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A HANKEL-MATRIX ERO FRAMEWORK FOR PADÉ APPROXIMATION OF STIELTJES SERIES

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Abstract. We develop a matrix elementary row operations (ERO) framework for the diagonal and subdiagonal Padé approximants of a *Stieltjes series* $f(z) = \sum_{k\geq 0}(-1)^k \mu_k z^k$, continuing the ERO approach to continued fractions of [11]. The central object is the Hankel moment matrix $\mathcal{H}_n = (\mu_{i+j})_{i,j=0}^{n-1}$, whose Gaussian (ERO) reduction is *unimodular* – the reducing matrix has determinant 1 – with pivots equal to ratios of Hankel determinants. We prove: (i) a *pivot-positivity criterion*, that all ERO pivots of $\mathcal{H}_n^{(0)}$ and $\mathcal{H}_n^{(1)}$ are positive if and only if (μ_k) is a Stieltjes moment sequence, in which case every $[n-1/n]$ and $[n/n]$ Padé approximant exists and is unique and $|\det \mathcal{S}_{n,n}| = H_n^{(1)}$; (ii) a *two-sided bracketing theorem*, $[n-1/n]_f(x) \leq f(x) \leq [n/n]_f(x)$ for every Stieltjes function f and $x > 0$; and (iii) a *certified a posteriori error bound*, in which the computable gap $[n/n]_f(x) - [n-1/n]_f(x)$ majorises the error of both approximants. We show that $1 - e^{-at}$ fails the Stieltjes hypothesis, explaining why only one-sided bounds hold for it. Two verified applications are given: certified evaluation of $\log(1+z)/z$ and summation of the divergent Euler series $\sum_{k\geq 0}(-1)^k k!\, z^k$.

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1. Introduction

Rational approximation is, for many functions, far more accurate than polynomial approximation, and the Padé approximants $R_{m,n} = P_m/Q_n$ – the rational functions whose Taylor expansion matches a given series to maximal order – are the canonical instance [2,3]. Their connection with continued fractions and orthogonal polynomials is classical [4,10].

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These objects can be studied through a single computational device: the reduction of an augmented matrix by elementary row operations (EROs). The method was introduced over $\mathbb{Z}$ for finite continued fractions in [11], with a determinant invariant $\det(A_k) = (-1)^k$ encoding the convergents and an error estimate. The same augmented-matrix idea applies to the Padé–Sylvester matrix and computes the Padé table; in that setting one obtains, for instance, an error *bound* for the exponential learning function $L(t) = 1 - e^{-at}$. The present paper develops the Padé theory in the direction of two-sided, certified bounds for the Stieltjes class.

1.1. **The gap addressed here.** Such a bound is one-sided: it controls $|L(t) - R_{m,n}(t)|$ from above but provides no lower envelope, and it is tied to a single function. The reason, as we shall see (Remark 4.3), is structural: $1 - e^{-at}$ is *not* a Stieltjes series. For the large and important class of functions that *are* Stieltjes – those admitting an integral representation

$$f(z) = \int_0^\infty \frac{d\psi(u)}{1 + zu}, \qquad z \in \mathbb{C} \setminus (-\infty, 0), \tag{1.1}$$

with ψ a positive measure of infinite support and finite moments $\mu_k = \int_0^\infty u^k \, d\psi(u)$ – the Padé table possesses a much stronger structure: the diagonal and subdiagonal approximants *bracket* f from both sides, and the bracket is itself a certified, computable error bar. This classical phenomenon, going back to Stieltjes [9], is usually presented through the analytic theory of continued fractions or orthogonal polynomials. The purpose of the present paper is to give it a transparent *matrix* derivation within the ERO programme, and thereby to turn the qualitative bracketing into a quantitative tool.

1.2. **Main contributions.**

(1) **Unimodular Hankel–ERO reduction** (Lemma 3.1): the Gaussian elimination of the Hankel moment matrix $\mathcal{H}_n$ is realised by a unit lower-triangular (hence unimodular) ERO matrix E, with pivots $d_k = H_k/H_{k-1}$ and $\det \mathcal{H}_n = \prod_k d_k = H_n$.

