THEORETICAL INVESTIGATIONS ON INTERVAL-VALUED INTUITIONISTIC NEUTROSOPHIC $\hat{Z}$-IDEALS IN $\hat{Z}$-ALGEBRAS

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Abstract. Uncertainty in algebraic systems has been widely studied via fuzzy, intuitionistic fuzzy, and neutrosophic frameworks. In this paper, we introduce interval-valued intuitionistic neutrosophic (IVIN) $\hat{Z}$-ideals in $\hat{Z}$-algebras, extending the classical notion of $\hat{Z}$-ideals by using interval-valued membership and non-membership functions together with the neutrosophic components of truth, indeterminacy, and falsity. We develop basic properties of IVIN $\hat{Z}$-ideals, including closure under intersection, and investigate their behavior under homomorphisms (images and inverse images). Moreover, we study Cartesian products of IVIN $\hat{Z}$-ideals within the product $\hat{Z}$-algebra. These results enrich the ideal theory of $\hat{Z}$-algebras in an interval-valued intuitionistic neutrosophic setting and provide a foundation for further work on uncertainty-aware algebraic structures.

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1. Introduction

Mathematical models of real-world phenomena frequently involve uncertainty, vagueness, and incomplete information, which cannot be adequately described by classical (crisp) set theory. To overcome this limitation, Zadeh [19,20] introduced fuzzy sets, allowing graded membership in $[0, 1]$. Extending this idea, Atanassov [1] proposed intuitionistic fuzzy sets (IFS), in which each element is described by both a membership degree and a non-membership degree, leaving a margin of hesitation. Later, Atanassov and Gargov [2] developed interval-valued intuitionistic fuzzy sets (IVIFS), replacing these degrees with intervals to better reflect imprecise assessments.

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In parallel, Smarandache [13] introduced neutrosophic sets, characterized by three independent components: truth, indeterminacy, and falsity. Interval-based extensions were subsequently studied (e.g., [18]), and intuitionistic neutrosophic variants were introduced to combine intuitionistic and neutrosophic viewpoints (e.g., [3]). These generalizations provide richer tools for representing higher-order uncertainty.

Alongside set-theoretic developments, fuzzy and neutrosophic concepts have been incorporated into algebraic structures, in which ideals and subalgebras play central roles in structural analysis and homomorphic constructions. For instance, fuzzy ideals and their variants have been investigated in several algebraic systems, including BCK/BCI-algebras and their neutrosophic extensions [6,7,9,10]. In a related direction, Songsaeng and Iampan [14] studied neutrosophic sets in UP-algebras by employing interval-valued fuzzy sets, further illustrating how interval-based uncertainty models can be effectively integrated into algebraic frameworks.

The notion of $\hat{Z}$-algebra was introduced by Chandramouleeswaran et al. [4], and uncertainty-based structures in this setting have been actively explored. Fuzzy and intuitionistic fuzzy structures in $\hat{Z}$-algebras were studied in [15,16], and fuzzy $\hat{Z}$-ideals were introduced in [17]. Further developments include interval-valued and neutrosophic-type subalgebras and related notions in $\hat{Z}$-algebras (see, for example, [5,8,11,12]).

Motivated by these works, we introduce the concept of an interval-valued intuitionistic neutrosophic (IVIN) $\hat{Z}$-ideal in a $\hat{Z}$-algebra. This notion extends $\hat{Z}$-ideals by incorporating interval-valued membership and non-membership information together with neutrosophic truth, indeterminacy, and falsity components. We investigate fundamental properties of IVIN $\hat{Z}$-ideals, including closure under intersection, their behavior under homomorphisms (images and inverse images), and Cartesian products within the product $\hat{Z}$-algebra. These results enrich the ideal theory of $\hat{Z}$-algebras in an interval-valued intuitionistic neutrosophic setting and provide a basis for further developments in uncertainty-aware algebraic systems.

2. Preliminaries

In this section, we recall the basic definitions and notation used throughout the paper. Unless stated otherwise, let $(\mathfrak{V}, *, 0)$ denote a $\hat{Z}$-algebra, and let α, β represent arbitrary elements of $\mathfrak{V}$. When needed, we also consider a mapping $h : \mathfrak{V} \to \mathfrak{F}$ between two nonempty sets $\mathfrak{V}$ and $\mathfrak{F}$.

Definition 2.1 ([19]). *Let $\mathfrak{V}$ be a nonempty set. A fuzzy set ν on $\mathfrak{V}$ is a mapping*

$$\mu_\nu : \mathfrak{V} \to [0, 1],$$

called the membership function of ν. For each $\alpha \in \mathfrak{V}$, the value $\mu_\nu(\alpha)$ represents the degree of membership of α in ν. Equivalently, one may write

$$\nu = \{(\alpha, \mu_\nu(\alpha)) : \alpha \in \mathfrak{V}\}.$$

Definition 2.2 ([20]). *Let $\mathfrak{V}$ be a nonempty set and let $\mathbb{D}[0,1]$ denote the family of all closed subintervals of $[0,1]$. An interval-valued fuzzy set (IVFS) ν on $\mathfrak{V}$ is specified by two membership functions*

$$\mu_\nu^L, \mu_\nu^U : \mathfrak{V} \to [0,1]$$

such that

$$\mu_\nu^L(\alpha) \le \mu_\nu^U(\alpha), \quad \text{for all } \alpha \in \mathfrak{V}.$$

Equivalently, ν can be described by the interval-valued mapping $\bar{\mu}_\nu : \mathfrak{V} \to \mathbb{D}[0,1]$ defined by

$$\bar{\mu}_\nu(\alpha) = \left[\mu_\nu^L(\alpha),\ \mu_\nu^U(\alpha)\right], \quad \text{for all } \alpha \in \mathfrak{V},$$

and we may write

$$\nu = \{(\alpha, \bar{\mu}_\nu(\alpha)) : \alpha \in \mathfrak{V}\}.$$

If $\mu_\nu^L(\alpha) = \mu_\nu^U(\alpha) = c$ for some $c \in [0,1]$, then $\bar{\mu}_\nu(\alpha) = [c, c]$.

For $\mathbb{D}_1 = [\gamma_1, \delta_1]$ and $\mathbb{D}_2 = [\gamma_2, \delta_2]$ in $\mathbb{D}[0,1]$, define

$$\mathrm{rmin}(\mathbb{D}_1, \mathbb{D}_2) = \left[\min\{\gamma_1, \gamma_2\},\ \min\{\delta_1, \delta_2\}\right].$$

Moreover, we use the componentwise order on $\mathbb{D}[0,1]$ given as follows:

$$\mathbb{D}_1 \ge \mathbb{D}_2 \text{ if and only if } \gamma_1 \ge \gamma_2 \text{ and } \delta_1 \ge \delta_2,$$

and similarly,

$$\mathbb{D}_1 \le \mathbb{D}_2 \text{ if and only if } \gamma_1 \le \gamma_2 \text{ and } \delta_1 \le \delta_2.$$

Also, $\mathbb{D}_1 = \mathbb{D}_2$ holds if and only if $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$.

Definition 2.3 ([1]). *Let $\mathfrak{V}$ be a nonempty set. An intuitionistic fuzzy set (IFS) ν on $\mathfrak{V}$ is characterized by two functions*

$$\mu_\nu, \vartheta_\nu : \mathfrak{V} \to [0,1],$$

called the membership and non-membership functions of ν, respectively, such that

$$0 \le \mu_\nu(\alpha) + \vartheta_\nu(\alpha) \le 1, \quad \text{for all } \alpha \in \mathfrak{V}.$$

Equivalently, ν may be written as

$$\nu = \{(\alpha, \mu_\nu(\alpha), \vartheta_\nu(\alpha)) : \alpha \in \mathfrak{V}\}.$$

Definition 2.4 ([2]). *Let $\mathfrak{V}$ be a nonempty set and let $\mathbb{D}[0,1]$ denote the family of all closed subintervals of $[0,1]$. An interval-valued intuitionistic fuzzy set (IVIFS) ξ on $\mathfrak{V}$ is characterized by two mappings*

$$\bar{\mu}_\xi, \bar{\vartheta}_\xi : \mathfrak{V} \to \mathbb{D}[0,1],$$

called the interval-valued membership and interval-valued non-membership functions of ξ, respectively. For each $\alpha \in \mathfrak{V}$,

$$\bar{\mu}_\xi(\alpha) = \left[\mu_\xi^L(\alpha),\, \mu_\xi^U(\alpha)\right], \quad \bar{\vartheta}_\xi(\alpha) = \left[\vartheta_\xi^L(\alpha),\, \vartheta_\xi^U(\alpha)\right],$$

where $\mu_\xi^L, \mu_\xi^U, \vartheta_\xi^L, \vartheta_\xi^U : \mathfrak{V} \to [0,1]$ satisfy

$$0 \le \mu_\xi^L(\alpha) + \vartheta_\xi^L(\alpha) \le 1, \quad 0 \le \mu_\xi^U(\alpha) + \vartheta_\xi^U(\alpha) \le 1, \quad \textit{for all } \alpha \in \mathfrak{V}.$$

Equivalently, one may write

$$\xi = \{(\alpha, \bar{\mu}_\xi(\alpha), \bar{\vartheta}_\xi(\alpha)) : \alpha \in \mathfrak{V}\}.$$

The complement of ξ, denoted by ξ^c, is defined by

$$\bar{\mu}_{\xi^c}(\alpha) = \left[1 - \mu_\xi^U(\alpha),\, 1 - \mu_\xi^L(\alpha)\right], \quad \bar{\vartheta}_{\xi^c}(\alpha) = \left[1 - \vartheta_\xi^U(\alpha),\, 1 - \vartheta_\xi^L(\alpha)\right],$$

for all $\alpha \in \mathfrak{V}$.

Definition 2.5 ([13])**.** *Let $\mathfrak{V}$ be a nonempty set. A neutrosophic set ν on $\mathfrak{V}$ is characterized by three functions*

$$\mu_{\nu_T}, \mu_{\nu_I}, \mu_{\nu_F} : \mathfrak{V} \to [0,1],$$

called the truth-membership, indeterminacy-membership, and falsity-membership functions of ν, respectively. For each $\alpha \in \mathfrak{V}$, the triple

$$\left(\mu_{\nu_T}(\alpha),\, \mu_{\nu_I}(\alpha),\, \mu_{\nu_F}(\alpha)\right)$$

represents the degrees of truth, indeterminacy, and falsity of α in ν. Equivalently, one may write

$$\nu = \{(\alpha, \mu_{\nu_T}(\alpha), \mu_{\nu_I}(\alpha), \mu_{\nu_F}(\alpha)) : \alpha \in \mathfrak{V}\}.$$

Definition 2.6 ([4])**.** *Let $\mathfrak{V}$ be a nonempty set equipped with a binary operation $* : \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ and a distinguished element $0 \in \mathfrak{V}$. The triple $(\mathfrak{V}, *, 0)$ is called a $\hat{Z}$-algebra if, for all $\alpha, \beta \in \mathfrak{V}$, the following axioms hold:*

(*i*) $\alpha * 0 = 0$,
(*ii*) $0 * \alpha = \alpha$,
(*iii*) $\alpha * \alpha = \alpha$,
(*iv*) $\alpha * \beta = \beta * \alpha$ *whenever* $\alpha \neq 0$ *and* $\beta \neq 0$.

