X^*$, there exists a constant $C_{x^*,E} > 0$ such that $|T_{n,E}(x^*)| \leq C_{x^*,E}$ for all $n \in \mathbb{N}$ and all compact subintervals $E \subset [a,b]$;

(ii) the sequence of the norms $\langle \|T_{n,E}\| \rangle$ is *uniformly bounded* if there exists a constant $C > 0$ such that $\|T_{n,E}\| \leq C$ for all $n \in \mathbb{N}$ and all compact subintervals $E \subset [a,b]$.

Moreover, the same definition applies to the uniform boundedness of the HKPS integral.

Theorem 3.13. Let X be a Banach space, $f_n : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKDS-integrable functions with respect to g on $[a,b]$ satisfying the following conditions:

(i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a,b] \to X$ on $[a,b]$;

(ii) For every compact subinterval $E \subset [a,b]$, the sequence $\langle \|T_{n,E}\| \rangle_{n=1}^{\infty}$ is uniformly bounded in X^{**}.

Then f is HKDS-integrable with respect to g over $[a,b]$ and

$$\lim_{n\to\infty} T_{n,E} = T_E$$

for all compact subintervals $E \subset [a,b]$.

Proof. Let $\varepsilon > 0$ be given and fix $x^* \in X^*$. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence in HKDS$[a,b]$. By Theorem 3.1.5 in [7], for all $n \in \mathbb{N}$ and for all $x^* \in X^*$, $x^*(f_n) : [a,b] \to \mathbb{R}$ is HKS-integrable with respect to g on $[a,b]$. Thus, for every $\varepsilon > 0$ and for each $n \in \mathbb{N}$, there exists a gauge δ_n on $[a,b]$ such that for every δ_n-fine Perron partition P of $[a,b]$,

$$\left| S(x^*(f_n); g; P) - \text{(HKS)} \int_a^b x^*(f_n)\, dg \right| < \frac{\varepsilon}{3}.$$

Define a decreasing sequence of gauges recursively by

$$\gamma_1 = \delta_1, \quad \gamma_2 = \min\{\gamma_1, \delta_2\}, \quad \gamma_3 = \min\{\gamma_2, \delta_3\}, \ldots, \quad \gamma_n = \min\{\gamma_{n-1}, \delta_n\}, \ldots$$

By the Completeness Axiom, $\inf\{\gamma_n\} \in \mathbb{R}$. Fix $\gamma = \inf\{\gamma_n\} > 0$. Then, for any γ-fine Perron partition $P = \{\,(t_k, [u_k, v_k])\,\}_{k=1}^{p}$ of $[a,b]$ and for each $n \in \mathbb{N}$,

$$\left| S(x^*(f_n); g; P) - \text{(HKS)} \int_a^b x^*(f_n)\, dg \right| < \frac{\varepsilon}{3}.$$

Let $M = \sup\left\{ \sum_{[u_k,v_k]\in D} |g(v_k) - g(u_k)| \right\} < \infty$. By assumption, $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to f on $[a,b]$. Therefore, for every $\varepsilon > 0$ and for every tag $t_k \in [a,b]$, there exists $N_1 \in \mathbb{N}$ such that

$$\|f_n(t_k) - f(t_k)\| < \frac{\varepsilon}{6(\|x^*\| + 1)M}, \quad \text{for all } n \geq N_1.$$

Let $m, n \geq N_1$. Then for each tag t_k in P,

$$\begin{aligned}
\|f_n(t_k) - f_m(t_k)\| &= \|f_n(t_k) - f(t_k) + f(t_k) - f_m(t_k)\| \\
&\leq \|f_n(t_k) - f(t_k)\| + \|f(t_k) - f_m(t_k)\| \\
&< \frac{\varepsilon}{6(\|x^*\| + 1)M} + \frac{\varepsilon}{6(\|x^*\| + 1)M} \\
&= \frac{\varepsilon}{3(\|x^*\| + 1)M}.
\end{aligned}$$

Consequently, for all $m, n \geq N_1$,

$$
\begin{aligned}
&|S(x^*(f_n);g;P) - S(x^*(f_m);g;P)| = |S(x^*(f_n - f_m);g;P)| \\
&= \left| \sum_{(t_k,[u_k,v_k])\in P} \big[x^*(f_n(t_k)) - x^*(f_m(t_k))\big]\big(g(v_k) - g(u_k)\big) \right| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \Big| \big[x^*(f_n(t_k)) - x^*(f_m(t_k))\big]\big(g(v_k) - g(u_k)\big) \Big| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \big|x^*(f_n(t_k)) - x^*(f_m(t_k))\big| \, \big|g(v_k) - g(u_k)\big| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \|x^*\| \, \|f_n(t_k) - f_m(t_k)\| \, \big|g(v_k) - g(u_k)\big| \\
&= \|x^*\| \sum_{(t_k,[u_k,v_k])\in P} \|f_n(t_k) - f_m(t_k)\| \, \big|g(v_k) - g(u_k)\big| \\
&< \|x^*\| \sum_{(t_k,[u_k,v_k])\in P} \frac{\varepsilon}{3(\|x^*\|+1)M} \big|g(v_k) - g(u_k)\big| \\
&= \|x^*\| \frac{\varepsilon}{3(\|x^*\|+1)M} \sum_{(t_k,[u_k,v_k])\in P} \big|g(v_k) - g(u_k)\big| \\
&\leq \|x^*\| \frac{\varepsilon}{3(\|x^*\|+1)M} \, M \\
&< (\|x^*\|+1) \frac{\varepsilon}{3(\|x^*\|+1)M} \, M \\
&= \frac{\varepsilon}{3}.
\end{aligned}
$$