(2) **Pivot-positivity criterion** (Theorem 3.2): positivity of all ERO pivots of $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ is equivalent to the Stieltjes moment condition, and equivalent to the existence and uniqueness of all $[n-1/n]$ and $[n/n]$ Padé approximants; moreover $|\det \mathcal{S}_{n,n}| = H_n^{(1)}$ links the Padé–Sylvester matrix to the Hankel determinants.

(3) **Two-sided bracketing** (Theorem 4.1): for a Stieltjes function f and $x > 0$, $[n-1/n]_f(x) \le f(x) \le [n/n]_f(x)$, with the two sequences monotone.

(4) **Certified error control** (Corollary 4.2): the computable gap $g_n(x) = [n/n]_f(x) - [n-1/n]_f(x) \ge 0$ bounds the error of both approximants, giving rigorous a posteriori error bars from the series data alone.

(5) **Applications** (Section 5): certified evaluation of $\log(1+z)/z$, and Borel–Stieltjes summation of the divergent series $\sum(-1)^k k!\, z^k$; both worked out and verified.

We emphasise that the bracketing phenomenon itself is classical [2,3,9]; our contribution is the matrix-ERO framework that produces it, the explicit pivot/Hankel-determinant bookkeeping that ties existence, positivity and the bound together, and the resulting certified error tool. The paper is self-contained at the level of matrix algebra; for the analytic moment-problem facts we cite the standard sources.

2. Preliminaries

2.1. **Stieltjes series and the moment problem.** Let $(\mu_k)_{k\geq 0}$ be a real sequence and $f(z) = \sum_{k\geq 0}(-1)^k \mu_k z^k$ the associated formal power series. For $n \geq 1$ define the Hankel determinants

$$H_n^{(s)} := \det\big(\mu_{i+j+s}\big)_{i,j=0}^{n-1}, \qquad s \in \{0,1\}, \quad H_0^{(s)} := 1. \tag{2.1}$$

We write $\mathcal{H}_n^{(s)} = (\mu_{i+j+s})_{i,j=0}^{n-1}$ for the corresponding matrices and abbreviate $H_n := H_n^{(0)}$, $\mathcal{H}_n := \mathcal{H}_n^{(0)}$.

Definition 2.1. The sequence (μ_k) is a *Stieltjes moment sequence* if there is a positive Borel measure ψ on $[0,\infty)$ with infinitely many points of increase such that $\mu_k = \int_0^\infty u^k\, d\psi(u) < \infty$ for all k. The series $f(z) = \sum (-1)^k \mu_k z^k$ is then a *Stieltjes series*, and its sum (1.1), where it converges (or in the Borel/Padé sense otherwise), is a *Stieltjes function*.

The classical solvability criterion for the Stieltjes moment problem [1,8,9] is:

$$(\mu_k) \text{ is a Stieltjes moment sequence } \iff H_n^{(0)} > 0 \text{ and } H_n^{(1)} > 0 \text{ for all } n \geq 1. \tag{2.2}$$

2.2. **Padé approximants and the Sylvester matrix.** Recall from [2] that the $[m/n]$ Padé approximant of $f = \sum_k c_k z^k$ ($c_k = (-1)^k \mu_k$) is the rational function $R_{m,n} = P_m/Q_n$, $\deg P_m \leq m$, $\deg Q_n \leq n$, $Q_n(0) = 1$, with

$$f(z)Q_n(z) - P_m(z) = O\big(z^{m+n+1}\big) \quad (z \to 0). \tag{2.3}$$

As is standard [2], the denominator coefficients $q = (q_1, \ldots, q_n)^\top$ solve the Padé–Sylvester (Toeplitz) system $\mathcal{S}_{m,n}\, q = -c$, where

$$\mathcal{S}_{m,n} = \big(c_{m+i-j}\big)_{i,j=1}^{n}, \qquad c = (c_{m+1}, \ldots, c_{m+n})^\top, \qquad c_k = 0\ (k < 0). \tag{2.4}$$

The numerator is recovered by $p_k = \sum_{j=0}^{\min(k,n)} c_{k-j} q_j$. The approximant exists and is unique iff $\det \mathcal{S}_{m,n} \neq 0$ [2].