Example 2.1 ([4])**.** *Let $\mathfrak{V} = \{0, \theta_1, \theta_2, \theta_3\}$. Define $*$ on $\mathfrak{V}$ by the following Cayley table:*

$*$	0	θ_1	θ_2	θ_3
0	0	θ_1	θ_2	θ_3
θ_1	0	θ_1	θ_3	θ_1
θ_2	0	θ_3	θ_2	θ_2
θ_3	0	θ_1	θ_2	θ_3

Then $(\mathfrak{V}, *, 0)$ *is a* $\hat{Z}$*-algebra.*

Definition 2.7 ([16]). *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra. A fuzzy set* ν *on* $\mathfrak{V}$ *with membership function* $\mu_\nu : \mathfrak{V} \to [0,1]$ *is called a fuzzy* $\hat{Z}$*-subalgebra of* $\mathfrak{V}$ *if, for all* $\alpha, \beta \in \mathfrak{V}$,

$$\mu_\nu(\alpha * \beta) \geq \min\{\mu_\nu(\alpha),\ \mu_\nu(\beta)\}.$$

Definition 2.8 ([5]). *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra. An interval-valued fuzzy set* ν *on* $\mathfrak{V}$ *with membership function*

$$\bar{\mu}_\nu : \mathfrak{V} \to \mathbb{D}[0,1]$$

is called an interval-valued fuzzy $\hat{Z}$*-subalgebra of* $\mathfrak{V}$ *if, for all* $\alpha, \beta \in \mathfrak{V}$,

$$\bar{\mu}_\nu(\alpha * \beta) \geq \mathrm{rmin}\big(\bar{\mu}_\nu(\alpha),\ \bar{\mu}_\nu(\beta)\big).$$

Definition 2.9 ([5]). *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra. An intuitionistic fuzzy set* (IFS) ν *on* $\mathfrak{V}$*, characterized by the membership and non-membership functions*

$$\mu_\nu, \vartheta_\nu : \mathfrak{V} \to [0,1],$$

is called an intuitionistic fuzzy $\hat{Z}$*-subalgebra of* $\mathfrak{V}$ *if, for all* $\alpha, \beta \in \mathfrak{V}$,

(*i*) $\mu_\nu(\alpha * \beta) \geq \min\{\mu_\nu(\alpha),\ \mu_\nu(\beta)\}$,
(*ii*) $\vartheta_\nu(\alpha * \beta) \leq \max\{\vartheta_\nu(\alpha),\ \vartheta_\nu(\beta)\}$.

Definition 2.10 ([12]). *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra. A neutrosophic set* ν *on* $\mathfrak{V}$*, characterized by the truth-, indeterminacy-, and falsity-membership functions*

$$\mu_{\nu_T}, \mu_{\nu_I}, \mu_{\nu_F} : \mathfrak{V} \to [0,1],$$

is called a neutrosophic fuzzy $\hat{Z}$*-subalgebra of* $\mathfrak{V}$ *if, for all* $\alpha, \beta \in \mathfrak{V}$,

(*i*) $\mu_{\nu_T}(\alpha * \beta) \geq \min\{\mu_{\nu_T}(\alpha),\ \mu_{\nu_T}(\beta)\}$,
(*ii*) $\mu_{\nu_I}(\alpha * \beta) \geq \min\{\mu_{\nu_I}(\alpha),\ \mu_{\nu_I}(\beta)\}$,
(*iii*) $\mu_{\nu_F}(\alpha * \beta) \leq \max\{\mu_{\nu_F}(\alpha),\ \mu_{\nu_F}(\beta)\}$.

Definition 2.11 ([11]). *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra and let* $\mathbb{D}[0,1]$ *denote the family of all closed subintervals of* $[0,1]$*. An interval-valued neutrosophic set* $(IVNS)$ $\bar{\nu}$ *on* $\mathfrak{V}$ *is characterized by three mappings*

$$\bar{\mu}_{\nu_T}, \bar{\mu}_{\nu_I}, \bar{\mu}_{\nu_F} : \mathfrak{V} \to \mathbb{D}[0,1],$$

called the interval-valued truth-, indeterminacy-, and falsity-membership functions, respectively. It is called an interval-valued neutrosophic fuzzy $\hat{Z}$-subalgebra of $\mathfrak{V}$ if, for all $\alpha, \beta \in \mathfrak{V}$,

(*i*) $\bar{\mu}_{\nu_T}(\alpha * \beta) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_T}(\alpha),\, \bar{\mu}_{\nu_T}(\beta)\big)$,

(*ii*) $\bar{\mu}_{\nu_I}(\alpha * \beta) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_I}(\alpha),\, \bar{\mu}_{\nu_I}(\beta)\big)$,

(*iii*) $\bar{\mu}_{\nu_F}(\alpha * \beta) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_F}(\alpha),\, \bar{\mu}_{\nu_F}(\beta)\big)$.

Definition 2.12. *Let $(\mathfrak{V}, *, 0)$ be a $\hat{Z}$-algebra. An intuitionistic neutrosophic set (INS) ν on $\mathfrak{V}$ is characterized by six functions*

$$\mu_{\nu_T}, \mu_{\nu_I}, \mu_{\nu_F}, \vartheta_{\nu_T}, \vartheta_{\nu_I}, \vartheta_{\nu_F} : \mathfrak{V} \to [0, 1],$$

where $\mu_{\nu_T}, \mu_{\nu_I}, \mu_{\nu_F}$ denote the degrees of truth-, indeterminacy-, and falsity-membership, respectively, and $\vartheta_{\nu_T}, \vartheta_{\nu_I}, \vartheta_{\nu_F}$ denote the corresponding degrees of non-membership. Equivalently, one may write

$$\nu = \{(\alpha, \mu_{\nu_T}(\alpha), \mu_{\nu_I}(\alpha), \mu_{\nu_F}(\alpha), \vartheta_{\nu_T}(\alpha), \vartheta_{\nu_I}(\alpha), \vartheta_{\nu_F}(\alpha)) : \alpha \in \mathfrak{V}\}.$$

The set ν is called an intuitionistic neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{V}$ if, for all $\alpha, \beta \in \mathfrak{V}$,

(*i*) $\mu_{\nu_T}(\alpha * \beta) \geq \min\{\mu_{\nu_T}(\alpha),\, \mu_{\nu_T}(\beta)\}$ *and* $\vartheta_{\nu_T}(\alpha * \beta) \leq \max\{\vartheta_{\nu_T}(\alpha),\, \vartheta_{\nu_T}(\beta)\}$,

(*ii*) $\mu_{\nu_I}(\alpha * \beta) \geq \min\{\mu_{\nu_I}(\alpha),\, \mu_{\nu_I}(\beta)\}$ *and* $\vartheta_{\nu_I}(\alpha * \beta) \leq \max\{\vartheta_{\nu_I}(\alpha),\, \vartheta_{\nu_I}(\beta)\}$,

(*iii*) $\mu_{\nu_F}(\alpha * \beta) \leq \max\{\mu_{\nu_F}(\alpha),\, \mu_{\nu_F}(\beta)\}$ *and* $\vartheta_{\nu_F}(\alpha * \beta) \geq \min\{\vartheta_{\nu_F}(\alpha),\, \vartheta_{\nu_F}(\beta)\}$.

Definition 2.13 ([17])**.** *Let $(\mathfrak{V}, *, 0)$ be a $\hat{Z}$-algebra and let $I \subseteq \mathfrak{V}$. The subset I is called a $\hat{Z}$-ideal of $\mathfrak{V}$ if it satisfies the following conditions:*

(*i*) $0 \in I$,

(*ii*) *for all $\alpha, \beta \in \mathfrak{V}$, if $\alpha * \beta \in I$ and $\beta \in I$, then $\alpha \in I$.*

Definition 2.14 ([17])**.** *Let $(\mathfrak{V}, *, 0)$ be a $\hat{Z}$-algebra. A fuzzy set ν on $\mathfrak{V}$ with membership function $\mu_\nu : \mathfrak{V} \to [0, 1]$ is called a fuzzy $\hat{Z}$-ideal of $\mathfrak{V}$ if, for all $\alpha, \beta \in \mathfrak{V}$, the following conditions hold:*

(*i*) $\mu_\nu(0) \geq \mu_\nu(\alpha)$,

(*ii*) $\mu_\nu(\alpha) \geq \min\{\mu_\nu(\alpha * \beta),\, \mu_\nu(\beta)\}$.

3. Interval-Valued Intuitionistic Neutrosophic $\hat{Z}$-Ideals

In this section, we introduce the notion of interval-valued intuitionistic neutrosophic (IVIN) $\hat{Z}$-ideals in $\hat{Z}$-algebras. We first formalize IVIN $\hat{Z}$-ideals by combining interval-valued membership and non-membership information with the neutrosophic components of truth, indeterminacy, and falsity, and then develop their basic ideal-theoretic properties. In particular, we establish useful characterizations and investigate fundamental closure and construction results that will be used throughout the remainder of the paper.

Definition 3.1. *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra and let* $\mathbb{D}[0,1]$ *denote the family of all closed subintervals of* $[0,1]$. *An interval-valued intuitionistic neutrosophic set* (*IVINS*) ν *on* $\mathfrak{V}$ *is given by the six mappings*

$$\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}} : \mathfrak{V} \to \mathbb{D}[0,1],$$

where $\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}$ *denote the interval-valued truth-, indeterminacy-, and falsity-membership functions, respectively, and* $\bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}$ *denote the corresponding interval-valued non-membership functions. Equivalently, one may write*

$$\nu = \{(\alpha, \bar{\mu}_{\nu_{\bar{T}}}(\alpha), \bar{\mu}_{\nu_{\bar{I}}}(\alpha), \bar{\mu}_{\nu_{\bar{F}}}(\alpha), \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha), \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha), \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha)) : \alpha \in \mathfrak{V}\}.$$

The IVINS ν *is called an interval-valued intuitionistic neutrosophic* $\hat{Z}$*-ideal* (*briefly, an IVIN* $\hat{Z}$*-ideal*) *of* $\mathfrak{V}$ *if, for all* $\alpha, \beta \in \mathfrak{V}$, *it satisfies:*

(*i*) $\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(\alpha)$ *and* $\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha)$,

(*ii*) $\bar{\mu}_{\nu_{\bar{I}}}(0) \geq \bar{\mu}_{\nu_{\bar{I}}}(\alpha)$ *and* $\bar{\vartheta}_{\nu_{\bar{I}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha)$,

(*iii*) $\bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(\alpha)$ *and* $\bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha)$,

(*iv*) $\bar{\mu}_{\nu_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha * \beta),\, \bar{\mu}_{\nu_{\bar{T}}}(\beta)\big)$ *and* $\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{T}}}(\beta)\big)$,

(*v*) $\bar{\mu}_{\nu_{\bar{I}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(\alpha * \beta),\, \bar{\mu}_{\nu_{\bar{I}}}(\beta)\big)$ *and* $\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{I}}}(\beta)\big)$,

(*vi*) $\bar{\mu}_{\nu_{\bar{F}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha * \beta),\, \bar{\mu}_{\nu_{\bar{F}}}(\beta)\big)$ *and* $\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{F}}}(\beta)\big)$.