Then, for each $m, n \geq N_1$,

$$
\begin{aligned}
&\left| (\text{HKS}) \int_a^b x^*(f_n)\, dg - (\text{HKS}) \int_a^b x^*(f_m)\, dg \right| \\
&= \Big| (\text{HKS}) \int_a^b x^*(f_n)\, dg - S(x^*(f_n);g;P) + S(x^*(f_n);g;P) - S(x^*(f_m);g;P) \\
&\quad + S(x^*(f_m);g;P) - (\text{HKS}) \int_a^b x^*(f_m)\, dg \Big| \\
&\leq \left| (\text{HKS}) \int_a^b x^*(f_n)\, dg - S(x^*(f_n);g;P) \right| + |S(x^*(f_n);g;P) - S(x^*(f_m);g;P)| \\
&\quad + \left| S(x^*(f_m);g;P) - (\text{HKS}) \int_a^b x^*(f_m)\, dg \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus, $\left\langle (\text{HKS}) \int_a^b x^*(f_n)\, dg \right\rangle_{n=1}^{\infty}$ is Cauchy and hence convergent. Suppose

$$
J = \lim_{n\to\infty} \left((\text{HKS}) \int_a^b x^*(f_n)\, dg \right)
$$

. Then, for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left|(\text{HKS})\int_a^b x^*(f_n)\,dg - J\right| < \frac{\varepsilon}{3}. \tag{8}$$

Set $N = \max\{N_1, N_2\}$. Note that $x^*(f_N)$ is HKS-integrable with respect to g on $[a,b]$, and we can choose a gauge δ_N on $[a,b]$ such that for every δ_N-fine Perron partition P of $[a,b]$,

$$\left|S(x^*(f_N); g; P) - (\text{HKS})\int_a^b x^*(f_N)\,dg\right| < \frac{\varepsilon}{3}. \tag{9}$$

Furthermore, by pointwise convergence of $\langle f_n\rangle_{n=1}^{\infty}$, for every $\varepsilon > 0$ and for every tag $t_k \in [a,b]$, there exists $N \in \mathbb{N}$ such that

$$\|f_n(t_k) - f(t_k)\| < \frac{\varepsilon}{3(\|x^*\|+1)M}, \quad \text{for all } n \geq N.$$

Thus, for each $n \geq N$,

$$\begin{aligned}
&|S(x^*(f); g; P) - S(x^*(f_N); g; P)| = |S(x^*(f - f_N); g; P)| \\
&= \left|\sum_{(t_k,[u_k,v_k])\in P} \big[x^*(f(t_k)) - x^*(f_N(t_k))\big]\big(g(v_k) - g(u_k)\big)\right| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \left|\big[x^*(f(t_k)) - x^*(f_N(t_k))\big]\big(g(v_k) - g(u_k)\big)\right| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \big|x^*(f(t_k)) - x^*(f_N(t_k))\big|\,\big|g(v_k) - g(u_k)\big| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \|x^*\|\,\|f(t_k) - f_N(t_k)\|\,\big|g(v_k) - g(u_k)\big| \\
&= \|x^*\| \sum_{(t_k,[u_k,v_k])\in P} \|f(t_k) - f_N(t_k)\|\,\big|g(v_k) - g(u_k)\big| \\
&< \|x^*\| \sum_{(t_k,[u_k,v_k])\in P} \frac{\varepsilon}{3(\|x^*\|+1)M}\,\big|g(v_k) - g(u_k)\big| \\
&= \|x^*\| \frac{\varepsilon}{3(\|x^*\|+1)M} \sum_{(t_k,[u_k,v_k])\in P} \big|g(v_k) - g(u_k)\big| \\
&\leq \|x^*\| \frac{\varepsilon}{3(\|x^*\|+1)M}\, M \\
&< (\|x^*\|+1) \frac{\varepsilon}{3(\|x^*\|+1)M}\, M \\
&= \frac{\varepsilon}{3}.
\end{aligned} \tag{10}$$

From (8), for $n \geq N$, we also have

$$\left|(\text{HKS})\int_a^b x^*(f_N)\,dg - J\right| < \frac{\varepsilon}{3}. \tag{11}$$