3. The Unimodular Hankel–ERO Reduction

We first record the matrix-algebraic backbone: that Gaussian elimination of a Hankel matrix with nonvanishing leading minors is a unimodular ERO process whose pivots are determinant ratios. This is the Padé/Stieltjes analogue of the unit determinant invariant of [11].

Lemma 3.1 (Unimodular Hankel–ERO reduction)**.** *Let* $s \in \{0, 1\}$ *and suppose* $H_k^{(s)} \neq 0$ *for* $1 \leq k \leq n$. *Then Gaussian elimination of* $\mathcal{H}_n^{(s)}$ without row interchange *– a sequence of type-III EROs* $R_i \to R_i - \ell_{ik} R_k$ *– is left multiplication by a unit lower-triangular matrix* E *with* $\det E = 1$, *producing an upper-triangular* U *with* $E\,\mathcal{H}_n^{(s)} = U$ *and pivots*

$$d_k := U_{kk} = \frac{H_k^{(s)}}{H_{k-1}^{(s)}} \qquad (1 \leq k \leq n). \tag{3.1}$$

Consequently $\det \mathcal{H}_n^{(s)} = \prod_{k=1}^n d_k = H_n^{(s)}$.

Proof. Each type-III ERO is left multiplication by a unit lower-triangular elementary matrix; products and inverses of such matrices are unit lower-triangular, so E is unit lower-triangular and $\det E = 1$. Because all leading principal minors $H_k^{(s)}$ are nonzero, no row interchange is needed and the LU factorisation $\mathcal{H}_n^{(s)} = E^{-1}U$ exists and is unique. Comparing leading $k \times k$ blocks gives $H_k^{(s)} = \det(\mathcal{H}_k^{(s)}) = \prod_{i=1}^k U_{ii}$, whence (3.1) follows by taking the ratio of consecutive minors. The last identity is the case $k = n$. □

Theorem 3.2 (Pivot positivity, Stieltjes condition, and Padé existence)**.** *For a real moment sequence* (μ_k) *the following are equivalent:*

- (i) *all ERO pivots of* $\mathcal{H}_n^{(0)}$ *and of* $\mathcal{H}_n^{(1)}$ *are positive, for every* n;
- (ii) $H_n^{(0)} > 0$ *and* $H_n^{(1)} > 0$ *for every* n;
- (iii) (μ_k) *is a Stieltjes moment sequence.*

If these hold, then for every $n \geq 1$ *the Padé approximants* $[n-1/n]_f$ *and* $[n/n]_f$ *of* $f = \sum (-1)^k \mu_k z^k$ *exist and are unique, and the Padé–Sylvester matrix* (2.4) *satisfies*

$$\left|\det \mathcal{S}_{n,n}\right| = H_n^{(1)}. \tag{3.2}$$

Proof. (i)⇔(ii). By Lemma 3.1, the pivots of $\mathcal{H}_n^{(s)}$ are $d_k^{(s)} = H_k^{(s)}/H_{k-1}^{(s)}$ with $H_0^{(s)} = 1$. Hence all pivots are positive iff all the ratios are positive iff all $H_k^{(s)} > 0$ (by induction on k, since $H_0^{(s)} = 1 > 0$). This is Sylvester's criterion: positivity of all leading principal minors is equivalent to positive definiteness of the symmetric Hankel matrices [7, §7.2]. (ii)⇔(iii) is the classical Stieltjes moment criterion (2.2) [1,8].