Example 3.1. *Consider the* $\hat{Z}$*-algebra* $(\mathfrak{V}, *, 0)$ *given in Example 2.1, where* $\mathfrak{V} = \{0, \theta_1, \theta_2, \theta_3\}$. *Define an IVIN set* ν *on* $\mathfrak{V}$ *by the following interval-valued functions.*

Truth component:

$$\bar{\mu}_{\nu_{\bar{T}}}(\alpha) = \begin{cases} [0.4, 0.5], & \alpha \in \{0, \theta_1\}, \\ [0.2, 0.6], & \alpha \in \{\theta_2, \theta_3\}, \end{cases} \qquad \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha) = \begin{cases} [0.1, 0.4], & \alpha \in \{0, \theta_1\}, \\ [0.3, 0.5], & \alpha \in \{\theta_2, \theta_3\}. \end{cases}$$

Indeterminacy component:

$$\bar{\mu}_{\nu_{\bar{I}}}(\alpha) = \begin{cases} [0.4, 0.5], & \alpha \in \{0, \theta_1\}, \\ [0.2, 0.6], & \alpha \in \{\theta_2, \theta_3\}, \end{cases} \qquad \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha) = \begin{cases} [0.1, 0.4], & \alpha \in \{0, \theta_1\}, \\ [0.3, 0.5], & \alpha \in \{\theta_2, \theta_3\}. \end{cases}$$

Falsity component:

$$\bar{\mu}_{\nu_{\bar{F}}}(\alpha) = \begin{cases} [0.1, 0.3], & \alpha \in \{0, \theta_1\}, \\ [0.3, 0.4], & \alpha \in \{\theta_2, \theta_3\}, \end{cases} \qquad \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) = \begin{cases} [0.3, 0.6], & \alpha \in \{0, \theta_1\}, \\ [0.2, 0.4], & \alpha \in \{\theta_2, \theta_3\}. \end{cases}$$

Theorem 3.1. *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra and let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}, \bar{I}, \bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}, \bar{I}, \bar{F}}}\big)$$

be an IVIN set on $\mathfrak{V}$, *where for each* $s \in \{T, I, F\}$ *and each* $\alpha \in \mathfrak{V}$,

$$\bar{\mu}_{\nu_{\bar{s}}}(\alpha) = \big[\mu^L_{\nu_s}(\alpha),\, \mu^U_{\nu_s}(\alpha)\big], \quad \bar{\vartheta}_{\nu_{\bar{s}}}(\alpha) = \big[\vartheta^L_{\nu_s}(\alpha),\, \vartheta^U_{\nu_s}(\alpha)\big].$$

Define the lower and upper endpoint (crisp) intuitionistic neutrosophic sets

$$\nu^L = (\mu^L_{\nu_T}, \mu^L_{\nu_I}, \mu^L_{\nu_F}, \vartheta^L_{\nu_T}, \vartheta^L_{\nu_I}, \vartheta^L_{\nu_F}), \quad \nu^U = (\mu^U_{\nu_T}, \mu^U_{\nu_I}, \mu^U_{\nu_F}, \vartheta^U_{\nu_T}, \vartheta^U_{\nu_I}, \vartheta^U_{\nu_F}).$$

Then ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$ if and only if both ν^L and ν^U are (crisp) intuitionistic neutrosophic $\hat{Z}$-ideals of $\mathfrak{V}$.

Proof. Assume first that ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. We use the componentwise order on $\mathbb{D}[0,1]$: for $[\gamma_1, \delta_1], [\gamma_2, \delta_2] \in \mathbb{D}[0,1]$,

$$[\gamma_1, \delta_1] \geq [\gamma_2, \delta_2] \text{ if and only if } \gamma_1 \geq \gamma_2 \text{ and } \delta_1 \geq \delta_2,$$

(and similarly for $\leq$). Also,

$$\operatorname{rmin}\big([\gamma_1, \delta_1], [\gamma_2, \delta_2]\big) = \big[\min\{\gamma_1, \gamma_2\},\ \min\{\delta_1, \delta_2\}\big],$$

$$\operatorname{rmax}\big([\gamma_1, \delta_1], [\gamma_2, \delta_2]\big) = \big[\max\{\gamma_1, \gamma_2\},\ \max\{\delta_1, \delta_2\}\big].$$

From the defining conditions of an IVIN $\hat{Z}$-ideal (applied to $s = T, I$ and to $s = F$), we have, for all $\alpha, \beta \in \mathfrak{V}$,

(a) $\bar{\mu}_{\nu_{\bar{s}}}(0) \geq \bar{\mu}_{\nu_{\bar{s}}}(\alpha)$ and $\bar{\vartheta}_{\nu_{\bar{s}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{s}}}(\alpha)$ for $s \in \{T, I\}$,

(b) $\bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(\alpha)$ and $\bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha)$,

(c) $\bar{\mu}_{\nu_{\bar{s}}}(\alpha) \geq \operatorname{rmin}\big(\bar{\mu}_{\nu_{\bar{s}}}(\alpha * \beta),\ \bar{\mu}_{\nu_{\bar{s}}}(\beta)\big)$ and $\bar{\vartheta}_{\nu_{\bar{s}}}(\alpha) \leq \operatorname{rmax}\big(\bar{\vartheta}_{\nu_{\bar{s}}}(\alpha * \beta),\ \bar{\vartheta}_{\nu_{\bar{s}}}(\beta)\big)$ for $s \in \{T, I\}$,

(d) $\bar{\mu}_{\nu_{\bar{F}}}(\alpha) \leq \operatorname{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha * \beta),\ \bar{\mu}_{\nu_{\bar{F}}}(\beta)\big)$ and $\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) \geq \operatorname{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha * \beta),\ \bar{\vartheta}_{\nu_{\bar{F}}}(\beta)\big)$.

Writing each interval in terms of its endpoints and using the componentwise order, (a) implies that for each $s \in \{T, I\}$,

$$\mu^L_{\nu_s}(0) \geq \mu^L_{\nu_s}(\alpha), \quad \mu^U_{\nu_s}(0) \geq \mu^U_{\nu_s}(\alpha), \quad \vartheta^L_{\nu_s}(0) \leq \vartheta^L_{\nu_s}(\alpha), \quad \vartheta^U_{\nu_s}(0) \leq \vartheta^U_{\nu_s}(\alpha).$$

Similarly, (b) yields

$$\mu^L_{\nu_F}(0) \leq \mu^L_{\nu_F}(\alpha), \quad \mu^U_{\nu_F}(0) \leq \mu^U_{\nu_F}(\alpha), \quad \vartheta^L_{\nu_F}(0) \geq \vartheta^L_{\nu_F}(\alpha), \quad \vartheta^U_{\nu_F}(0) \geq \vartheta^U_{\nu_F}(\alpha).$$

Next, using the explicit forms of rmin and rmax, (c) implies that for each $s \in \{T, I\}$,

$$\mu^L_{\nu_s}(\alpha) \geq \min\{\mu^L_{\nu_s}(\alpha * \beta),\ \mu^L_{\nu_s}(\beta)\}, \quad \mu^U_{\nu_s}(\alpha) \geq \min\{\mu^U_{\nu_s}(\alpha * \beta),\ \mu^U_{\nu_s}(\beta)\},$$

$$\vartheta^L_{\nu_s}(\alpha) \leq \max\{\vartheta^L_{\nu_s}(\alpha * \beta),\ \vartheta^L_{\nu_s}(\beta)\}, \quad \vartheta^U_{\nu_s}(\alpha) \leq \max\{\vartheta^U_{\nu_s}(\alpha * \beta),\ \vartheta^U_{\nu_s}(\beta)\}.$$

Likewise, (d) implies

$$\mu^L_{\nu_F}(\alpha) \leq \max\{\mu^L_{\nu_F}(\alpha * \beta),\ \mu^L_{\nu_F}(\beta)\}, \quad \mu^U_{\nu_F}(\alpha) \leq \max\{\mu^U_{\nu_F}(\alpha * \beta),\ \mu^U_{\nu_F}(\beta)\},$$

$$\vartheta^L_{\nu_F}(\alpha) \geq \min\{\vartheta^L_{\nu_F}(\alpha * \beta),\ \vartheta^L_{\nu_F}(\beta)\}, \quad \vartheta^U_{\nu_F}(\alpha) \geq \min\{\vartheta^U_{\nu_F}(\alpha * \beta),\ \vartheta^U_{\nu_F}(\beta)\}.$$

Hence ν^L and ν^U satisfy the defining conditions of an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{V}$.

Conversely, assume that both ν^L and ν^U are intuitionistic neutrosophic $\hat{Z}$-ideals of $\mathfrak{V}$. Then, for each $s \in \{T, I, F\}$, the endpoint inequalities above hold for $\mu^L_{\nu_s}, \mu^U_{\nu_s}$ and $\vartheta^L_{\nu_s}, \vartheta^U_{\nu_s}$. Combining the two endpoint inequalities yields the corresponding interval inequalities. For example, for $s \in \{T, I\}$,

$$\mu^L_{\nu_s}(\alpha) \geq \min\{\mu^L_{\nu_s}(\alpha * \beta),\, \mu^L_{\nu_s}(\beta)\} \text{ and } \mu^U_{\nu_s}(\alpha) \geq \min\{\mu^U_{\nu_s}(\alpha * \beta),\, \mu^U_{\nu_s}(\beta)\}$$

imply

$$\bar{\mu}_{\nu_{\bar{s}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{s}}}(\alpha * \beta),\, \bar{\mu}_{\nu_{\bar{s}}}(\beta)\big),$$

and similarly for the remaining conditions (including the $\bar{F}$-part with rmax and rmin). Therefore, ν satisfies all the axioms of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. □

Theorem 3.2. *Let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$$

be an IVIN set on a $\hat{Z}$-algebra $(\mathfrak{V}, *, 0)$. Then ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$ if and only if

$$\big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}\big) \quad \textit{and} \quad \big((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c\big)$$

are IVIN $\hat{Z}$-ideals of $\mathfrak{V}$.

Proof. Assume that ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. Then, by definition, for all $\alpha, \beta \in \mathfrak{V}$,

(i) $\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(\alpha)$, $\bar{\mu}_{\nu_{\bar{I}}}(0) \geq \bar{\mu}_{\nu_{\bar{I}}}(\alpha)$, and $\bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(\alpha)$,

(ii) $\bar{\mu}_{\nu_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{T}}}(\beta)\big)$, $\bar{\mu}_{\nu_{\bar{I}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{I}}}(\beta)\big)$, and $\bar{\mu}_{\nu_{\bar{F}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{F}}}(\beta)\big)$,

(iii) $\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha)$, $\bar{\vartheta}_{\nu_{\bar{I}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha)$, and $\bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha)$,

(iv) $\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{T}}}(\beta)\big)$, $\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{I}}}(\beta)\big)$, and $\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{F}}}(\beta)\big)$.

