Asia Pac. J. Math. 2026 **13**:119

STUDY OF SOME NONLINEAR ELLIPTIC PROBLEM IN A TWO-COMPONENT DOMAIN WITH HARDY POTENTIAL AND L^1-DATA

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Received Apr. 27, 2026

Abstract. This paper deals with a class of quasilinear elliptic equations of the form:

$$\begin{cases} -\text{div}\,(a(x,u_1,\nabla u_1)) = f(x) + \frac{|u_1|^{q-2}u_1}{|x|^q} & \text{in } \Omega_1, \\ -\text{div}\,(a(x,u_2,\nabla u_2)) = f(x) + \frac{|u_2|^{q-2}u_2}{|x|^q} & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ a(x,u_1,\nabla u_1)\cdot\nu_1 = a(x,u_2,\nabla u_2)\cdot\nu_1 & \text{on } \Gamma, \\ a(x,u_1,\nabla u_1)\cdot\nu_1 = -g(x)|u_1-u_2|^{p-2}(u_1-u_2) & \text{on } \Gamma, \end{cases}$$

where Ω is an open bounded and connected subset of $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz boundary $\partial\Omega$, such that Ω is decomposed as $\Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_2 is an open subset of Ω such that $\overline{\Omega_2} \subset \Omega$, and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ and $\Gamma = \partial\Omega_2$. We prove the existence of a renormalized solution for our quasilinear elliptic equation. Moreover, we conclude some regularity results.

2020 Mathematics Subject Classification. 35J60; 46E30; 46E35.

Key words and phrases. elliptic equations; Hardy potential; two-component domain; renormalized solutions.

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$. in [14] Boccardo and Gallouët have investigated the elliptic problem:

$$(1) \qquad \begin{cases} -\text{div}\; a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $a(x,u,\nabla u)$ is a Carathéodory function, and $f(x)$ is a bounded Radon measure. They have proved the existence and regularity of solutions. In recent years, there has been a notable concentration of research on elliptic equations incorporating the Hardy singular potential $\frac{\lambda u}{|x|^2}$ or, more generally, $\frac{\lambda |u|^{p-2}u}{|x|^p}$. Several authors have explored the implications of this potential on the existence and non-existence of

DOI: 10.28924/APJM/13-119

a solution, see for instance [2,3,8]. In particular, the authors of [1] have investigated the nonlinear elliptic problem

$$(2)\qquad \begin{cases} -\Delta u \pm |Du|^2 = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > 0$. They have studied the existence of positive solutions in the absorption case $(+|Du|^2)$ for $f \in L^1(\Omega)$, and they have proved the non-existence of a solution in the diffusion case $(-|Du|^2)$ even in a very weak sense, we refer the reader also to [4,10]. Porzio has studied in [26] the quasilinear elliptic problem

$$(3)\qquad \begin{cases} -\operatorname{div}(M(x,u)Du) + \nu|u|^{s-1}u = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $s > \dfrac{N}{N-2}$. They have established the existence of solution $u \in W_0^{1,q}(\Omega)$ for every $1 < q < 2s/(s+1)$, for all $f \in L^1(\Omega)$, $\nu > 0$, and $\lambda \geq 0$, we refer the reader also to [8].

Note that, the notion of renormalized solutions was first introduced by DiPerna and Lions [19] for the Boltzmann equation, and later adapted to elliptic problems with L^1-data or measure data [13,18]. For the parabolic equations with L^1 data, the theory was developed by Blanchard and Murat [12]. However, the notion of entropy solutions was independently proposed by Bénilan et al. for nonlinear elliptic equations. Further developments and various applications can be found in [5,6,15,17].

Donato and Raimondi have studied in [20] the following class of quasilinear elliptic problems

$$(4)\qquad \begin{cases} -\operatorname{div}(A(x,u)\nabla u) + \lambda u = f(x)\zeta(u) & \text{in } \Omega/\Gamma, \\ u = 0 & \text{on } \partial\Omega, \\ (A(x,u_1)\nabla u_1)\cdot \nu_1 = (A(x,u_2)\nabla u_2)\cdot \nu_1 & \text{on } \Gamma, \\ (A(x,u_1)\nabla u_1)\cdot \nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma, \end{cases}$$

where $A(x,s)$ is a bounded Carathéodory matrix field, and λ is a non-negative real number, $\zeta(s)$ is a non-negative real function with singularity at $s = 0$, such that the data $f(x)$ depends on the growth of $\zeta(s)$ near the singularity, with $h(x) \in L^\infty(\Gamma)$. They have proved the existence and uniqueness of the solution, also some regularity results for the solution of the problem (4) are concluded.