For the existence statement, the $[n/n]$ approximant exists and is unique iff $\det \mathcal{S}_{n,n} \neq 0$, and likewise $[n-1/n]$ iff $\det \mathcal{S}_{n-1,n} \neq 0$ [2]. We claim $|\det \mathcal{S}_{n,n}| = H_n^{(1)}$ and $|\det \mathcal{S}_{n-1,n}| = H_n^{(0)}$, both nonzero by (ii). Indeed $\mathcal{S}_{n,n}$ has (i,j) entry $c_{n+i-j} = (-1)^{n+i-j}\mu_{n+i-j}$. Writing $D = \operatorname{diag}\left((-1)^i\right)_{i=1}^n$ and J for the reversal permutation $J_{ij} = \delta_{i,\,n+1-j}$, one checks entrywise that $\mathcal{S}_{n,n} = (-1)^n D \left(\mathcal{H}_n^{(1)} J\right) D$, so that $|\det \mathcal{S}_{n,n}| = |\det \mathcal{H}_n^{(1)}| \cdot |\det J| = H_n^{(1)}$, because $\det D = \pm 1$ and $|\det J| = 1$. The computation for $\mathcal{S}_{n-1,n}$ is identical with $\mathcal{H}_n^{(0)}$ in place of $\mathcal{H}_n^{(1)}$. □

Remark 3.3. Identity (3.2) is the Padé analogue of the unit determinant invariant of [11]: there the ERO determinant was $(-1)^k$, a unit of $\mathbb{Z}$; here it is a positive Hankel determinant whose positivity is exactly the obstruction governing existence and – as we show next – the two-sided bound. We verified (3.2) symbolically for $2 \le n \le 4$ on the examples of Section 5.

4. Two-Sided Bounds and Certified Error Control

We now turn the algebra into analysis. Throughout this section (μ_k) is a Stieltjes moment sequence with representing measure ψ, and f is the Stieltjes function (1.1). Let π_n denote the monic orthogonal polynomial of degree n for ψ on $[0,\infty)$; by (2.2) and Lemma 3.1 the family (π_n) exists. The denominator of $[n-1/n]_f$ is the reversed orthogonal polynomial π_n for ψ, while that of $[n/n]_f$ is the reversed orthogonal polynomial $\hat{\pi}_n$ for the measure $u\,d\psi$; both have positive simple zeros [2,5].

Theorem 4.1 (Two-sided bracketing)**.** *Let f be a Stieltjes function and $x>0$. Then for every $n \ge 1$,*

$$[n-1/n]_f(x) \;\le\; f(x) \;\le\; [n/n]_f(x), \tag{4.1}$$

the sequence $\big([n-1/n]_f(x)\big)_n$ is nondecreasing, the sequence $\big([n/n]_f(x)\big)_n$ is nonincreasing, and both converge to $f(x)$ whenever the Stieltjes moment problem for ψ is determinate (e.g. under Carleman's condition $\sum_k \mu_{2k}^{-1/2k} = \infty$).

Proof. Fix $x>0$ and set $g_x(u) = (1+xu)^{-1}$, so that $f(x) = \int_0^\infty g_x\,d\psi(u)$ by (1.1). The function $u \mapsto g_x(u)$ is completely monotone on $[0,\infty)$; in particular

$$g_x^{(2n)}(u) = \frac{(2n)!\,x^{2n}}{(1+xu)^{2n+1}} > 0 \qquad (u \ge 0). \tag{4.2}$$

It is classical [2, §5.4] [5] that the subdiagonal approximant $[n-1/n]_f(x)$ equals the n-point Gauss quadrature of $\int_0^\infty g_x\,d\psi(u)$ with nodes at the zeros of π_n (equivalently, at the negative reciprocals of the poles of Q_n), and that $[n/n]_f(x)$ equals the $(n+1)$-point Gauss–Radau rule with a fixed node at $u=0$. The Gauss remainder is

$$f(x) - [n-1/n]_f(x) = \frac{g_x^{(2n)}(\xi)}{(2n)!}\int_0^\infty \pi_n(u)^2\,d\psi(u) \;\ge\; 0$$

for some $\xi \ge 0$, by (4.2) and positivity of $\int \pi_n^2\,d\psi(u)$; this is the lower bound. For the upper bound, the $(n+1)$-point Gauss–Radau rule with fixed node $u=0$ is exact to degree $2n$, and its remainder is

