Asia Pac. J. Math. 2026 **13**:115

WEAKLY CONNECTED GEODETIC DOMINATION IN GRAPHS UNDER SOME BINARY OPERATIONS

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Received May 25, 2026

Abstract. Let G be a simple connected graph. For $S \subseteq V(G)$, if $I_G[S] = V(G)$, where $I_G[S] = \cup_{u,v \in S} I_G[u,v]$, then S is a geodetic set. It is a dominating set if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$, i.e., $N_G[S] = V(G)$. A set S is a weakly connected geodetic dominating set in G if S is both a geodetic set and a dominating set, and the subgraph $\langle S \rangle_w = \langle N_G[S], E_w \rangle$ weakly induced by S is connected, where $E_w = \{uv \in E(G) : u \in S \text{ or } v \in S\}$. The minimum cardinality of a weakly connected geodetic dominating set in G is denoted by $\gamma_{wg}(G)$. In this paper, weakly connected geodetic dominating sets in graphs obtained from some graph operations are characterized, and their corresponding weakly connected geodetic domination numbers are determined. Moreover, the weakly connected geodetic domination numbers of some common graphs are determined.

2020 Mathematics Subject Classification. 05C12; 05C40; 05C69; 05C76.

Key words and phrases. domination; geodetic; geodetic domination; weakly connected domination; weakly connected geodetic domination.

1. Introduction

In a road network, the concept of domination focuses on monitoring all intersections using a strategically chosen set of cameras. However, it does not consider the overall structure of movement along routes. Geodetic domination addresses this by incorporating shortest-path considerations to ensure coverage along shortest routes. Additionally, weak connectivity further ensures that these monitored intersections are interconnected, forming a single, cohesive monitoring structure.

Motivated by these considerations, in this paper, we study the concept of weakly connected geodetic domination in graphs. This variant may be viewed as a combination of domination concept and geodetic properties, together with the additional requirement of weak connectivity.

DOI: 10.28924/APJM/13-115

A geodetic dominating set $S \subseteq V(G)$ is said to be weakly connected if the weakly induced subgraph $\langle S \rangle_w$ is connected. The corresponding parameter, called the weakly connected geodetic domination number of a graph G, is denoted by $\gamma_{wg}(G)$, and defined as the minimum cardinality of a weakly connected geodetic dominating set in G.

In particular, we provide some characterizations of the weakly connected geodetic dominating sets in the join, corona, and edge corona of two graphs, and determine the weakly connected geodetic domination numbers of these graph operations. Several related concepts and parameters have been studied, for example, by Dunbar et al. [5], Hansberg and Volkmann [8], Patangan et al. [11], and Hamja et al. [7].

2. Terminology and Notation

Let G be a simple connected graph and $S \subseteq V(G)$. The set of neighbors of a vertex v in G, denoted by $N_G(v)$, is called the *open neighborhood* of v in G. The open neighborhood of S in G is the set $N_G(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhood* of S in G is the set $N_G[S] = N_G(S) \cup S$. If $N_G[S] = V(G)$, i.e., for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$, then S is a *dominating set* of G [3]. The graph whose vertex set is $N_G[S]$ and whose edge set is $E_w = \{uv \in E(G) : u \in S \text{ or } v \in S\}$ is called a *weakly induced subgraph* of G and is denoted by $\langle S \rangle_w$ [11]. The graph whose vertex set is S and whose edge set comprises exactly the edges in G which join vertices in S is termed an *induced subgraph* of G and is denoted by $\langle S \rangle$ [3]. A graph is said to be *connected* if there is at least one path that connects every two of its vertices [3]. A dominating set S is a *weakly connected dominating set* if $\langle S \rangle_w$ is connected. The minimum cardinality among the weakly connected dominating sets in G is called the *weakly connected domination number* of G and is denoted by $\gamma_w(G)$. If $|S| = \gamma_w(G)$, we say that S is a γ_w-set of G [12].

For two vertices $u, v \in V(G)$, the closed interval $I_G[u, v]$ is the set consisting of u, v, and all vertices lying in some u-v *geodesic* (shortest u-v path) [2]. If $S \subseteq V(G)$, then $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$ is called the *geodetic closure* of S in G [11]. If $I_G[S] = V(G)$, then S is a *geodetic set* of G. If a dominating set S of G is a geodetic set of G, then S is a *geodetic dominating set* of G [12]. If S is a geodetic dominating set of G and $\langle S \rangle_w$ is connected, then S is a *connected geodetic dominating set* of G.

A vertex v in a graph G is an *extreme vertex* if $\langle N_G(v) \rangle$ is complete [4]. The 2-*path closure* $P_2[S]_G$ of $S \subseteq V(G)$ is the set $P_2[S]_G = S \cup \{x \in V(G) : x \in I_G(u, v) \text{ with } d_G(u, v) = 2 \text{ for some } u, v \in S\}$. If $P_2[S]_G = V(G)$, then S is called a 2-*path closure absorbing set* of G [1]. The minimum cardinality of a 2-path closure absorbing set in G is called the 2-*path closure absorbing number* of G and is denoted by $\eta_2(G)$. If $|S| = \eta_2(G)$, we say that S is an η_2-set of G.