Hence, using (9) - (11), we obtain

$$|S(x^*(f); g; P) - J| = \Big|S(x^*(f); g; P) - S(x^*(f_N); g; P) + S(x^*(f_N); g; P)$$

$$
\begin{aligned}
&\quad - (\mathrm{HKS}) \int_a^b x^*(f_N)\, dg + (\mathrm{HKS}) \int_a^b x^*(f_N)\, dg - J \Big| \\
&\le |S(x^*(f); g; P) - S(x^*(f_N); g; P)| \\
&\quad + \left| S(x^*(f_N); g; P) - (\mathrm{HKS}) \int_a^b x^*(f_N)\, dg \right| \\
&\quad + \left| (\mathrm{HKS}) \int_a^b x^*(f_N)\, dg - J \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus, $x^*(f)$ is HKS integrable with respect to g on $[a, b]$ and by Theorem 3.1.5 in [7], f is HKDS integrable with respect to g on $[a, b]$. Therefore,

$$
\lim_{n\to\infty} (\mathrm{HKS}) \int_a^b x^*(f_n)\, dg = (\mathrm{HKS}) \int_E x^*(f)\, dg = J
$$

for all $x^* \in X^*$ and $n \in \mathbb{N}$. Now fix a compact subinterval $E \subset [a, b]$. For each $n \in \mathbb{N}$, define $T_{n,E} : X^* \to \mathbb{R}$ by

$$
T_{n,E}(x^*) = (\mathrm{HKS}) \int_E x^*(f_n)\, dg, \quad \text{for all } x^* \in X^*,
$$

so that $T_{n,E} \in X^{**}$. Define

$$
\|T_{n,E}\| = \sup_{n\in\mathbb{N}} \left\{ \frac{|T_{n,E}(x^*)|}{\|x^*\|} : \|x^*\| \neq 0 \right\}.
$$

By linearity and continuity of x^*,

$$
T_{n,E}(x^*) = x^* \left((\mathrm{HKS}) \int_E f_n\, dg \right).
$$

For each fixed $x^* \in X^*$, choose a constant $C_{x^*,E} > 0$ such that

$$
\begin{aligned}
|T_{n,E}(x^*)| &= \left| x^* \left((\mathrm{HKS}) \int_E f_n\, dg \right) \right| \le \|x^*\| \left\| (\mathrm{HKS}) \int_E f_n\, dg \right\| \\
&\le \|x^*\| \sup_{n\in\mathbb{N}} \left\| (\mathrm{HKS}) \int_E f_n\, dg \right\| \le C_{x^*,E}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Such constant exists because each f_n is HKDS-integrable over the compact interval E, so the integrals are finite. In particular, $\sup_{n\in\mathbb{N}} |T_{n,E}(x^*)| \le C_{x^*,E}$. This shows that the sequence $\langle |T_{n,E}(x^*)| \rangle_{n=1}^{\infty}$ is pointwise bounded for each fixed $x^* \in X^*$. Now, observe that $\|T_{n,E}\| = \sup_{n\in\mathbb{N}} \left\{ \frac{|T_{n,E}(x^*)|}{\|x^*\|} : \|x^*\| \neq 0 \right\}$. So,

$$
\frac{|T_{n,E}(x^*)|}{\|x^*\|} = \frac{|x^*((\mathrm{HKS}) \int_E f_n\, dg)|}{\|x^*\|} \le \frac{\|x^*\| \, \|(\mathrm{HKS}) \int_E f_n\, dg\|}{\|x^*\|} \le \sup_{n\in\mathbb{N}} \left\| (\mathrm{HKS}) \int_E f_n\, dg \right\|
$$

for all $x^* \in X^* \setminus \{0\}$. By the Uniform Boundedness Theorem, pointwise boundedness implies operator norms are uniformly bounded. That is,

$$
\|T_{n,E}\| \le \sup_{n\in\mathbb{N}} \|T_{n,E}\| = C < \infty.
$$

Hence, the sequence $\langle \|T_{n,E}\| \rangle_{n=1}^{\infty}$ is uniformly bounded. This ensures that the limit operator is bounded in X^{**}. Since

$$\lim_{n\to\infty} T_{n,E}(x^*) = T_E(x^*) = (\text{HKS}) \int_E x^*(f)\, dg = \lim_{n\to\infty} (\text{HKS}) \int_E x^*(f_n)\, dg,$$

we have

$$\lim_{n\to\infty} T_{n,E} = T_E$$

for all compact subinterval $E \subset [a,b]$. □

In the following remark, we consider $\langle \|T_{n,E}\| \rangle_{n=1}^{\infty}$ as the sequence of norms of HKPS-integral operators.

Remark 3.14. Let X be a Banach space, $f_n : [a,b] \to X$, and $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of HKPS-integrable functions with respect to g on $[a,b]$ satisfying the following conditions:

(i) $\langle f_n \rangle_{n=1}^{\infty}$ converges pointwise to $f : [a,b] \to X$ on $[a,b]$;

(ii) For every compact subinterval $E \subset [a,b]$, the sequence $\langle \|T_{n,E}\| \rangle_{n=1}^{\infty}$ is uniformly bounded in X.

Then f is HKPS-integrable with respect to g over $[a,b]$ and

$$\lim_{n\to\infty} T_{n,E} = T_E$$

for all compact subintervals $E \subset [a,b]$.

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