Hence $\big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}\big)$ satisfies the defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.

It remains to show that $\big((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c\big)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. Recall that for any $\gamma \in \mathbb{D}[0, 1]$, the complement is defined by $\gamma^c = [1, 1] - \gamma$, and for all $\gamma_1, \gamma_2 \in \mathbb{D}[0, 1]$,

$$(\mathrm{rmax}(\gamma_1, \gamma_2))^c = \mathrm{rmin}(\gamma_1^c, \gamma_2^c) \quad \text{and} \quad (\mathrm{rmin}(\gamma_1, \gamma_2))^c = \mathrm{rmax}(\gamma_1^c, \gamma_2^c).$$

From (iii) and the order-reversing property of the complement, for all $\alpha \in \mathfrak{V}$,

$$(\bar{\vartheta}_{\nu_{\bar{T}}})^c(0) \geq (\bar{\vartheta}_{\nu_{\bar{T}}})^c(\alpha), \quad (\bar{\vartheta}_{\nu_{\bar{I}}})^c(0) \geq (\bar{\vartheta}_{\nu_{\bar{I}}})^c(\alpha), \quad (\bar{\vartheta}_{\nu_{\bar{F}}})^c(0) \leq (\bar{\vartheta}_{\nu_{\bar{F}}})^c(\alpha).$$

Also, applying the complement to (iv) and using the identities above, we obtain for all $\alpha, \beta \in \mathfrak{V}$,

$$(\bar{\vartheta}_{\nu_{\bar{T}}})^c(\alpha) \geq \mathrm{rmin}\big((\bar{\vartheta}_{\nu_{\bar{T}}})^c(\alpha * \beta),\, (\bar{\vartheta}_{\nu_{\bar{T}}})^c(\beta)\big),$$

$$(\bar{\vartheta}_{\nu_{\bar{I}}})^c(\alpha) \geq \mathrm{rmin}\big((\bar{\vartheta}_{\nu_{\bar{I}}})^c(\alpha * \beta),\, (\bar{\vartheta}_{\nu_{\bar{I}}})^c(\beta)\big),$$

$$(\bar{\vartheta}_{\nu_{\bar{F}}})^c(\alpha) \leq \mathrm{rmax}\big((\bar{\vartheta}_{\nu_{\bar{F}}})^c(\alpha * \beta),\, (\bar{\vartheta}_{\nu_{\bar{F}}})^c(\beta)\big).$$

Therefore $\left((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c\right)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.

Conversely, assume that $\left(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}\right)$ and $\left((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c\right)$ are IVIN $\hat{Z}$-ideals of $\mathfrak{V}$. Then the defining conditions (i) and (ii) hold for $\bar{\mu}$. Moreover, since $(\bar{\vartheta}_{\nu_{\bar{T}}})^c$ and $(\bar{\vartheta}_{\nu_{\bar{I}}})^c$ are ideals of the "$\bar{T}$-type" and $(\bar{\vartheta}_{\nu_{\bar{F}}})^c$ is an ideal of the "$\bar{F}$-type", we have for all $\alpha, \beta \in \mathfrak{V}$,

$$(\bar{\vartheta}_{\nu_{\bar{T}}})^c(0) \geq (\bar{\vartheta}_{\nu_{\bar{T}}})^c(\alpha), \quad (\bar{\vartheta}_{\nu_{\bar{I}}})^c(0) \geq (\bar{\vartheta}_{\nu_{\bar{I}}})^c(\alpha), \quad (\bar{\vartheta}_{\nu_{\bar{F}}})^c(0) \leq (\bar{\vartheta}_{\nu_{\bar{F}}})^c(\alpha),$$

which implies (by taking complements) that

$$\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha), \quad \bar{\vartheta}_{\nu_{\bar{I}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha), \quad \bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha).$$

Similarly, from the ideal conditions on $(\bar{\vartheta}_{\nu_{\bar{T}}})^c$, $(\bar{\vartheta}_{\nu_{\bar{I}}})^c$, and $(\bar{\vartheta}_{\nu_{\bar{F}}})^c$, and the complement identities for $\mathrm{rmin}/\mathrm{rmax}$, we recover

$$\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\left(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{T}}}(\beta)\right), \quad \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\left(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{I}}}(\beta)\right),$$

$$\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\left(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha * \beta),\, \bar{\vartheta}_{\nu_{\bar{F}}}(\beta)\right).$$

Thus, $\nu = \left(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\right)$ satisfies all defining conditions of an IVIN $\hat{Z}$-ideal, completing the proof. □

Theorem 3.3. *Let ν and ε be two IVIN $\hat{Z}$-ideals of $\mathfrak{V}$. Then $\nu \cap \varepsilon$ is also an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.*

Proof. Let

$$\nu = \left(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\right) \quad \text{and} \quad \varepsilon = \left(\bar{\mu}_{\varepsilon_{\bar{T}}}, \bar{\mu}_{\varepsilon_{\bar{I}}}, \bar{\mu}_{\varepsilon_{\bar{F}}}, \bar{\vartheta}_{\varepsilon_{\bar{T}}}, \bar{\vartheta}_{\varepsilon_{\bar{I}}}, \bar{\vartheta}_{\varepsilon_{\bar{F}}}\right)$$

be IVIN $\hat{Z}$-ideals of $\mathfrak{V}$. Define $\nu \cap \varepsilon$ by, for all $\alpha \in \mathfrak{V}$,

$$\begin{aligned}
\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha) &= \mathrm{rmin}\left(\bar{\mu}_{\nu_{\bar{T}}}(\alpha),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha)\right), & \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha) &= \mathrm{rmax}\left(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha),\, \bar{\vartheta}_{\varepsilon_{\bar{T}}}(\alpha)\right),\\
\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha) &= \mathrm{rmin}\left(\bar{\mu}_{\nu_{\bar{I}}}(\alpha),\, \bar{\mu}_{\varepsilon_{\bar{I}}}(\alpha)\right), & \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha) &= \mathrm{rmax}\left(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha),\, \bar{\vartheta}_{\varepsilon_{\bar{I}}}(\alpha)\right),\\
\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha) &= \mathrm{rmax}\left(\bar{\mu}_{\nu_{\bar{F}}}(\alpha),\, \bar{\mu}_{\varepsilon_{\bar{F}}}(\alpha)\right), & \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha) &= \mathrm{rmin}\left(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha),\, \bar{\vartheta}_{\varepsilon_{\bar{F}}}(\alpha)\right).
\end{aligned}$$

We verify that $\nu \cap \varepsilon$ satisfies the axioms of an IVIN $\hat{Z}$-ideal. Let $\alpha, \beta \in \mathfrak{V}$.

(i) Since ν and ε are IVIN $\hat{Z}$-ideals, we have $\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(\alpha)$ and $\bar{\mu}_{\varepsilon_{\bar{T}}}(0) \geq \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha)$. Thus,

$$\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(0) = \mathrm{rmin}\left(\bar{\mu}_{\nu_{\bar{T}}}(0),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(0)\right) \geq \mathrm{rmin}\left(\bar{\mu}_{\nu_{\bar{T}}}(\alpha),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha)\right) = \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha).$$

The same argument gives

$$\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(0) \geq \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha), \quad \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(0) \leq \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha).$$

(ii) Using $\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha)$ and $\bar{\vartheta}_{\varepsilon_{\bar{T}}}(0) \leq \bar{\vartheta}_{\varepsilon_{\bar{T}}}(\alpha)$, we obtain

$$\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(0) = \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(0),\, \bar{\vartheta}_{\varepsilon_{\bar{T}}}(0)\big) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha),\, \bar{\vartheta}_{\varepsilon_{\bar{T}}}(\alpha)\big) = \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha).$$

Similarly,

$$\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(0) \leq \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha), \quad \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(0) \geq \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha).$$

(iii) Since ν and ε are IVIN $\hat{Z}$-ideals,

$$\bar{\mu}_{\nu_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha*\beta),\, \bar{\mu}_{\nu_{\bar{T}}}(\beta)\big), \quad \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha*\beta),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta)\big).$$

Taking rmin of both sides and using the monotonicity of rmin on $\mathbb{D}[0,1]$, we get

$$\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha*\beta),\, \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{T}}}(\beta)\big).$$

The same argument yields

$$\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha*\beta),\, \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{I}}}(\beta)\big),$$

and, using the $\bar{F}$-conditions for ν and ε together with the monotonicity of rmax,

$$\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha*\beta),\, \bar{\mu}_{(\nu\cap\varepsilon)_{\bar{F}}}(\beta)\big).$$

(iv) The verification for $\bar{\vartheta}$ is analogous. Using the ideal conditions for $\bar{\vartheta}$ in ν and ε and the monotonicity of rmax and rmin, we obtain

$$\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(\alpha*\beta),\, \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{T}}}(\beta)\big),$$

$$\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(\alpha*\beta),\, \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{I}}}(\beta)\big),$$

and

$$\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(\alpha*\beta),\, \bar{\vartheta}_{(\nu\cap\varepsilon)_{\bar{F}}}(\beta)\big).$$

Thus $\nu\cap\varepsilon$ satisfies all defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. □

Theorem 3.4. *Let* $(\mathfrak{V}, *, 0)$ *be a* $\hat{Z}$*-algebra and let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$$

be an IVIN set on $\mathfrak{V}$*. Define the IVIN sets*

$$\circledast\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, (\bar{\mu}_{\nu_{\bar{T}}})^c, (\bar{\mu}_{\nu_{\bar{I}}})^c, (\bar{\mu}_{\nu_{\bar{F}}})^c\big),$$

$$\circledcirc\nu = \big((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big),$$

where the complement on $\mathbb{D}[0,1]$ *is given by* $\gamma^c = [1,1] - \gamma$*. Then* ν *is an IVIN* $\hat{Z}$*-ideal of* $\mathfrak{V}$ *if and only if both* $\circledast\nu$ *and* $\circledcirc\nu$ *are IVIN* $\hat{Z}$*-ideals of* $\mathfrak{V}$*.*

Proof. Assume that ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. Then, for all $\alpha, \beta \in \mathfrak{V}$, we have

(i) $\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(\alpha)$, $\bar{\mu}_{\nu_{\bar{I}}}(0) \geq \bar{\mu}_{\nu_{\bar{I}}}(\alpha)$, and $\bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(\alpha)$,

(ii) $\bar{\mu}_{\nu_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{T}}}(\beta)\big)$, $\bar{\mu}_{\nu_{\bar{I}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{I}}}(\beta)\big)$, and $\bar{\mu}_{\nu_{\bar{F}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha * \beta), \bar{\mu}_{\nu_{\bar{F}}}(\beta)\big)$,

(iii) $\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha)$, $\bar{\vartheta}_{\nu_{\bar{I}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha)$, and $\bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha)$,

(iv) $\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{T}}}(\beta)\big)$, $\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{I}}}(\beta)\big)$, and $\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha * \beta), \bar{\vartheta}_{\nu_{\bar{F}}}(\beta)\big)$.