3. Results

Definition 1. *A set $S \subseteq V(G)$ is a weakly connected geodetic dominating set of G if S is a geodetic dominating set of G and $\langle S \rangle_w$ is connected. The minimum cardinality among the weakly connected geodetic dominating sets in G is called the weakly connected geodetic domination number of G and is denoted by $\gamma_{wg}(G)$. If $|S| = \gamma_{wg}(G)$, we say that S is a γ_{wg}-set of G.*

Example 1. Consider the graph G in Figure 1. The set $S = \{v_4, v_5, v_6, v_7\}$ is a weakly connected geodetic dominating set in G. Indeed, $N_G[S] = V(G) = I_G[S]$ and $\langle S \rangle_w$ is connected. On the other hand, the set $T = \{v_1, v_4, v_7\}$ is not weakly connected geodetic dominating set in G because $\langle T \rangle_w$ is disconnected. It can easily be verified that there is no weakly connected geodetic dominating set in G of cardinality strictly less than 4. Therefore, S is a γ_{wg}-set of G and $\gamma_{wg}(G) = 4$.

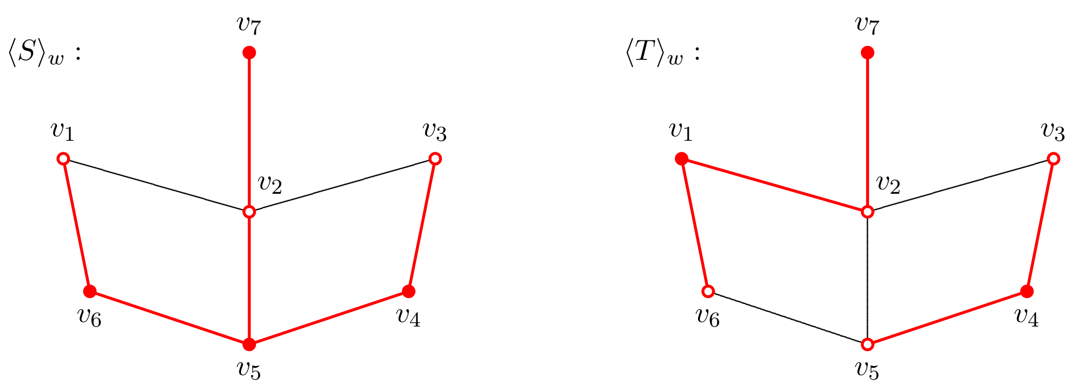

Figure 1. A connected graph G with $\gamma_{wg}(G) = 4$

Theorem 1. *Every connected geodetic dominating set of a graph G is a weakly connected geodetic dominating set of G.*

Proof. Assume that $\langle S \rangle$ is connected. Let $x, y \in N_G[S]$ such that $x \neq y$. If $x, y \in S$, then $[x, y]$ is an x-y path in $\langle S \rangle_w$. Suppose that $x \in S$ and $y \notin S$. Then $y \in N_G(S)$. Hence, there exists $z \in S \cap N_G(y)$. Since $\langle S \rangle$ is connected, there exists an x-z path $P(x, z) = [x_1, x_2, ..., x_k]$, where $x_1 = x$ and $x_k = z$, in $\langle S \rangle$. It follows that $P(x, y) = [x_1, x_2, ..., x_k, y]$ is an x-y path in $\langle S \rangle_w$. Finally, suppose that $x, y \in N_G(S)$. Then there exist $s, t \in S$ such that $xs, yt \in E(G)$. Let $P(s, t) = [s_1, s_2, ..., s_m]$, where $s_1 = s$ and $s_m = t$, be an s-t path in $\langle S \rangle$. Then $P(x, y) = [x, s_1, s_2, ..., s_m, y]$ is an x-y path in $\langle S \rangle_w$. □

The next result is a consequence of Theorem 1.

Corollary 1. *For any connected graph G, $\gamma_{wg}(G) \leq \gamma_{cg}(G)$.*

Theorem 2. *For any connected graph G,*

$$\max\{\gamma_w(G), \gamma_g(G)\} \leq \gamma_{wg}(G) \leq \gamma_{cg}(G).$$

Proof. Let S be a weakly connected geodetic dominating set of G. Then S is both weakly connected dominating and a geodetic dominating set. Hence,

$$\max\{\gamma_w(G), \gamma_g(G)\} \leq \gamma_{wg}(G).$$

Moreover, by Corollary 1,

$$\gamma_{wg}(G) \leq \gamma_{cg}(G).$$

This proves the assertion. □

Remark 1. *Let G be a connected graph and let S be a weakly connected geodetic dominating set of G. Then $Ext(G) \subseteq S$, where $Ext(G)$ is the set containing all the extreme vertices of G.*

Theorem 3. *Let G be a connected graph of order n. Then each of the following holds:*

(i) $\gamma_{wg}(G) = 1$ *if and only if* $G = K_1$.