In [23], Guibé and Oropeza have studied a class of nonlinear elliptic equations of the type:

$$(5)\qquad \begin{cases} -\operatorname{div}(A(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_0, \\ B(x,u)\nabla u \cdot \vec{n} + \gamma(x)h(u) = g & \text{on } \partial\Omega_1, \end{cases}$$

where Ω is a perforated domain in $\mathbb{R}^N$ $(N \geq 2)$, and the matrix field $A(x,s)$ is coercive but lacks any growth condition concerning the second variable. The source term f belongs to $L^1(\Omega)$ with Dirichlet boundary conditions on $\partial\Omega_0$, while nonlinear and nonhomogeneous Robin conditions are assumed on $\partial\Omega_1$, with g belongs to $L^1(\partial\Omega)$, and γ is a nonnegative function in $L^\infty(\partial\Omega_1)$, the function $h(\cdot)$ is a nondecreasing and continuous such that $h(0) = 0$, and $\vec{n}$ denotes the outer unit normal to $\partial\Omega_1$.

In this work, we study the following degenerated quasilinear elliptic problem:

$$(6) \qquad \begin{cases} -\mathrm{div}\,(a(x,u_1,\nabla u_1)) = f(x) + \frac{|u_1|^{q-2}u_1}{|x|^q} & \text{in } \Omega_1, \\ -\mathrm{div}\,(a(x,u_2,\nabla u_2)) = f(x) + \frac{|u_2|^{q-2}u_2}{|x|^q} & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ a(x,u_1,\nabla u_1)\cdot\nu_1 = a(x,u_2,\nabla u_2)\cdot\nu_1 & \text{on } \Gamma, \\ a(x,u_1,\nabla u_1)\cdot\nu_1 = -g(x)|u_1-u_2|^{p-2}(u_1-u_2) & \text{on } \Gamma, \end{cases}$$

where ν_1 is the unit outward normal on Γ, and $a(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function (i.e. measurable with respect to x in Ω for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, and continuous with respect to $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) and verifying the following assumptions

$$(7) \qquad |a(x,s,\xi)| \leq \beta\big(d(x) + |s|^{p-1} + |\xi|^{p-1}\big),$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, with $\beta > 0$.

$$(8) \qquad \big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi-\xi') > 0 \quad \text{for any } \xi \neq \xi',$$

$$(9) \qquad a(x,s,\xi)\cdot\xi \geq b(|s|)|\xi|^p,$$

such that $b(|\cdot|) : I\!R^+ \longmapsto I\!R^+$ is a decreasing positive function such that

$$(10) \qquad \frac{b_0}{(1+|s|)^\lambda} \leq b(|s|) \qquad \text{for any} \quad s \in I\!R,$$

with $0 \leq \lambda < p-1$. Moreover, we assume that $1 < q < \dfrac{N(p-\lambda)}{N+p-\lambda-1}$ and $f(x) \in L^1(\Omega\setminus\Gamma)$ and $g(\cdot) \in L^\infty(\Gamma)$ with $0 < g_0 \leq g(x)$ a.e. on Γ.

This paper aims to investigate the existence and regularity of renormalized solutions for the non-coercive quasilinear elliptic equation (6). The combination of these features gives rise to several substantial analytical challenges. First, the absence of coercivity makes standard energy estimates inapplicable. Second, the low integrability of the data, namely $f(x) \in L^1(\Omega\setminus\Gamma)$, forces us to adopt the framework of renormalized solutions.

This paper is organized as follows: In section 2, we recall some definitions and results concerning the framework of Sobolev spaces V. In section 3, we introduce the definition of a renormalized solution for the problem (6) and we present our main result. The last section 4 is dedicated to proving the existence of renormalized solutions for our degenerated quasilinear elliptic equation (6), and we conclude some regularity results.

2. Some Preliminary Results

This section recalls a few definitions and basic facts about the functional space V.

Let $1 < p < N$, we set $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_2 is an open subset of Ω such that $\overline{\Omega_2} \subset \Omega$, and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ and $\Gamma = \partial\Omega_2$. We consider the Sobolev spaces

$$V_1 = \{v \in W^{1,p}(\Omega_1) : \ v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^p(\Omega_1)},$$

and

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in W^{1,p}(\Omega_2)\},$$

equipped with the norm

$$\|v\|_V^p := \|\nabla v_1\|^p_{L^p(\Omega_1)} + \|\nabla v_2\|^p_{L^p(\Omega_2)} + \|v_1 - v_2\|^p_{L^p(\Gamma)}. \tag{11}$$

Remark 2.1. *There exist two positive constants c_1 and c_2 such that:*

$$c_1|v|_{1,p,\Omega_2} \le \|v\|_{W^{1,p}(\Omega_2)} \le c_2|v|_{1,p,\Omega_2} \qquad \text{for any} \quad v \in W^{1,p}(\Omega_2), \tag{12}$$

where $|v|_{1,p,\Omega_2} := \|v\|_{L^p(\Gamma)} + \|\nabla v\|_{L^p(\Omega_2)}$.