We show that $\circledast\nu$ is an IVIN $\hat{Z}$-ideal. Its "membership part" $\big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}\big)$ already satisfies the required conditions by (i) and (ii). It remains to check the "non-membership part" $\big((\bar{\mu}_{\nu_{\bar{T}}})^c, (\bar{\mu}_{\nu_{\bar{I}}})^c, (\bar{\mu}_{\nu_{\bar{F}}})^c\big)$.

Recall that the complement reverses the order on $\mathbb{D}[0,1]$, and for all $\gamma_1, \gamma_2 \in \mathbb{D}[0,1]$,

$$(\mathrm{rmin}(\gamma_1, \gamma_2))^c = \mathrm{rmax}(\gamma_1^c, \gamma_2^c) \quad \text{and} \quad (\mathrm{rmax}(\gamma_1, \gamma_2))^c = \mathrm{rmin}(\gamma_1^c, \gamma_2^c).$$

Applying the complement to (i), for all $\alpha \in \mathfrak{V}$ we obtain

$$(\bar{\mu}_{\nu_{\bar{T}}})^c(0) \leq (\bar{\mu}_{\nu_{\bar{T}}})^c(\alpha), \quad (\bar{\mu}_{\nu_{\bar{I}}})^c(0) \leq (\bar{\mu}_{\nu_{\bar{I}}})^c(\alpha), \quad (\bar{\mu}_{\nu_{\bar{F}}})^c(0) \geq (\bar{\mu}_{\nu_{\bar{F}}})^c(\alpha).$$

Similarly, applying the complement to (ii) yields, for all $\alpha, \beta \in \mathfrak{V}$,

$$(\bar{\mu}_{\nu_{\bar{T}}})^c(\alpha) \leq \mathrm{rmax}\big((\bar{\mu}_{\nu_{\bar{T}}})^c(\alpha * \beta), (\bar{\mu}_{\nu_{\bar{T}}})^c(\beta)\big),$$

$$(\bar{\mu}_{\nu_{\bar{I}}})^c(\alpha) \leq \mathrm{rmax}\big((\bar{\mu}_{\nu_{\bar{I}}})^c(\alpha * \beta), (\bar{\mu}_{\nu_{\bar{I}}})^c(\beta)\big),$$

$$(\bar{\mu}_{\nu_{\bar{F}}})^c(\alpha) \geq \mathrm{rmin}\big((\bar{\mu}_{\nu_{\bar{F}}})^c(\alpha * \beta), (\bar{\mu}_{\nu_{\bar{F}}})^c(\beta)\big).$$

Thus $\circledast\nu$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.

Next, we show that $\circledcirc\nu$ is an IVIN $\hat{Z}$-ideal. Its "non-membership part" $\big(\bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$ satisfies the required conditions by (iii) and (iv). Applying the same complement rules to (iii) and (iv), we conclude that $\big((\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c\big)$ satisfies the corresponding conditions required for the "membership part" of an IVIN $\hat{Z}$-ideal. Hence $\circledcirc\nu$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.

Conversely, assume that both $\circledast\nu$ and $\circledcirc\nu$ are IVIN $\hat{Z}$-ideals of $\mathfrak{V}$. From $\circledast\nu$ being an ideal, we obtain the defining conditions (i) and (ii) for $\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}$. From $\circledcirc\nu$ being an ideal, we obtain the defining conditions for $(\bar{\vartheta}_{\nu_{\bar{T}}})^c, (\bar{\vartheta}_{\nu_{\bar{I}}})^c, (\bar{\vartheta}_{\nu_{\bar{F}}})^c$, and by taking complements (using the same identities) we recover (iii) and (iv) for $\bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}$. Hence ν satisfies all the axioms of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. □

4. Homomorphism of Interval-Valued Intuitionistic Neutrosophic $\hat{Z}$-Ideals

In this section, we study how IVIN $\hat{Z}$-ideals behave under $\hat{Z}$-homomorphisms. To this end, we define the image and inverse image of an IVIN set with respect to a mapping, and then investigate the preservation of the IVIN $\hat{Z}$-ideal structure through these transformations. In particular, we establish conditions under which homomorphic images of IVIN $\hat{Z}$-ideals remain IVIN $\hat{Z}$-ideals (including a suitable attainment property for interval-valued operators), and we show that inverse images preserve

IVIN $\hat{Z}$-ideals without additional assumptions. These results connect IVIN $\hat{Z}$-ideals to standard homomorphism techniques and provide tools for constructing new examples and transferring properties between $\hat{Z}$-algebras.

Definition 4.1. *Let $\mathfrak{V}$ and $\mathfrak{F}$ be nonempty sets and let $h : \mathfrak{V} \to \mathfrak{F}$ be a mapping. Let $\mathbb{D}[0,1]$ denote the family of all closed subintervals of $[0,1]$, and let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$$

be an IVIN set on $\mathfrak{V}$, where each component maps $\mathfrak{V}$ into $\mathbb{D}[0,1]$. The image of ν under h, denoted by $h(\nu)$, is the IVIN set on $\mathfrak{F}$ defined by

$$h(\nu) = \big(h_{\mathrm{rsup}}\bar{\mu}_{\nu_{\bar{T}}},\, h_{\mathrm{rsup}}\bar{\mu}_{\nu_{\bar{I}}},\, h_{\mathrm{rinf}}\bar{\mu}_{\nu_{\bar{F}}},\, h_{\mathrm{rinf}}\bar{\vartheta}_{\nu_{\bar{T}}},\, h_{\mathrm{rinf}}\bar{\vartheta}_{\nu_{\bar{I}}},\, h_{\mathrm{rsup}}\bar{\vartheta}_{\nu_{\bar{F}}}\big),$$

where, for each $\beta \in \mathfrak{F}$,

$$\big(h_{\mathrm{rsup}}\bar{\mu}_{\nu_{\bar{T}}}\big)(\beta) = \begin{cases} \operatorname*{rsup}_{\alpha \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{T}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [0,0], & \text{otherwise}, \end{cases}$$

$$\big(h_{\mathrm{rsup}}\bar{\mu}_{\nu_{\bar{I}}}\big)(\beta) = \begin{cases} \operatorname*{rsup}_{\alpha \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{I}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [0,0], & \text{otherwise}, \end{cases}$$

$$\big(h_{\mathrm{rinf}}\bar{\mu}_{\nu_{\bar{F}}}\big)(\beta) = \begin{cases} \operatorname*{rinf}_{\alpha \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{F}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [1,1], & \text{otherwise}, \end{cases}$$

$$\big(h_{\mathrm{rinf}}\bar{\vartheta}_{\nu_{\bar{T}}}\big)(\beta) = \begin{cases} \operatorname*{rinf}_{\alpha \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{T}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [1,1], & \text{otherwise}, \end{cases}$$

$$\big(h_{\mathrm{rinf}}\bar{\vartheta}_{\nu_{\bar{I}}}\big)(\beta) = \begin{cases} \operatorname*{rinf}_{\alpha \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [1,1], & \text{otherwise}, \end{cases}$$

$$\big(h_{\mathrm{rsup}}\bar{\vartheta}_{\nu_{\bar{F}}}\big)(\beta) = \begin{cases} \operatorname*{rsup}_{\alpha \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha), & \text{if } h^{-1}(\beta) \neq \varnothing, \\ [0,0], & \text{otherwise}. \end{cases}$$

Definition 4.2. *Let $h : (\mathfrak{V}, *, 0) \to (\mathfrak{F}, *', 0')$ be a surjective mapping and let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$$

be an IVIN set on $\mathfrak{V}$. We say that ν has the sup–sup–inf property with respect to h if, for every $\beta \in \mathfrak{F}$, there exist elements

$$\alpha_T, \alpha_I, \alpha_F, \alpha'_T, \alpha'_I, \alpha'_F \in h^{-1}(\beta)$$

such that

$$\bar{\mu}_{\nu_{\bar{T}}}(\alpha_T) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{T}}}(x), \quad \bar{\mu}_{\nu_{\bar{I}}}(\alpha_I) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{I}}}(x), \quad \bar{\mu}_{\nu_{\bar{F}}}(\alpha_F) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{F}}}(x),$$

and

$$\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha'_T) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{T}}}(x), \quad \bar{\vartheta}_{\nu_{\bar{I}}}(\alpha'_I) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{I}}}(x), \quad \bar{\vartheta}_{\nu_{\bar{F}}}(\alpha'_F) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{F}}}(x).$$

Theorem 4.1. *Let $h : (\mathfrak{V}, *, 0) \to (\mathfrak{F}, *', 0')$ be a surjective homomorphism of $\hat{Z}$-algebras, and let*

$$\nu = \left(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\right)$$

be an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$ having the sup–sup–inf property. Then the image $h(\nu)$ (defined as in Definition 4.1) is an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$.

Proof. Since h is surjective, for every $\beta \in \mathfrak{F}$ the set $h^{-1}(\beta)$ is nonempty. By Definition 4.1, for each $\beta \in \mathfrak{F}$,

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(\beta) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{T}}}(x), \quad \bar{\mu}_{h(\nu)_{\bar{I}}}(\beta) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{I}}}(x), \quad \bar{\mu}_{h(\nu)_{\bar{F}}}(\beta) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\mu}_{\nu_{\bar{F}}}(x),$$

$$\bar{\vartheta}_{h(\nu)_{\bar{T}}}(\beta) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{T}}}(x), \quad \bar{\vartheta}_{h(\nu)_{\bar{I}}}(\beta) = \operatorname*{rinf}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{I}}}(x), \quad \bar{\vartheta}_{h(\nu)_{\bar{F}}}(\beta) = \operatorname*{rsup}_{x \in h^{-1}(\beta)} \bar{\vartheta}_{\nu_{\bar{F}}}(x).$$

We verify that $h(\nu)$ satisfies the defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$. Let $a, b \in \mathfrak{F}$ and choose $a_0 \in h^{-1}(a)$ and $b_0 \in h^{-1}(b)$. Since h is a homomorphism, $h(0) = 0'$ and

$$h(a_0 * b_0) = h(a_0) *' h(b_0) = a *' b,$$

hence $a_0 * b_0 \in h^{-1}(a *' b)$.