(ii) $\gamma_{wg}(G) = 2$ *if and only if* $G = K_2$ *or* $G = \overline{K}_2 + H$ *for some graph* H *of order* $n-2$,*where* $n \geq 3$.

(iii) $\gamma_{wg}(G) = n$ *if and only if* $G = K_n$.

Proof. (i) Suppose $\gamma_{wg}(G) = 1$, say $S = \{v\}$ is a γ_{wg}-set of G. Since S is a geodetic set of G, we must have $G = K_1$.

Conversely, if $G = K_1$, then $\gamma_{wg}(G) = 1$.

(ii) Suppose that $\gamma_{wg}(G) = 2$. Then there exists a γ_{wg}-set $S = \{u, v\}$. Since S is a geodetic set of G, $I_G[S] = V(G)$. If $uv \in E(G)$, then $I_G[S] = \{u, v\} = V(G)$. Hence, $G = K_2$. Now, suppose that $uv \notin E(G)$ and let $w \in V(G) \setminus \{u, v\}$. Since S is weakly connected, it follows that $|V(G)| = n \geq 3$. Also, since S is a geodetic dominating set of G, it follows that $w \in N_G(u) \cap N_G(v)$. Thus, $G = \overline{K}_2 + H$, where $H = \langle V(G) \setminus \{u, v\}\rangle$.

Conversely, suppose $G = K_2$. Then $V(G)$ is a weakly connected geodetic dominating set of G, and so $\gamma_{wg}(G) = 2$. Let $G = \overline{K}_2 + H$ for some graph H of order $n-2$, where $n \geq 3$ and let $S = V(\overline{K}_2)$. Then S is a weakly connected geodetic dominating set of G. Therefore, $\gamma_{wg}(G) = 2$.

(iii) Suppose that $\gamma_{wg}(G) = n$ and assume that $G \neq K_n$. Then there exist $u, v \in V(G)$ such that $d_G(u, v) = 2$. Let $x \in N_G(u) \cap N_G(v)$. Then $S = V(G) \setminus \{x\}$ is clearly a weakly connected geodetic dominating set in G. Hence, $\gamma_{wg}(G) \leq |S| = n - 1$, a contradiction. Thus, $G = K_n$.

Conversely, let $G = K_n$ and let S be a γ_{wg}-set of G. If $n = 1$, then $\gamma_{wg}(G) = 1$ by part (i). Let $n \geq 2$. Since $V(G) = Ext(G)$, it follows from Remark 1 that $V(G) \subseteq S$. Therefore, $S = V(G)$ and $\gamma_{wg}(G) = n$. □

Observation 1. *Each of the following holds:*

(i) $\gamma_{wg}(P_n) = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$ *for all* $n \geq 1$.

(ii) $\gamma_{wg}(C_n) = \lfloor \frac{n}{2} \rfloor$ *for all* $n \geq 6$.

The *join* $G+H$ of two graphs G and H is the graph with $V(G+H) = V(G) \dot{\cup} V(H)$ and $E(G+H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G) \text{ and } v \in V(H)\}$ [9].

Theorem 4. *Let G and H be non-complete graphs, and let $S \subseteq V(G+H)$. Then S is a weakly connected geodetic dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ and one of the following holds:*

(i) *S_G is a 2-path closure absorbing set in G and $S_H = \varnothing$ or $\langle S_H \rangle$ is complete in H;*

(ii) *S_H is a 2-path closure absorbing set in H and $S_G = \varnothing$ or $\langle S_G \rangle$ is complete in G; or*

(iii) *$\langle S_G \rangle$ and $\langle S_H \rangle$ are non-complete subgraphs of G and H, respectively.*

Proof. Assume that S is a weakly connected geodetic dominating set of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$. Suppose that $\langle S_G \rangle$ and $\langle S_H \rangle$ are non-complete. Then (iii) holds. Next, assume that (iii) does not hold. Suppose that $S_H = \varnothing$ or $\langle S_H \rangle$ is complete. Let $x \in V(G) \setminus S_G$. Since S is a geodetic set in $G+H$ and $S_H = \varnothing$ or $\langle S_H \rangle$ is complete, there exists $p, q \in S_G \subseteq S$ such that $x \in I_{G+H}(p,q)$ where $d_{G+H}(p,q) = 2$. It follows that $p, q \in S_G$, $d_G(p,q) = 2$ and $x \in I_G(p,q)$. Hence, S_G is a 2-path closure absorbing set in G and (i) holds. Similarly, if $S_G = \varnothing$ or $\langle S_G \rangle$ is complete, then S_H is a 2-path closure absorbing set in H and (ii) holds.