Proposition 2.1. *There exist two positive constants c_1 and c_2 such that*

$$c_1\|v\|_V \le \|v\|_{V_1 \times W^{1,p}(\Omega_2)} \le c_2\|v\|_V \qquad \text{for any} \quad v \in V,$$

where $\|\cdot\|_{V_1 \times W^{1,p}(\Omega_2)}$ denotes the norm of $V_1 \times W^{1,p}(\Omega_2)$.

The proof of Proposition 2.1 is based on Remark 2.1 and the trace theorem.

Remark 2.2. *The space V allows searching for a solution that is defined separately on Ω_1 and Ω_2, but a connection condition (continuity or controlled jump) will be imposed via the norm (11). The space V is tailored for interface problems on $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. Its natural norm includes the jump term $\|v_1 - v_2\|_{L^p(\Gamma)}$, which measures the difference of traces on Γ and allows imposing transmission or Robin conditions.*

Remark 2.3. *In view of the Proposition 2.1, There exists a constant $C > 0$ such that:*

$$\|v_1\|_{L^p(\Omega_1)} + \|v_2\|_{L^p(\Omega_2)} \le C\|v\|_V \qquad \text{for any} \quad v = (v_1, v_2) \in V. \tag{13}$$

Definition 2.1. *Let $k > 0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, given by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Proposition 2.2. *(see [11,22,23]) Let $u := (u_1, u_2) : \Omega \setminus \Gamma \mapsto \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k \geq 1$, we have*

(1) *There exists a unique measurable function $v_i : \Omega_i \mapsto \mathbb{R}^N$ such that*

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i|<k\}} \quad \text{a.e. in } \Omega_i \qquad \text{for all} \quad k \geq 1,$$

where $\chi_{\{|u_i|<k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |u_i(x)| < k\}$. We define v_i as the gradient of u_i and we write $v_i = \nabla u_i$ for $i = 1, 2$.

(2) *Let $0 \leq \lambda < p - 1$, if $\sup_{k \geq 1} \frac{1}{k^{1+\lambda}} \|T_k(u)\|_V^p < \infty$, then there exists a unique measurable function $w_i : \Gamma \mapsto \mathbb{R}$, for $i = 1, 2$ such that*

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma \qquad \text{for any} \quad k \geq 1,$$

where $\gamma_i : W^{1,p}(\Omega_i) \longmapsto L^p(\Gamma)$ is the trace operator, we define the function w_i as the trace of u_i on Γ and we set

$$\gamma_i(u_i) = w_i.$$

Now, we will recall some lemma essential to establish the existence of renormalized solutions for our degenerated quasilinear elliptic equation (6).

Lemma 2.1. *(see. Lemma 4.4, [7]) Let $h > 0$, assuming that conditions (7) $-$ (9) hold true, and $(v_n := (v_{1,n}, v_{2,n}))_n \subset V$ be a sequence that satisfies the following convergence:*

(i) *$v_n \rightharpoonup v$ weakly in V,*

(ii) $\displaystyle\sum_{i=1}^{2} \int_{\Omega_i} \Big(a(x, T_h(v_{i,n}), \nabla v_{i,n}) - a(x, T_h(v_{i,n}), \nabla v_i) \Big) \cdot (\nabla v_{i,n} - \nabla v_i)\, dx \longrightarrow 0 \text{ as } n \to \infty.$

Then $v_n \to v$ strongly in V for a subsequence.

3. Main Result

Now, we present the definition of renormalized solutions for our degenerated quasilinear elliptic problem.