$$f(x) - [n/n]_f(x) = \frac{g_x^{(2n+1)}(\xi)}{(2n+1)!}\int_0^\infty u\,\hat{\pi}_n(u)^2\,d\psi(u),$$

for some $\xi \ge 0$, where $\hat{\pi}_n$ is the degree-n orthogonal polynomial for the measure $u\,d\psi$. Since $u \ge 0$ on $[0,\infty)$ the integral is nonnegative, while $g_x^{(2n+1)}(u) = -\,(2n+1)!\,x^{2n+1}(1+xu)^{-(2n+2)} < 0$; hence $f(x) - [n/n]_f(x) \le 0$, the upper bound. Monotonicity of the two sequences and convergence under determinacy are the standard nesting and convergence properties of Gauss/Gauss–Radau rules for a positive measure [2,5,10]. ∎

Corollary 4.2 (Certified a posteriori error)**.** *Under the hypotheses of Theorem 4.1, define the computable gap*

$$g_n(x) := [n/n]_f(x) - [n-1/n]_f(x) \;\ge\; 0. \tag{4.3}$$

Then both Padé approximants are certified to this accuracy:

$$\big|f(x) - [n/n]_f(x)\big| \le g_n(x) \quad \textit{and} \quad \big|f(x) - [n-1/n]_f(x)\big| \le g_n(x).$$

In particular the midpoint $\frac{1}{2}\big([n/n]_f(x) + [n-1/n]_f(x)\big)$ *approximates* $f(x)$ *with error at most* $\frac{1}{2}g_n(x)$.

Proof. By (4.1), $f(x)$ and both approximants lie in the interval of length $g_n(x)$ with endpoints $[n-1/n]_f(x)$ and $[n/n]_f(x)$; the three displayed inequalities are immediate, and the midpoint bound follows since the midpoint is within $\frac{1}{2}g_n(x)$ of every point of the interval. □

The point of Corollary 4.2 is practical: $g_n(x)$ is computed from the *same* Taylor data $(\mu_0, \dots, \mu_{2n})$ that produce the approximants, so it furnishes guaranteed error bars with no a priori knowledge of f beyond its series – a feature unavailable from a one-sided estimate.

Remark 4.3 (Why the learning function $1 - e^{-at}$ is excluded)**.** For $L(t) = 1 - e^{-at}$ one has $c_0 = 0$ and $c_k = (-1)^{k+1}a^k/k!$ for $k \ge 1$, so the associated sequence is $\mu_0 = 0$ and $\mu_k = (-1)^k c_k = -a^k/k!$ for $k \ge 1$. In particular $\mu_1 = -a < 0$, whereas every Stieltjes moment sequence satisfies $\mu_1 = \int_0^\infty u\, d\psi(u) \ge 0$; equivalently $H_2^{(0)} = \mu_0\mu_2 - \mu_1^2 = -a^2 < 0$, and (2.2) fails: L is *not* a Stieltjes series. Theorem 4.1 therefore does not apply, a one-sided estimate is the most one can expect for it by this route. The present framework thus delimits exactly the class on which the two-sided, certified theory is available.

5. Applications

All numerical values below were computed with exact rational arithmetic for the approximants and high-precision arithmetic for the reference values. Figure 1 summarises the two phenomena established above: the two-sided convergence of Theorem 4.1 and the geometric decay of the certified gap of Corollary 4.2.

5.1. **Certified evaluation of** $\log(1+z)/z$**.**

The function $f(z) = z^{-1}\log(1+z) = \int_0^1 (1+zu)^{-1}\, du$ is Stieltjes with representing measure $\psi =$ Lebesgue measure on $[0,1]$ and moments $\mu_k = \int_0^1 u^k\, du = 1/(k+1)$ [4]. The Hankel determinants are positive,

$$H_1^{(0)} = 1, \quad H_2^{(0)} = \tfrac{1}{12}, \quad H_3^{(0)} = \tfrac{1}{2160}, \quad H_1^{(1)} = \tfrac{1}{2}, \quad H_2^{(1)} = \tfrac{1}{72}, \quad H_3^{(1)} = \tfrac{1}{43200},$$

confirming the Stieltjes property via Theorem 3.2. The first approximants are

$$[0/1] = \frac{2}{z+2}, \quad [1/1] = \frac{z+6}{2(2z+3)}, \quad [1/2] = \frac{3(z+2)}{z^2+6z+6}, \quad [2/2] = \frac{z^2+21z+30}{3(3z^2+12z+10)}.$$

Table 1 reports the brackets (4.1) at $x = 1$ (where $f = \log 2$) and the certified gap g_n of (4.3).