For the converse, suppose first that (i) holds, i.e., suppose that S_G is a 2-path closure absorbing set in G and $S_H = \varnothing$ or $\langle S_H \rangle$ is complete. Then clearly, S is a weakly connected dominating set in $G+H$. Note that since S_G is a 2-path closure absorbing set in G and G is non-complete, $\langle S_G \rangle$ is not complete. Let $a, b \in S_G$ such that $d_G(a,b) \neq 1$. Then $d_{G+H}(a,b) = 2$ and $q \in I_{G+H}(a,b)$ for all $q \in V(H) \setminus S_H$. Now, let $p \in V(G) \setminus S_G$. Since S_G is 2-path closure absorbing set in G, it follows that there exist $v, w \in S_G$ such that $d_G(v,w) = 2$ and $q \in I_G(v,w)$. Thus, S is a geodetic set in $G+H$. Similarly, S is a weakly connected geodetic dominating set of $G+H$ if (ii) holds. Finally, suppose that (iii) holds. Then $S = S_G \cup S_H$ is a weakly connected dominating set of $G+H$. Let $c, d \in S_G$ and $s, t \in S_H$ such that $d_G(c,d) \neq 1$ and $d_H(s,t) \neq 1$. Then $d_{G+H}(c,d) = 2$, $d_{G+H}(s,t) = 2$, $g \in I_{G+H}(s,t)$, and $h \in I_{G+H}(c,d)$ for all $g \in V(G) \setminus S_G$ and for all $h \in V(H) \setminus S_H$. This shows that S is a geodetic set in $G+H$. $\square$

Corollary 2. *Let G and H be non-complete graphs. Then*

$$\gamma_{wg}(G+H) = \min\{\eta_2(G), \eta_2(H), 4\}.$$

Proof. Let S_G be an η_2-set of G and let $S_H = \varnothing$. Then $S = S_G$ is a weakly connected geodetic dominating set of $G+H$ by Theorem 4 (i). Thus, $\gamma_{wg}(G+H) \leq |S| = |S_G| = \eta_2(G)$. Similarly, if S_H is an η_2-set of

H and $S_G = \varnothing$, then $\gamma_{wg}(G+H) \leq |S| = |S_H| = \eta_2(H)$. Since G and H are non-complete, there exist two nonadjacent vertices $u, v \in V(G)$ and $x, y \in V(H)$. Let $S_G = \{u, v\}$ and $S_H = \{x, y\}$. Then $\langle S_G \rangle$ and $\langle S_H \rangle$ are non-complete subgraphs of G and H, respectively. By Theorem 4 (iii), $S = S_G \cup S_H$ is a weakly connected geodetic dominating set of $G+H$. Thus, $\gamma_{wg}(G+H) \leq |S| = 4$. Hence,

$$\gamma_{wg}(G+H) \leq \min\{\eta_2(G), \eta_2(H), 4\}. \tag{1}$$

Now, let S be a γ_{wg}-set of $G+H$, so that $|S| = \gamma_{wg}(G+H)$. By Theorem 4, $S = S_G \cup S_H$, where $S_G \subseteq V(G)$, $S_H \subseteq V(H)$, and satisfy (i), (ii), or (iii). If (i) is satisfied, then $\eta_2(G) \leq |S_G| \leq |S| = \gamma_{wg}(G+H)$. If (ii) is satisfied, then $\eta_2(H) \leq |S_H| \leq |S| = \gamma_{wg}(G+H)$. If (iii) is satisfied, then $4 \leq |S_G| + |S_H| = |S| = \gamma_{wg}(G+H)$. Therefore,

$$\min\{\eta_2(G), \eta_2(H), 4\} \leq \gamma_{wg}(G+H). \tag{2}$$

Combining the inequalities in (1) and (2), we obtain the desired equality. □

The next result follows from Theorem 4.

Corollary 3. *Let G and H be non-complete graphs. Then each of the following statements holds:*

(i) $\gamma_{wg}(G+H) = 2$ *if and only if* $\eta_2(G) = 2$ *or* $\eta_2(H) = 2$.

(ii) $\gamma_{wg}(G+H) = 3$ *if and only if* $\eta_2(G) = 3$ *and* $\eta_2(H) \geq 3$ *or* $\eta_2(H) = 3$ *and* $\eta_2(G) \geq 3$.

Example 2. Consider the join $C_5 + P_4$ in Figure 2. Let $S_{C_5} = \{a, c, e\}$ and $S_{P_4} = \{v, y\}$. Clearly, $S = S_{C_5} \cup S_{P_4} = \{a, c, e, v, y\}$ is a weakly connected geodetic dominating set of $C_5 + P_4$. However, since $\eta_2(C_5) = \eta_2(P_4) = 3$, it follows from Corollary 3 that $\gamma_{wg}(C_5 + P_4) = 3$.