(i) Conditions at $0'$. Because $0 \in h^{-1}(0')$ and ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$, for all $x \in \mathfrak{V}$,

$$\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(x), \quad \bar{\mu}_{\nu_{\bar{I}}}(0) \geq \bar{\mu}_{\nu_{\bar{I}}}(x), \quad \bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(x),$$

$$\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(x), \quad \bar{\vartheta}_{\nu_{\bar{I}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{I}}}(x), \quad \bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(x).$$

Taking rsup or rinf over $h^{-1}(a)$ gives

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(0') = \operatorname*{rsup}_{x \in h^{-1}(0')} \bar{\mu}_{\nu_{\bar{T}}}(x) \geq \bar{\mu}_{\nu_{\bar{T}}}(0) \geq \operatorname*{rsup}_{x \in h^{-1}(a)} \bar{\mu}_{\nu_{\bar{T}}}(x) = \bar{\mu}_{h(\nu)_{\bar{T}}}(a),$$

and similarly,

$$\bar{\mu}_{h(\nu)_{\bar{I}}}(0') \geq \bar{\mu}_{h(\nu)_{\bar{I}}}(a), \quad \bar{\mu}_{h(\nu)_{\bar{F}}}(0') \leq \bar{\mu}_{h(\nu)_{\bar{F}}}(a).$$

Likewise,

$$\bar{\vartheta}_{h(\nu)_{\bar{T}}}(0') \leq \bar{\vartheta}_{h(\nu)_{\bar{T}}}(a), \quad \bar{\vartheta}_{h(\nu)_{\bar{I}}}(0') \leq \bar{\vartheta}_{h(\nu)_{\bar{I}}}(a), \quad \bar{\vartheta}_{h(\nu)_{\bar{F}}}(0') \geq \bar{\vartheta}_{h(\nu)_{\bar{F}}}(a).$$

(ii) The $\hat{Z}$-ideal inequalities. We prove the inequality for $\bar{\mu}_{h(\nu)_{\bar{T}}}$; the proofs for $\bar{I}$, $\bar{F}$ and for $\bar{\vartheta}$ are analogous. Since $a_0 * b_0 \in h^{-1}(a *' b)$ and $b_0 \in h^{-1}(b)$, we have

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(a *' b) = \operatorname*{rsup}_{x \in h^{-1}(a*'b)} \bar{\mu}_{\nu_{\bar{T}}}(x) \geq \bar{\mu}_{\nu_{\bar{T}}}(a_0 * b_0), \quad \bar{\mu}_{h(\nu)_{\bar{T}}}(b) \geq \bar{\mu}_{\nu_{\bar{T}}}(b_0).$$

Hence, by monotonicity of rmin,

$$\operatorname{rmin}\big(\bar{\mu}_{h(\nu)_{\bar{T}}}(a *' b),\, \bar{\mu}_{h(\nu)_{\bar{T}}}(b)\big) \geq \operatorname{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(a_0 * b_0),\, \bar{\mu}_{\nu_{\bar{T}}}(b_0)\big) \geq \bar{\mu}_{\nu_{\bar{T}}}(a_0),$$

because ν is an IVIN $\hat{Z}$-ideal in $\mathfrak{V}$. Now, by the sup–sup–inf property, the element $a_0 \in h^{-1}(a)$ can be chosen so that

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(a) = \operatorname*{rsup}_{x \in h^{-1}(a)} \bar{\mu}_{\nu_{\bar{T}}}(x) = \bar{\mu}_{\nu_{\bar{T}}}(a_0),$$

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(b) = \bar{\mu}_{\nu_{\bar{T}}}(b_0),$$

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(a *' b) = \bar{\mu}_{\nu_{\bar{T}}}(a_0 * b_0).$$

Therefore,

$$\bar{\mu}_{h(\nu)_{\bar{T}}}(a) \geq \operatorname{rmin}\big(\bar{\mu}_{h(\nu)_{\bar{T}}}(a *' b),\, \bar{\mu}_{h(\nu)_{\bar{T}}}(b)\big).$$

The same argument applies to $\bar{\mu}_{h(\nu)_{\bar{I}}}$. For the $\bar{F}$-part, using rinf in the definition of $\bar{\mu}_{h(\nu)_{\bar{F}}}$ and the $\bar{F}$-inequality for ν, we similarly obtain

$$\bar{\mu}_{h(\nu)_{\bar{F}}}(a) \leq \operatorname{rmax}\big(\bar{\mu}_{h(\nu)_{\bar{F}}}(a *' b),\, \bar{\mu}_{h(\nu)_{\bar{F}}}(b)\big).$$

The verification for $\bar{\vartheta}_{h(\nu)_{\bar{T}}}, \bar{\vartheta}_{h(\nu)_{\bar{I}}}$ and $\bar{\vartheta}_{h(\nu)_{\bar{F}}}$ is analogous, using the defining inequalities for $\bar{\vartheta}$ in ν and the monotonicity of rmax and rmin.

Hence $h(\nu)$ satisfies all axioms of an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$. □

Theorem 4.2. *Let $h : (\mathfrak{V}, *, 0) \to (\mathfrak{F}, *', 0')$ be a homomorphism of $\hat{Z}$-algebras, and let*

$$\varepsilon = \big(\bar{\mu}_{\varepsilon_{\bar{T}}}, \bar{\mu}_{\varepsilon_{\bar{I}}}, \bar{\mu}_{\varepsilon_{\bar{F}}}, \bar{\vartheta}_{\varepsilon_{\bar{T}}}, \bar{\vartheta}_{\varepsilon_{\bar{I}}}, \bar{\vartheta}_{\varepsilon_{\bar{F}}}\big)$$

be an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$. Define the inverse image $h^{-1}(\varepsilon)$ on $\mathfrak{V}$ by

$$\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha) = \bar{\mu}_{\varepsilon_{\bar{T}}}\big(h(\alpha)\big), \quad \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha) = \bar{\mu}_{\varepsilon_{\bar{I}}}\big(h(\alpha)\big), \quad \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha) = \bar{\mu}_{\varepsilon_{\bar{F}}}\big(h(\alpha)\big),$$

$$\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha) = \bar{\vartheta}_{\varepsilon_{\bar{T}}}\big(h(\alpha)\big), \quad \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha) = \bar{\vartheta}_{\varepsilon_{\bar{I}}}\big(h(\alpha)\big), \quad \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha) = \bar{\vartheta}_{\varepsilon_{\bar{F}}}\big(h(\alpha)\big),$$

for all $\alpha \in \mathfrak{V}$. Then $h^{-1}(\varepsilon)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$.

Proof. Let $\alpha, \beta \in \mathfrak{V}$. Since h is a homomorphism, $h(0) = 0'$ and $h(\alpha * \beta) = h(\alpha) *' h(\beta)$.

(i) Since ε is an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$, for all $y \in \mathfrak{F}$ we have

$$\bar{\mu}_{\varepsilon_{\bar{T}}}(0') \geq \bar{\mu}_{\varepsilon_{\bar{T}}}(y), \quad \bar{\mu}_{\varepsilon_{\bar{I}}}(0') \geq \bar{\mu}_{\varepsilon_{\bar{I}}}(y), \quad \bar{\mu}_{\varepsilon_{\bar{F}}}(0') \leq \bar{\mu}_{\varepsilon_{\bar{F}}}(y),$$

$$\bar{\vartheta}_{\varepsilon_{\bar{T}}}(0') \leq \bar{\vartheta}_{\varepsilon_{\bar{T}}}(y), \quad \bar{\vartheta}_{\varepsilon_{\bar{I}}}(0') \leq \bar{\vartheta}_{\varepsilon_{\bar{I}}}(y), \quad \bar{\vartheta}_{\varepsilon_{\bar{F}}}(0') \geq \bar{\vartheta}_{\varepsilon_{\bar{F}}}(y).$$

Taking $y = h(\alpha)$ and using the definition of $h^{-1}(\varepsilon)$ (together with $h(0) = 0'$) yields

$$\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(0) \geq \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha), \quad \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(0) \geq \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha), \quad \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(0) \leq \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha),$$

$$\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(0) \leq \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha), \quad \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(0) \leq \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha), \quad \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(0) \geq \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha).$$

(ii) Again, since ε is an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$, we have

$$\bar{\mu}_{\varepsilon_{\bar{T}}}(h(\alpha)) \geq \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{T}}}(h(\alpha) *' h(\beta)),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(h(\beta))\big),$$

$$\bar{\mu}_{\varepsilon_{\bar{I}}}(h(\alpha)) \geq \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{I}}}(h(\alpha) *' h(\beta)),\, \bar{\mu}_{\varepsilon_{\bar{I}}}(h(\beta))\big),$$

$$\bar{\mu}_{\varepsilon_{\bar{F}}}(h(\alpha)) \leq \mathrm{rmax}\big(\bar{\mu}_{\varepsilon_{\bar{F}}}(h(\alpha) *' h(\beta)),\, \bar{\mu}_{\varepsilon_{\bar{F}}}(h(\beta))\big),$$

$$\bar{\vartheta}_{\varepsilon_{\bar{T}}}(h(\alpha)) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\varepsilon_{\bar{T}}}(h(\alpha) *' h(\beta)),\, \bar{\vartheta}_{\varepsilon_{\bar{T}}}(h(\beta))\big),$$

$$\bar{\vartheta}_{\varepsilon_{\bar{I}}}(h(\alpha)) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\varepsilon_{\bar{I}}}(h(\alpha) *' h(\beta)),\, \bar{\vartheta}_{\varepsilon_{\bar{I}}}(h(\beta))\big),$$

$$\bar{\vartheta}_{\varepsilon_{\bar{F}}}(h(\alpha)) \geq \mathrm{rmin}\big(\bar{\vartheta}_{\varepsilon_{\bar{F}}}(h(\alpha) *' h(\beta)),\, \bar{\vartheta}_{\varepsilon_{\bar{F}}}(h(\beta))\big).$$

Using $h(\alpha * \beta) = h(\alpha) *' h(\beta)$ and the definition of $h^{-1}(\varepsilon)$, we obtain

$$\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha * \beta),\, \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{T}}}(\beta)\big),$$

$$\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha * \beta),\, \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{I}}}(\beta)\big),$$

$$\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha * \beta),\, \bar{\mu}_{h^{-1}(\varepsilon)_{\bar{F}}}(\beta)\big),$$

$$\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(\alpha * \beta),\, \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{T}}}(\beta)\big),$$

$$\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha) \leq \mathrm{rmax}\big(\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(\alpha * \beta),\, \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{I}}}(\beta)\big),$$

$$\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha) \geq \mathrm{rmin}\big(\bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(\alpha * \beta),\, \bar{\vartheta}_{h^{-1}(\varepsilon)_{\bar{F}}}(\beta)\big).$$

Therefore $h^{-1}(\varepsilon)$ satisfies all defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$. □

Theorem 4.3. *Let $h : (\mathfrak{V}, *, 0) \to (\mathfrak{F}, *', 0')$ be a surjective homomorphism of $\hat{Z}$-algebras, and let*

$$\nu = \big(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\big)$$

be an IVIN set on $\mathfrak{F}$. If $h^{-1}(\nu)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$, then ν is an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$.

Proof. Since h is surjective, for every $\beta \in \mathfrak{F}$ there exists $\alpha \in \mathfrak{V}$ such that $h(\alpha) = \beta$. Also, $h(0) = 0'$ and $h(x * y) = h(x) *' h(y)$ for all $x, y \in \mathfrak{V}$.