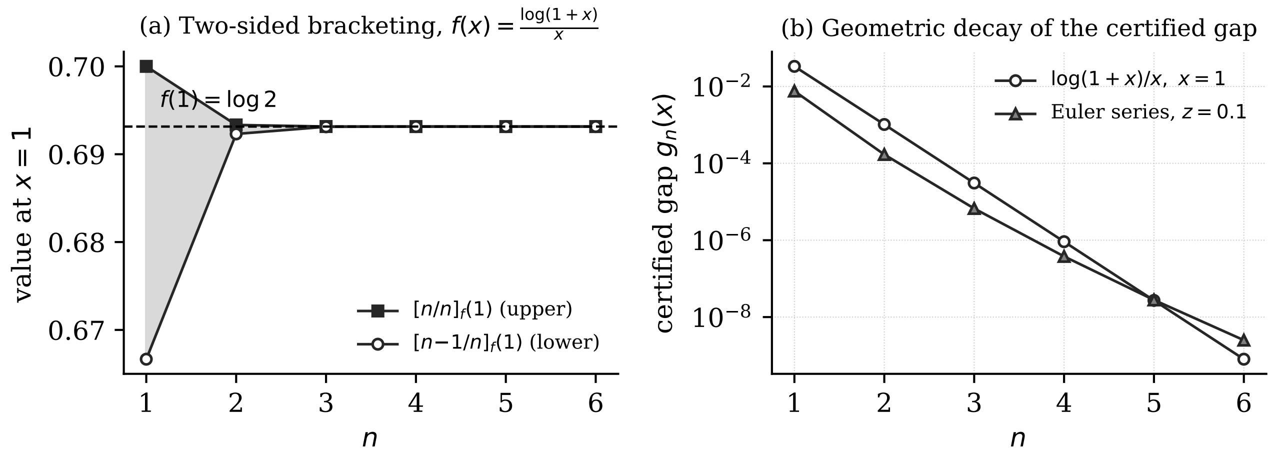

FIGURE 1. (a) The diagonal $[n/n]_f$ and subdiagonal $[n-1/n]_f$ Padé approximants bracket $f(x) = x^{-1}\log(1+x)$ from above and below at $x = 1$ (Theorem 4.1); the shaded region is the certified enclosure. (b) The certified gap $g_n(x) = [n/n]_f(x) - [n-1/n]_f(x)$ of Corollary 4.2 decays geometrically in n for both worked examples (semilogarithmic scale).

TABLE 1. Two-sided Padé bounds and certified gap for $f(x) = x^{-1}\log(1+x)$ at $x = 1$ ($f = \log 2 = 0.6931472\ldots$).

n	$[n-1/n]_f(1)$	$f(1)$	$[n/n]_f(1)$	$g_n(1)$
1	0.6666667	0.6931472	0.7000000	3.3×10^{-2}
2	0.6923077	0.6931472	0.6933333	1.0×10^{-3}
3	0.6931217	0.6931472	0.6931525	3.1×10^{-5}
4	0.9156330	0.9156333	0.9156334	3.8×10^{-7}
5	0.9156333	0.9156333	0.9156333	2.8×10^{-8}

At $x = 5$ ($f = \frac{1}{5}\log 6 = 0.3583519$) the same scheme gives, for $n = 3$, the bracket $[0.3557423,\ 0.3598843]$ with $g_3 = 4.1 \times 10^{-3}$; the bracket contains f as guaranteed by Theorem 4.1.