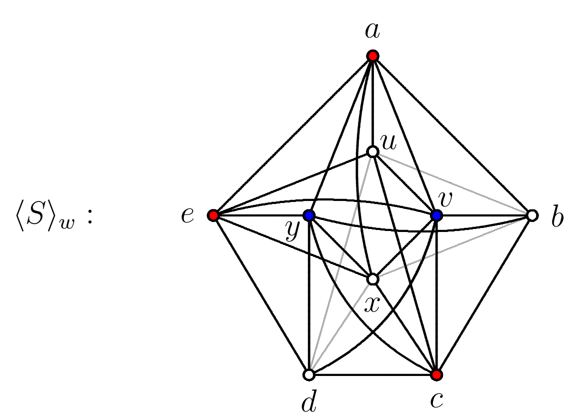

FIGURE 2. The join $C_5 + P_4$ with $\gamma_{wg}(C_5 + P_4) = 3$

Theorem 5. *Let $G = K_n$ and H be a non-complete graph. Then $S \subseteq V(G+H)$ is a weakly connected geodetic dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and S_H is a 2-path closure absorbing set of H.*

Proof. Let S be a weakly connected geodetic dominating set of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$. Let $x \in V(H) \setminus S_H$. Since S is a geodetic set of $G+H$, it follows that there exist $u, v \in S$ such that $x \in I_{G+H}(u,v)$. Since S_G induces a complete graph or $S_G = \varnothing$, we must have $u, v \in S_H$. Also, since $d_{G+H}(u,v) = 2$ and $x \in I_{G+H}(u,v)$, it follows that $d_H(u,v) = 2$ and $x \in I_H(u,v)$. Hence, S_H is a 2-path closure absorbing set in H.

Conversely, suppose $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and S_H is a 2-path closure absorbing set of H. Suppose S_H induces a complete graph. Since H is non-complete, choose $a, b \in V(H)$ such that $d_H(a,b) \neq 1$. Then at least one of a and b, say a, is not in S_H. This implies that $a \notin I_H(x,y)$ for all $x, y \in S_H$. This shows that S_H is not 2-path closure absorbing, a contradiction. Thus, S_H induces a non-complete graph. Let $s, t \in S_H$ such that $d_H(s,t) \neq 1$. Then $p \in I_{G+H}(s,t)$ for all $p \in V(G) \setminus S_G$. Since S_H is also 2-path closure absorbing in H, it follows that S is a geodetic set of $G+H$. Moreover, since every 2-path closure absorbing set is a dominating set, it follows that S is a dominating set of $G+H$. Now, let $c, d \in N_{G+H}[S] = V(G+H)$ such that $c \neq d$. Suppose $c, d \in V(G)$. If $c \in S_G$ or $d \in S_G$, then $[c,d]$ is a c-d geodesic in $\langle S \rangle_w$. Suppose $c, d \notin S_G$. Let $q \in S_H$. Then $[c,q,d]$ is a c-d geodesic in $\langle S \rangle_w$. Suppose now that $c, d \in V(H)$. If $c \in S_H$ or $d \in S_H$, then $[c,d]$ is a c-d geodesic in $\langle S \rangle_w$ if $cd \in E(H)$. Suppose $cd \notin E(H)$. If $c, d \in S_H$, then $[c,q,d]$ is a c-d geodesic in $\langle S \rangle_w$ for $q \in V(G)$. Suppose $c \in S_H$ and $d \notin S_H$. If $S_G \neq \varnothing$, then $[c,u,d]$ is a c-d geodesic in $\langle S \rangle_w$ for $u \in S_G$. Suppose $S_G = \varnothing$. Since S_H is dominating in H, there exists $w \in S_H \cap N_H(d)$. Hence, $[c,t,w,d]$ is a c-d path in $\langle S \rangle_w$ for $t \in V(G)$. Suppose now that $c, d \notin S_H$ (c and d are not necessarily adjacent). If $S_G \neq \varnothing$, then we may choose any $s \in S_G$. Clearly, $[c,s,d]$ is a c-d geodesic in $\langle S \rangle_w$. So suppose $S_G = \varnothing$. If there exists $p \in S_H$ such that $p \in N_H(c) \cap N_H(d)$, then $[c,p,d]$ is a c-d geodesic in $\langle S \rangle_w$. Suppose $[N_H(c) \cap N_H(d)] \cap S_H = \varnothing$. Since S_H is a dominating set in H, choose $r, q \in S_H$ such that $r \in N_G(c)$ and $q \in N_G(d)$. Then $[c,r,g,q,d]$, where $g \in V(G)$, is a c-d path in $\langle S \rangle_w$. Therefore, $\langle S \rangle_w$ is connected. □

The next result is a consequence of Theorem 5.

Corollary 4. *$\gamma_{wg}(K_n + H) = \eta_2(H)$ for any non-complete graph H. In particular, each of the following holds:*

(i) $\gamma_{wg}(S_n) = \gamma_{wg}(K_{1,n-1}) = n-1$ *for all* $n \geq 3$.
(ii) $\gamma_{wg}(W_n) = \gamma_{wg}(K_1 + C_{n-1}) = \lfloor \frac{n}{2} \rfloor$ *for all* $n \geq 5$.
(iii) $\gamma_{wg}(F_n) = \gamma_{wg}(K_1 + P_{n-1}) = \lceil \frac{n}{2} \rceil$ *for all* $n \geq 4$.

Let G and H be graphs of orders m and n, respectively. The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order m and m copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$ [9].