(i) Conditions at $0'$. Let $\beta \in \mathfrak{F}$ and choose $\alpha \in \mathfrak{V}$ with $h(\alpha) = \beta$. Since $h^{-1}(\nu)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$, we have

$$\bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(0) \geq \bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(\alpha), \quad \bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(0) \geq \bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(\alpha), \quad \bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(0) \leq \bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(\alpha),$$

$$\bar{\vartheta}_{h^{-1}(\nu)_{\bar{T}}}(0) \leq \bar{\vartheta}_{h^{-1}(\nu)_{\bar{T}}}(\alpha), \quad \bar{\vartheta}_{h^{-1}(\nu)_{\bar{I}}}(0) \leq \bar{\vartheta}_{h^{-1}(\nu)_{\bar{I}}}(\alpha), \quad \bar{\vartheta}_{h^{-1}(\nu)_{\bar{F}}}(0) \geq \bar{\vartheta}_{h^{-1}(\nu)_{\bar{F}}}(\alpha).$$

By the definition of inverse image,

$$\bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(\alpha) = \bar{\mu}_{\nu_{\bar{T}}}(h(\alpha)) = \bar{\mu}_{\nu_{\bar{T}}}(\beta), \quad \bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(\alpha) = \bar{\mu}_{\nu_{\bar{I}}}(\beta), \quad \bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(\alpha) = \bar{\mu}_{\nu_{\bar{F}}}(\beta),$$

and similarly for $\bar{\vartheta}$. Using $h(0) = 0'$, the inequalities above become

$$\bar{\mu}_{\nu_{\bar{T}}}(0') \geq \bar{\mu}_{\nu_{\bar{T}}}(\beta), \quad \bar{\mu}_{\nu_{\bar{I}}}(0') \geq \bar{\mu}_{\nu_{\bar{I}}}(\beta), \quad \bar{\mu}_{\nu_{\bar{F}}}(0') \leq \bar{\mu}_{\nu_{\bar{F}}}(\beta),$$

$$\bar{\vartheta}_{\nu_{\bar{T}}}(0') \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\beta), \quad \bar{\vartheta}_{\nu_{\bar{I}}}(0') \leq \bar{\vartheta}_{\nu_{\bar{I}}}(\beta), \quad \bar{\vartheta}_{\nu_{\bar{F}}}(0') \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\beta).$$

Thus the $0'$-conditions hold for ν in $\mathfrak{F}$.

(ii) The $\hat{Z}$-ideal inequalities. Let $a, b \in \mathfrak{F}$. Choose $x, y \in \mathfrak{V}$ such that $h(x) = a$ and $h(y) = b$. Since $h^{-1}(\nu)$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}$, we have

$$\bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(x) \geq \mathrm{rmin}\big(\bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(x * y),\, \bar{\mu}_{h^{-1}(\nu)_{\bar{T}}}(y)\big),$$

$$\bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(x) \geq \mathrm{rmin}\big(\bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(x * y),\, \bar{\mu}_{h^{-1}(\nu)_{\bar{I}}}(y)\big),$$

$$\bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(x) \leq \mathrm{rmax}\big(\bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(x * y),\, \bar{\mu}_{h^{-1}(\nu)_{\bar{F}}}(y)\big),$$

$$\bar{\vartheta}_{h^{-1}(\nu)_{\bar{T}}}(x) \leq \mathrm{rmax}\big(\bar{\vartheta}_{h^{-1}(\nu)_{\bar{T}}}(x * y),\, \bar{\vartheta}_{h^{-1}(\nu)_{\bar{T}}}(y)\big),$$

$$\bar{\vartheta}_{h^{-1}(\nu)_{\bar{I}}}(x) \leq \mathrm{rmax}\big(\bar{\vartheta}_{h^{-1}(\nu)_{\bar{I}}}(x * y),\, \bar{\vartheta}_{h^{-1}(\nu)_{\bar{I}}}(y)\big),$$

$$\bar{\vartheta}_{h^{-1}(\nu)_{\bar{F}}}(x) \geq \mathrm{rmin}\big(\bar{\vartheta}_{h^{-1}(\nu)_{\bar{F}}}(x * y),\, \bar{\vartheta}_{h^{-1}(\nu)_{\bar{F}}}(y)\big).$$

Using the definition of inverse image and $h(x * y) = h(x) *' h(y) = a *' b$, these inequalities become

$$\bar{\mu}_{\nu_{\bar{T}}}(a) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(a *' b),\, \bar{\mu}_{\nu_{\bar{T}}}(b)\big),$$

$$\bar{\mu}_{\nu_{\bar{I}}}(a) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(a *' b),\, \bar{\mu}_{\nu_{\bar{I}}}(b)\big),$$

$$\bar{\mu}_{\nu_{\bar{F}}}(a) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(a *' b),\, \bar{\mu}_{\nu_{\bar{F}}}(b)\big),$$

$$\bar{\vartheta}_{\nu_{\bar{T}}}(a) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(a *' b),\, \bar{\vartheta}_{\nu_{\bar{T}}}(b)\big),$$

$$\bar{\vartheta}_{\nu_{\bar{I}}}(a) \leq \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(a *' b),\, \bar{\vartheta}_{\nu_{\bar{I}}}(b)\big),$$

$$\bar{\vartheta}_{\nu_{\bar{F}}}(a) \geq \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(a *' b),\, \bar{\vartheta}_{\nu_{\bar{F}}}(b)\big).$$

Hence the $\hat{Z}$-ideal inequalities hold for ν in $\mathfrak{F}$.

Therefore, ν satisfies all defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{F}$. □

5. Cartesian Product of Interval-Valued Intuitionistic Neutrosophic $\hat{Z}$-Ideals

In this section, we develop the Cartesian product construction for IVIN $\hat{Z}$-ideals and show how it generates new IVIN $\hat{Z}$-ideals from given ones. After defining the product $\hat{Z}$-algebra on $\mathfrak{V} \times \mathfrak{F}$, we introduce the Cartesian product of IVIN sets by combining the truth, indeterminacy, and falsity components through the interval operators rmin and rmax. We then prove that the product of two IVIN $\hat{Z}$-ideals is again an IVIN $\hat{Z}$-ideal in the product algebra, providing a systematic method for constructing examples and extending the ideal theory in the interval-valued intuitionistic neutrosophic setting.

Definition 5.1. *Let* $(\mathfrak{V}, *, 0)$ *and* $(\mathfrak{F}, *', 0')$ *be two* $\hat{Z}$*-algebras. Let*

$$\nu = \left(\bar{\mu}_{\nu_{\bar{T}}}, \bar{\mu}_{\nu_{\bar{I}}}, \bar{\mu}_{\nu_{\bar{F}}}, \bar{\vartheta}_{\nu_{\bar{T}}}, \bar{\vartheta}_{\nu_{\bar{I}}}, \bar{\vartheta}_{\nu_{\bar{F}}}\right) \quad \text{and} \quad \varepsilon = \left(\bar{\mu}_{\varepsilon_{\bar{T}}}, \bar{\mu}_{\varepsilon_{\bar{I}}}, \bar{\mu}_{\varepsilon_{\bar{F}}}, \bar{\vartheta}_{\varepsilon_{\bar{T}}}, \bar{\vartheta}_{\varepsilon_{\bar{I}}}, \bar{\vartheta}_{\varepsilon_{\bar{F}}}\right)$$

be IVIN sets on $\mathfrak{V}$ *and* $\mathfrak{F}$*, respectively. The Cartesian product* $\nu \times \varepsilon$ *is the IVIN set on* $\mathfrak{V} \times \mathfrak{F}$ *defined for each* $(\alpha, \beta) \in \mathfrak{V} \times \mathfrak{F}$ *by*

(i) $\bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(\alpha, \beta) = \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta)\big)$,
(ii) $\bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(\alpha, \beta) = \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(\alpha),\ \bar{\mu}_{\varepsilon_{\bar{I}}}(\beta)\big)$,
(iii) $\bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(\alpha, \beta) = \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha),\ \bar{\mu}_{\varepsilon_{\bar{F}}}(\beta)\big)$,
(iv) $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}(\alpha, \beta) = \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(\alpha),\ \bar{\vartheta}_{\varepsilon_{\bar{T}}}(\beta)\big)$,
(v) $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}(\alpha, \beta) = \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(\alpha),\ \bar{\vartheta}_{\varepsilon_{\bar{I}}}(\beta)\big)$,
(vi) $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}(\alpha, \beta) = \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(\alpha),\ \bar{\vartheta}_{\varepsilon_{\bar{F}}}(\beta)\big)$.

Theorem 5.1. *Let* $(\mathfrak{V}, *, 0)$ *and* $(\mathfrak{F}, *', 0')$ *be* $\hat{Z}$*-algebras. Let* ν *and* ε *be IVIN* $\hat{Z}$*-ideals of* $\mathfrak{V}$ *and* $\mathfrak{F}$*, respectively. Define the* $\hat{Z}$*-algebra structure on* $\mathfrak{V} \times \mathfrak{F}$ *by*

$$(\alpha_1, \alpha_2) \odot (\beta_1, \beta_2) = (\alpha_1 * \beta_1,\ \alpha_2 *' \beta_2), \quad (0, 0') \in \mathfrak{V} \times \mathfrak{F}.$$

Then $\nu \times \varepsilon$ *is an IVIN* $\hat{Z}$*-ideal of* $\mathfrak{V} \times \mathfrak{F}$*.*

Proof. Let $(\alpha_1, \alpha_2) \in \mathfrak{V} \times \mathfrak{F}$. By the definition of the Cartesian product IVIN set,

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(0, 0') = \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(0),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(0')\big) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2)\big) = \bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(\alpha_1, \alpha_2),$$

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(0, 0') = \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(0),\ \bar{\mu}_{\varepsilon_{\bar{I}}}(0')\big) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{I}}}(\alpha_1),\ \bar{\mu}_{\varepsilon_{\bar{I}}}(\alpha_2)\big) = \bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(\alpha_1, \alpha_2),$$

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(0, 0') = \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(0),\ \bar{\mu}_{\varepsilon_{\bar{F}}}(0')\big) \leq \mathrm{rmax}\big(\bar{\mu}_{\nu_{\bar{F}}}(\alpha_1),\ \bar{\mu}_{\varepsilon_{\bar{F}}}(\alpha_2)\big) = \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(\alpha_1, \alpha_2),$$

and similarly

$$\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}(0, 0') = \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{T}}}(0),\ \bar{\vartheta}_{\varepsilon_{\bar{T}}}(0')\big) \leq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}(\alpha_1, \alpha_2),$$
$$\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}(0, 0') = \mathrm{rmax}\big(\bar{\vartheta}_{\nu_{\bar{I}}}(0),\ \bar{\vartheta}_{\varepsilon_{\bar{I}}}(0')\big) \leq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}(\alpha_1, \alpha_2),$$

$$\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}(0,0') = \mathrm{rmin}\big(\bar{\vartheta}_{\nu_{\bar{F}}}(0),\ \bar{\vartheta}_{\varepsilon_{\bar{F}}}(0')\big) \geq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}(\alpha_1,\alpha_2).$$

Now let $(\alpha_1,\alpha_2),(\beta_1,\beta_2) \in \mathfrak{V}\times\mathfrak{F}$. Since ν and ε are IVIN $\hat{Z}$-ideals in $\mathfrak{V}$ and $\mathfrak{F}$, respectively, we have

$$\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1 * \beta_1),\ \bar{\mu}_{\nu_{\bar{T}}}(\beta_1)\big), \quad \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2) \geq \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2 *' \beta_2),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta_2)\big),$$

and analogous inequalities for $\bar{I}$, $\bar{F}$, and for $\bar{\vartheta}$.