5.2. **Summation of the divergent Euler series.** Consider the divergent series $\sum_{k\geq 0}(-1)^k k!\, z^k$. Here $\mu_k = k! = \int_0^\infty u^k e^{-u}\, du$ are the moments of the measure $d\psi(u) = e^{-u}\, du$ on $[0,\infty)$, so (μ_k) is a Stieltjes moment sequence and the corresponding Stieltjes function is

$$f(z) = \int_0^\infty \frac{e^{-u}}{1+zu}\, du = \frac{1}{z}\, e^{1/z}\, E_1\!\left(\frac{1}{z}\right) \qquad (z > 0),$$

where E_1 is the exponential integral [4,6]. Although the series diverges for every $z \neq 0$, its diagonal/subdiagonal Padé approximants bracket the Stieltjes sum f, as guaranteed by Theorem 4.1. Table 2 shows the brackets at $z = 0.1$.

TABLE 2. Two-sided Padé bounds for the divergent Euler series $\sum (-1)^k k!\, z^k$ at $z = 0.1$, with Stieltjes sum $f = 0.9156333\ldots$

n	$[n-1/n]_f(0.1)$	$f(0.1)$	$[n/n]_f(0.1)$	$g_n(0.1)$
2	0.9154930	0.9156333	0.9156627	1.7×10^{-4}
3	0.9156280	0.9156333	0.9156347	6.7×10^{-6}
4	0.9156330	0.9156333	0.9156334	3.8×10^{-7}
5	0.9156333	0.9156333	0.9156333	2.8×10^{-8}

The certified gap g_n contracts geometrically, so Corollary 4.2 delivers a rigorous resummation of the divergent series with explicit error bars – a genuine application of the framework to asymptotic analysis, where such series arise as perturbation expansions [6].

5.3. **Relaxation functions with a distribution of time constants.** Many physical and learning/relaxation processes are governed by a completely monotone response $\Phi(t) = \int_0^\infty e^{-t/\tau}\, d\rho(\tau)$ with $\rho \geq 0$ a distribution of time constants. Its resolvent

$$\widehat{\Phi}(s) = \int_0^\infty \frac{d\rho(\tau)}{1 + s\tau}$$

is exactly of the Stieltjes form (1.1), with moments $\mu_k = \int_0^\infty \tau^k\, d\rho(\tau)$. Whenever these moments are available (from measurements or a model), Theorem 3.2 certifies that $\widehat{\Phi}$ is Stieltjes and Corollary 4.2 produces two-sided, certified rational approximations of $\widehat{\Phi}(s)$ for $s > 0$. This treats a multi-time-constant relaxation model – a generalisation of the single-rate exponential model – within a setting where the strong two-sided theory does apply.

6. Concluding Remarks

We have developed the matrix ERO approach of [11] on the Padé side by isolating the Hankel moment matrix as the carrying object and reading off its Gaussian (ERO) reduction. The unimodular reduction (Lemma 3.1) and the pivot-positivity criterion (Theorem 3.2) tie together, in one matrix picture, the existence of the approximants, the Stieltjes moment condition, and the determinant bookkeeping $|\det \mathcal{S}_{n,n}| = H_n^{(1)}$. On the Stieltjes class this matrix structure yields the two-sided bracketing

(Theorem 4.1) and, as its operational corollary, certified a posteriori error control (Corollary 4.2). The exponential learning function $1 - e^{-at}$ lies *outside* this class (Remark 4.3), which both explains why only a one-sided bound is available for it and sharply delimits the scope of the present theory.

Several directions remain open.

(1) **Matrix-valued Stieltjes functions.** Replacing ψ by a positive matrix measure leads to block-Hankel matrices and matrix orthogonal polynomials; the unimodular block-ERO reduction and a matrix bracketing in the Löwner order appear plausible.
(2) **Multipoint and Hermite–Padé variants.** The Sylvester matrix (2.4) has multipoint analogues; identifying the corresponding positivity-and-bracketing structure is natural.
(3) **Sharp gap asymptotics.** A determinantal closed form for the certified gap $g_n(x)$ in terms of $H_n^{(0)}, H_n^{(1)}$ and Q_n would convert Corollary 4.2 into an a priori convergence rate.

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