Theorem 6. *Let G be a connected graph of order $m \geq 2$ and let H be any graph. Then $S \subseteq V(G \circ H)$ is a weakly connected geodetic dominating set of $G \circ H$ if and only if*

$$S = S^* \cup \bigcup_{v \in V(G)} S^v, \tag{3}$$

where $S^* \subseteq V(G)$ and $S^v \subseteq V(H^v)$ for each $v \in V(G)$ such that the following conditions hold:

(i) S^* is a weakly connected dominating set of G; and

(ii) for every $v \in V(G)$, S^v is a 2-path closure absorbing set of H^v.

Proof. Assume that $S \subseteq V(G \circ H)$ is a weakly connected geodetic dominating set of $G \circ H$. Let $S^* = S \cap V(G)$ and $S^v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = S^* \cup \bigcup_{v \in V(G)} S^v$. Note that since S is a geodetic set, it follows that $S^v \neq \varnothing$ for every $v \in V(G)$. Suppose there exists $v \in V(G) \setminus N_G[S^*]$. Let $u \in V(G) \setminus \{v\}$. Then $\langle S^v \rangle_w$ and $\langle S^u \rangle_w$ are in distinct components of $\langle S \rangle_w$. Thus, $\langle S \rangle_w$ is not connected, a contradiction. Hence, S^* is a dominating set of $G \circ H$. Next, suppose that $\langle S^* \rangle_w$ is not connected. Then there exist distinct vertices x and y of G such that there exists no x-y path in $\langle S^* \rangle_w$. Choose any $p \in S^x$ and $q \in S^y$. Then there is no p-q path in $\langle S \rangle_w$, contrary to the assumption that S is weakly connected. Therefore, S^* is weakly connected dominating in G, showing that (i) holds. Moreover, since S is a geodetic set of $G \circ H$, it follows that S^v is a 2-path closure absorbing set of H^v for each $v \in V(G)$. This shows that (ii) also holds.

For the converse, suppose that S has the form (3) satisfying (i) and (ii). Let $x \in V(G \circ H) \setminus S$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Suppose $x \in V(H^v) \setminus S^v$. Since S^v is 2-path closure absorbing by (ii), there exists $p, q \in S^x$ such that $d_{H^v}(p,q) = d_{G \circ H}(p,q) = 2$ and $x \in I_{H^v}(p,q)$, where $I_{H^v}(p,q) = I_{G \circ H}(p,q)$. Suppose $x = v$. By (i), choose any $z \in S^* \cap N_G(x)$. Also, pick any $t \in S^v$. Then $t, z \in S$ and $x \in I_{G \circ H}(z,t)$. Therefore, S is a geodetic dominating set of $G \circ H$. Finally, since S^* is weakly connected, it is routine to show that S is also weakly connected. $\square$

Corollary 5. *Let G be a connected graph of order $m \geq 2$, and let H be any graph of order n. Then*

$$\gamma_{wg}(G \circ H) = \gamma_w(G) + m \cdot \eta_2(H).$$

Proof. Let S be a γ_{wg}-set of $G \circ H$. By Theorem 6, $S = S^* \cup \bigcup_{v \in V(G)} S^v$, where S^* is a weakly connected dominating set of G, and each S^v is a 2-path closure absorbing set of $H^v \cong H$. Thus,

$$\gamma_{wg}(G \circ H) = |S| = |S^*| + \sum_{v \in V(G)} |S^v| \geq \gamma_w(G) + m \cdot \eta_2(H). \tag{4}$$

Now, let D^* be a γ_w-set of G, and let D^v be an η_2-set of H^v for each $v \in V(G)$. Then $D = D^* \cup \bigcup_{v\in V(G)} D^v$ is a weakly connected geodetic dominating set of $G \circ H$ by Theorem 6 This implies that

$$\gamma_{wg}(G \circ H) \leq |D| = \gamma_w(G) + m \cdot \eta_2(H). \tag{5}$$

Combining (4) and (5), we obtain

$$\gamma_{wg}(G \circ H) = \gamma_w(G) + (m \cdot \eta_2(H)).$$

This proves the assertion. □

The next result directly follows from Corollary 5 because $\gamma_w(K_m) = 1$ and $\eta_2(K_n) = n$ for all positive integers m and n.

Corollary 6. *Let G and H be complete graphs of orders m and n, respectively. Then*

$$\gamma_{wg}(G \circ H) = 1 + mn.$$

Example 3. Consider the corona $P_4 \circ C_4$ in Figure 3. The set $S = \{u, x, a^u, c^u, a^v, c^v, a^x, c^x, a^y, c^y\}$ is a weakly connected geodetic dominating set in the corona $P_4 \circ C_4$. Since $S^* = \{u, x\}$ is a γ_w-set of P_4 and the sets $S^u = \{a^u, c^u\}$, $S^v = \{a^v, c^v\}$, $S^x = \{a^x, c^x\}$, and $S^y = \{a^y, c^y\}$ are η_2-sets of C_4^u, C_4^v, C_4^x, and C_4^y, respectively, it follows that S is a γ_{wg}-set of $P_4 \circ C_4$. Hence, $\gamma_{wg}(P_4 \circ C_4) = \gamma_w(P_4) + 4 \cdot \eta_2(C_4) = 2 + 4(2) = 10$.