Using the definition of $\nu\times\varepsilon$ and the monotonicity of rmin, we obtain

$$\begin{aligned}
&\bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(\alpha_1,\alpha_2)\\
&= \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2)\big)\\
&\geq \mathrm{rmin}\Big(\mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1 * \beta_1),\ \bar{\mu}_{\nu_{\bar{T}}}(\beta_1)\big),\ \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2 *' \beta_2),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta_2)\big)\Big)\\
&= \mathrm{rmin}\Big(\mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1 * \beta_1),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2 *' \beta_2)\big),\ \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\beta_1),\ \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta_2)\big)\Big)\\
&= \mathrm{rmin}\big(\bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}((\alpha_1,\alpha_2)\odot(\beta_1,\beta_2)),\ \bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(\beta_1,\beta_2)\big).
\end{aligned}$$

The same argument gives

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(\alpha_1,\alpha_2) \geq \mathrm{rmin}\big(\bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}((\alpha_1,\alpha_2)\odot(\beta_1,\beta_2)),\ \bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(\beta_1,\beta_2)\big),$$

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(\alpha_1,\alpha_2) \leq \mathrm{rmax}\big(\bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}((\alpha_1,\alpha_2)\odot(\beta_1,\beta_2)),\ \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(\beta_1,\beta_2)\big),$$

and similarly for $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}$, $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}$, and $\bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}$.

Therefore $\nu\times\varepsilon$ satisfies all defining conditions of an IVIN $\hat{Z}$-ideal of $\mathfrak{V}\times\mathfrak{F}$. □

Theorem 5.2. *Let $(\mathfrak{V},*,0)$ be a $\hat{Z}$-algebra and let ν and ε be two IVIN $\hat{Z}$-ideals of $\mathfrak{V}$. Consider the product $\hat{Z}$-algebra $(\mathfrak{V}\times\mathfrak{V},\odot,(0,0))$ where*

$$(\alpha_1,\alpha_2)\odot(\beta_1,\beta_2) = (\alpha_1 * \beta_1, \alpha_2 * \beta_2).$$

Then $\nu\times\varepsilon$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V}\times\mathfrak{V}$. In particular, for all $\beta\in\mathfrak{V}$, the following inequalities hold:

$$\begin{aligned}
&(i) \quad \bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(0,\beta) \leq \bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(0,0), \quad \bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(0,\beta) \leq \bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(0,0),\\
&(ii) \quad \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(0,\beta) \geq \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(0,0),\\
&(iii) \quad \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}(0,\beta) \geq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{T}}}(0,0), \quad \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}(0,\beta) \geq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{I}}}(0,0),\\
&(iv) \quad \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}(0,\beta) \leq \bar{\vartheta}_{(\nu\times\varepsilon)_{\bar{F}}}(0,0).
\end{aligned}$$

Proof. Step 1: Inequalities at $(0,0)$. Let $\beta\in\mathfrak{V}$. Since ν and ε are IVIN $\hat{Z}$-ideals of $\mathfrak{V}$, we have

$$\bar{\mu}_{\nu_{\bar{T}}}(0) \geq \bar{\mu}_{\nu_{\bar{T}}}(\beta), \quad \bar{\mu}_{\varepsilon_{\bar{T}}}(0) \geq \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta),$$

and similarly for $\bar{I}$, while for $\bar{F}$ the inequality is reversed:

$$\bar{\mu}_{\nu_{\bar{F}}}(0) \leq \bar{\mu}_{\nu_{\bar{F}}}(\beta), \quad \bar{\mu}_{\varepsilon_{\bar{F}}}(0) \leq \bar{\mu}_{\varepsilon_{\bar{F}}}(\beta).$$

Using the definitions above and the monotonicity of rmin and rmax gives

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(0,\beta) = \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(0),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta)\big) \leq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(0),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(0)\big) = \bar{\mu}_{(\nu\times\varepsilon)_{\bar{T}}}(0,0),$$

and similarly,

$$\bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(0,\beta) \leq \bar{\mu}_{(\nu\times\varepsilon)_{\bar{I}}}(0,0), \quad \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(0,\beta) \geq \bar{\mu}_{(\nu\times\varepsilon)_{\bar{F}}}(0,0).$$

The corresponding relations for $\bar{\vartheta}$ follow from

$$\bar{\vartheta}_{\nu_{\bar{T}}}(0) \leq \bar{\vartheta}_{\nu_{\bar{T}}}(\beta), \quad \bar{\vartheta}_{\varepsilon_{\bar{T}}}(0) \leq \bar{\vartheta}_{\varepsilon_{\bar{T}}}(\beta),$$

(and similarly for $\bar{I}$), and

$$\bar{\vartheta}_{\nu_{\bar{F}}}(0) \geq \bar{\vartheta}_{\nu_{\bar{F}}}(\beta), \quad \bar{\vartheta}_{\varepsilon_{\bar{F}}}(0) \geq \bar{\vartheta}_{\varepsilon_{\bar{F}}}(\beta).$$

Hence (i)–(iv) hold.

Step 2: $\hat{Z}$-ideal inequalities. Let $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathfrak{V} \times \mathfrak{V}$. Since ν and ε are IVIN $\hat{Z}$-ideals in $\mathfrak{V}$, we have (for example)

$$\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1) \geq \mathrm{rmin}\big(\bar{\mu}_{\nu_{\bar{T}}}(\alpha_1 * \beta_1),\, \bar{\mu}_{\nu_{\bar{T}}}(\beta_1)\big), \quad \bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2) \geq \mathrm{rmin}\big(\bar{\mu}_{\varepsilon_{\bar{T}}}(\alpha_2 * \beta_2),\, \bar{\mu}_{\varepsilon_{\bar{T}}}(\beta_2)\big),$$

and analogous inequalities for $\bar{I}$, $\bar{F}$ and for $\bar{\vartheta}$. Combining these with the definition of $\nu \times \varepsilon$ and the monotonicity of rmin and rmax yields the required ideal inequalities for $\nu \times \varepsilon$ on $\mathfrak{V} \times \mathfrak{V}$. Therefore $\nu \times \varepsilon$ is an IVIN $\hat{Z}$-ideal of $\mathfrak{V} \times \mathfrak{V}$. □

6. Conclusion

In this paper, we introduced the notion of interval-valued intuitionistic neutrosophic (IVIN) $\hat{Z}$-ideals in $\hat{Z}$-algebras, extending the classical theory of $\hat{Z}$-ideals by incorporating interval-valued membership and non-membership functions together with neutrosophic truth, indeterminacy, and falsity components. We established fundamental properties of IVIN $\hat{Z}$-ideals, including closure under intersection, preservation under homomorphisms (images and inverse images), and construction via Cartesian products in the product $\hat{Z}$-algebra. These results enrich the ideal theory of $\hat{Z}$-algebras in an interval-valued intuitionistic neutrosophic setting and provide a foundation for uncertainty-aware algebraic modeling. Future work may include developing homomorphism and isomorphism theorems and quotient-type constructions for IVIN $\hat{Z}$-ideals, studying further ideal-theoretic notions such as prime and maximal IVIN $\hat{Z}$-ideals and the lattice structure they form, extending the framework to related algebraic systems (e.g., semigroups, rings, and lattices), and designing computational procedures for constructing and verifying IVIN $\hat{Z}$-ideals on finite $\hat{Z}$-algebras.

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References

[1] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[2] K. Atanassov, G. Gargov, Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 31 (1989), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4.

[3] M. Bhowmik, M. Pal, Intuitionistic Neutrosophic Set, J. Inf. Comput. Sci. 4 (2009), 142–152.

[4] M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha, S. Sabarinathan, A Note on Z-Algebra, Ital. J. Pure Appl. Math. 38 (2017), 707–714.

[5] P. Hemavathi, P. Muralikrishna, K. Palanivel, On Interval Valued Intuitionistic Fuzzy β-Subalgebras, Afrika Mat. 29 (2017), 249–262. https://doi.org/10.1007/s13370-017-0539-z.

[6] Y.B. Jun, E.H. Roh, Fuzzy Commutative Ideals of BCK-Algebras, Fuzzy Sets Syst. 64 (1994), 401–405. https://doi.org/10.1016/0165-0114(94)90163-5.

[7] Y.B. Jun, K.H. Kim, Intuitionistic Fuzzy Ideals of BCK-Algebras, Int. J. Math. Math. Sci. 24 (2000), 839–849. https://doi.org/10.1155/S0161171200004610.

[8] S. Manokaran, M. Prakasam, On MBJ–Neutrosophic β–Subalgebra, Neutrosophic Sets Syst. 28 (2019), 216–227. https://doi.org/10.5281/zenodo.3382544.

[9] J. Meng, X. Guo, On Fuzzy Ideals in BCK/BCI-Algebras, Fuzzy Sets Syst. 149 (2005), 509–525. https://doi.org/10.1016/j.fss.2003.11.014.

[10] G. Muhiuddin, H. Bordbar, F. Smarandache, Y.B. Jun, Further Results on $(\in, \in)$-Neutrosophic Subalgebras and Ideals in BCK/BCI-Algebras, Neutrosoph. Sets Syst. 20 (2018), 36–43.

[11] K.P. Shanmugapriya, P. Hemavathi, A New Framework on Interval-Valued Neutrosophic in $\hat{Z}$-Algebra, Neutrosoph. Sets Syst. 60 (2023), 300–316. https://doi.org/10.5281/zenodo.10224204.

[12] K.P. Shanmugapriya, P. Hemavathi, A Novel Concept of Neutrosophic Fuzzy Sets in $\hat{Z}$-Algebra, in: Trends in Mathematics, Springer, Cham, 2024: pp. 47–56. https://doi.org/10.1007/978-3-031-37538-5_5.

[13] F. Smarandache, Neutrosophic Set - A Generalization of the Intuitionistic Fuzzy Set, in: 2006 IEEE International Conference on Granular Computing, IEEE, pp. 38–42. https://doi.org/10.1109/GRC.2006.1635754.

[14] M. Songsaeng, A. Iampan, Neutrosophic Sets in UP-Algebras by Means of Interval-Valued Fuzzy Sets, J. Int. Math. Virtual Inst. 10 (2020), 93–122.

[15] S. Sowmiya, P. Jeyalakshmi, Fuzzy Algebraic Structures in Z-Algebras, World J. Eng. Res. Technol. 5 (2019), 74–88.

[16] S. Sowmiya, P. Jeyalakshmi, Intuitionistic Fuzzy Sets in $\hat{Z}$-Algebras, J. Adv. Math. Stud. 13 (2020), 302–310.

[17] S. Jeyalakshmi, On Fuzzy Z-Ideals in Z-Algebras, Glob. J. Pure Appl. Math. 15 (2019), 505–516. https://doi.org/10.37622/GJPAM/15.4.2019.505-516.

[18] H. Wang, P. Madiraju, Y. Zhang, R. Sunderraman, Interval Neutrosophic Sets, Int. J. Appl. Math. Stat. 3 (2005), 1–18.

[19] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[20] L.A. Zadeh, The Concept of a Linguistic Variable and Its Application to Approximate Reasoning—I, Inf. Sci. 8 (1975), 199–249. https://doi.org/10.1016/0020-0255(75)90036-5.