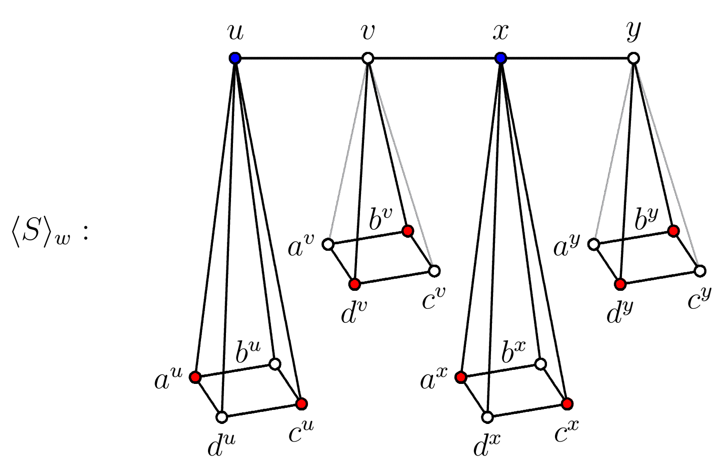

FIGURE 3. The corona $P_4 \circ C_4$ with $\gamma_{wg}(P_4 \circ C_4) = 10$

Let G and H be two graphs on disjoint sets of p_1 and p_2 vertices, q_1 and q_2 edges, respectively. The *edge corona* $G \diamond H$ of G and H is defined as the graph obtained by taking one copy of G and $q_1 = |E(G)|$ copies of H, and then joining the two end-vertices of the ith edge of G to every vertex in the ith copy of H [10]. If $uv \in E(G)$, then the copy H whose vertices are connected one by one to both u and v in $G \diamond H$ is called the *uv-copy* of H and is denoted by H^{uv}. Moreover, if $V(H) = \{v_1, v_2, ..., v_{p_2}\}$, then the vertices of H^{uv} may be denoted by $v_1^{uv}, v_2^{uv}, ..., v_{p_2}^{uv}$. Note that $\langle\{u,v\}\rangle + H^{uv} \cong P_2 + H$ for every $uv \in E(G)$.

Theorem 7. *Let G be a connected graph of size $p_1 \geq 1$, and let H be any non-complete graph. Then $S \subseteq V(G \diamond H)$ is a weakly connected geodetic dominating set of $G \diamond H$ if and only if, $S = A \cup \bigcup_{v \in V(G)} D^{uv}$, where $A \subseteq V(G)$ and D^{uv} is a 2-path closure absorbing set of H^{uv} for every $uv \in E(G)$.*

Proof. Suppose that $S \subseteq V(G \diamond H)$ is a weakly connected geodetic dominating set in $G \diamond H$. Let $A = S \cap V(G)$ and let $D^{uv} = S \cap V(H^{uv})$ for every $uv \in E(G)$. Then $S = A \cup \bigcup_{v \in V(G)} D^{uv}$. Let $uv \in E(G)$ and let $a^{uv} \in V(H^{uv}) \setminus D^{uv}$. Since S is a geodetic set in $G \diamond H$, there exist $x, y \in S$ such that $a^{uv} \in I_{G \diamond H}(x, y)$. This implies that $x, y \in D^{uv}$, $d_{H^{uv}+uv}(x, y) = 2$, and $a^{uv} \in I_{H^{uv}}(x, y)$. Hence, D^{uv} is a 2-path closure absorbing set in H^{uv}.

For the converse, suppose that S is as described and that D^{uv} is a 2-path closure absorbing set in H^{uv} for every $uv \in E(G)$. Let $x \in V(G \diamond H) \setminus S$ and let $ab \in E(G)$ such that $x \in V(\langle\{u,v\}\rangle + H^{uv})$. Consider the following cases:

Case 1. $x \in V(H^{uv}) \setminus D^{uv}$.

Since D^{uv} is a 2-path closure absorbing set in H^{uv}, there exist $a^{uv}, b^{uv} \in D^{uv} \subseteq S$ such that $x \in I_{G \diamond H}(a^{uv}, b^{uv})$ with $d_{G \diamond H}(a^{uv}, b^{uv}) = 2$.

Case 2. $x \in V(G) \setminus A$, say $x = u$.

Since D^{uv} is a 2-path closure absorbing set in H^{uv} and H is non-complete, D^{uv} induces a non-complete graph. Let $p, q \in D^{uv}$ such that $d_{H^{uv}}(p, q) \neq 1$. Then $x \in I_{G \diamond H}(p, q)$. Clearly, $x \in N_{G \diamond H}(p) \cap N_{G \diamond H}(q)$.

Therefore, S is a geodetic dominating set of $G \diamond H$. Moreover, since G is connected, the definition of $G \diamond H$ would imply that $\langle S\rangle_w$ is connected. Therefore, S is weakly connected geodetic dominating set in $G \diamond H$. □

Corollary 7. *Let G be a non-trivial graph of size $p_1 \geq 1$, and let H be a non-complete graph. Then*

$$\gamma_{wg}(G \diamond H) = p_1 \eta_2(H).$$

Proof. Suppose that S is a γ_{wg}-set of $G \diamond H$. By Theorem 7, $S = A \cup (\bigcup_{uv \in E(G)} D^{uv})$, where $A \subseteq V(G)$ and D^{uv} is 2-path closure absorbing set of H^{uv} for every $uv \in E(G)$. Hence,

$$\gamma_{wg}(G \diamond H) = |S| \geq p_1 \eta_2(H).$$

Next, let S^{uv} be an η_2-set of H^{uv} for each $uv \in E(G)$. Then $C = \bigcup_{uv \in E(G)} S^{uv}$ is a weakly connected geodetic dominating set in $G \diamond H$ by Theorem 7. This implies that

$$\gamma_{wg}(G \diamond H) \geq |C| = p_1 \eta_2(H).$$

This establishes the desired equality. □

Example 4. Consider the edge corona $P_4 \diamond C_4$ in Figure 4. The set $S = \{a^{uv}, c^{uv}, a^{vx}, c^{vx}, a^{xy}, c^{xy}\}$ is a weakly connected geodetic dominating set in $P_4 \diamond C_4$. Since $S^{uv} = \{a^{uv}, c^{uv}\}$, $S^{vx} = \{a^{vx}, c^{vx}\}$, and $S^{xy} = \{a^{xy}, c^{xy}\}$ are η_2-sets of C_4^{uv}, C_4^{vx}, and C_4^{xy}, respectively, it follows that $\gamma_{wg}(P_4 \diamond C_4) = 3 \cdot 2 = 6$.

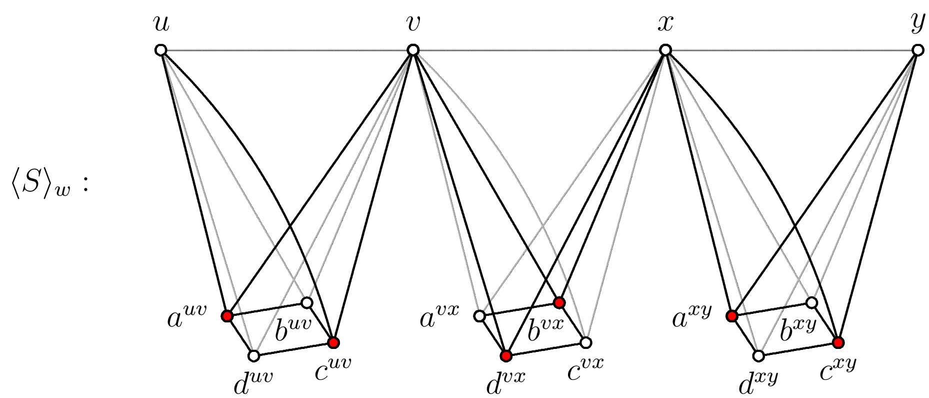

FIGURE 4. The edge corona $P_4 \diamond C_4$ with $\gamma_{wg}(P_4 \diamond C_4) = 6$

Theorem 8. *Let G be a connected graph of size q_1 with $\delta(G) \geq 2$ and let H be a complete graph of order $n \geq 2$. Then $\gamma_{wg}(G \diamond H) = nq_1$.*

Proof. Consider the set $S = \bigcup_{uv \in E(G)} V(H^{uv})$. Clearly, S is a weakly connected dominating set of $G \diamond H$. Let $v \in V(G)$. Since $\delta(G) \geq 2$, there exist distinct vertices $y, z \in N_G(v)$. Pick any $p \in V(H^{vy})$ and $q \in V(H^{vy})$. Then $p, q \in S$, $d_{G \diamond H}(p, q) = 2$, and $v \in I_{G \diamond H}(p, q)$. Thus, S is a geodetic set in $G \diamond H$. It follows that

$$\gamma_{wg}(G \diamond H) \leq nq_1.$$

Next, suppose that C is be a γ_{wg}-set of $G \diamond H$. Let $A = V(G) \cap C$ and let $R^{uv} = C \cap V(H^{uv})$ for each $uv \in E(G)$. Since C is a geodetic set in $G \diamond H$, we must have $S^{uv} = V(H^{uv})$ for every $uv \in E(G)$. Thus,

$$\gamma_{wg}(G \diamond H) = |C| \geq nq_1.$$

Therefore, $\gamma_{wg}(G \diamond H) = nq_1$. □

Acknowledgment. The authors would like to thank the referees for reading the initial manuscript and, subsequently, for giving their comments and suggestions. The authors are grateful to the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines and the MSU-Iligan Institute of Technology, Iligan City for funding this research.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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