Definition 3.1. *A measurable function $u = (u_1, u_2) : \Omega \setminus \Gamma \mapsto \mathbb{R}$ is a renormalized solution for the non-coercive quasilinear elliptic problem (6) if $T_k(u) \in V$ for any $k > 0$, with $\frac{|u_i|^{q-2} u_i}{|x|^q} \in L^1(\Omega_i)$ for $i = 1, 2$, and*

$$|u_1 - u_2|^{p-2}(u_1 - u_2) \cdot (T_k(u_1) - T_k(u_2)) \in L^1(\Gamma) \tag{14}$$

and

$$\lim_{h \to \infty} \frac{1}{h} \int_\Gamma g(x) |u_1 - u_2|^{p-2} (u_1 - u_2)(T_h(u_1) - T_h(u_2))\, d\sigma = 0, \tag{15}$$

with

$$\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{2}\int_{\{|u_i|\le h\}} a(x,u_i,\nabla u_i)\cdot\nabla u_i\,dx = 0, \tag{16}$$

such that u verifies the following equality

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i} a(x,u_i,\nabla u_i)\cdot(S'(u_i)\varphi_i\nabla u_i + S(u_i)\nabla\varphi_i)\,dx\\
&\qquad+\int_\Gamma g(x)|u_1-u_2|^{p-2}(u_1-u_2)(S(u_1)\varphi_1 - S(u_2)\varphi_2)\,d\sigma\\
&=\sum_{i=1}^{2}\int_{\Omega_i} f(x)S(u_i)\varphi_i\,dx + \sum_{i=1}^{2}\int_{\Omega_i}\frac{|u_i|^{q-2}u_i}{|x|^q}S(u_i)\varphi_i\,dx,
\end{aligned}\tag{17}$$

for any $\varphi=(\varphi_1,\varphi_2)\in V\cap(L^\infty(\Omega_1)\times L^\infty(\Omega_2))$, and any $S(\cdot)\in W^{1,\infty}(\mathbb{R})$ with compact support.

Lemma 3.1.

Let $S(\cdot)\in W^{1,\infty}(\mathbb{R})$ with a compact support. In view of (15) we have

$$g(x)|u_1-u_2|^{p-2}(u_1-u_2)S(u_i)\in L^1(\Gamma)\quad \textit{for } i=1,2. \tag{18}$$

Proof of Lemma 3.1. (We use some similar arguments as in [24])

Let $S(\cdot)\in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}(S(\cdot))\subset[-M,M]$ for $M>0$.

Taking $h\ge M$, we set

$$\psi_h(r) = 1-\frac{|T_{2h}(r)-T_h(r)|}{h}\qquad\text{for any}\quad r\in\mathbb{R}.$$

It's clear that $S(r)=S(r)\psi_h(r)\psi_{2h}(r)$ for any $r\in I\!R$, then

$$\begin{aligned}
&\int_\Gamma g(x)|u_1-u_2|^{p-1}S(u_1)\,d\sigma\\
&=\int_\Gamma g(x)|u_1-u_2|^{p-1}S(u_1)\psi_{2h}(u_1)\psi_h(u_1)\,d\sigma\\
&\le\int_\Gamma g(x)|u_1-u_2|^{p-1}S(u_1)\psi_{2h}(u_1)(\psi_h(u_1)-\psi_h(u_2))\,d\sigma\\
&\qquad+\int_\Gamma g(x)|u_1-u_2|^{p-1}S(u_1)\psi_{2h}(u_1)\psi_h(u_2)\,d\sigma\\
&\le\frac{\|S(\cdot)\|_{L^\infty(I\!R)}}{h}\int_\Gamma g(x)|u_1-u_2|^{p-1}|T_{2h}(u_1)-T_{2h}(u_2)|\,d\sigma\\
&\qquad+\|S(\cdot)\|_{L^\infty(I\!R)}\int_\Gamma g(x)|T_{4h}(u_1)-T_{2h}(u_2)|^{p-1}\,d\sigma.
\end{aligned}\tag{19}$$

In view of (15), there exists $h>0$ large enough such that

$$\frac{\|S(\cdot)\|_{L^\infty(I\!R)}}{h}\int_\Gamma g(x)|u_1-u_2|^{p-1}|T_{2h}(u_1)-T_{2h}(u_2)|\,d\sigma\le 1. \tag{20}$$

On the other hand, we have $g(x)|T_{4h}(u_1)-T_{2h}(u_2)|^{p-1}\in L^\infty(\Gamma)$, then there exists a positive constant $C>0$ such that

$$\|S(\cdot)\|_{L^\infty(I\!R)}\int_\Gamma g(x)|T_{4h}(u_1)-T_{2h}(u_2)|^{p-1}\,d\sigma\le C_0. \tag{21}$$

By combining (19) and (20) – (21), we conclude that

$$\int_\Gamma g(x)|u_1 - u_2|^{p-1} S(u_1)\, d\sigma \leq C_1. \tag{22}$$

Similarly, we can show that

$$\int_\Gamma g(x)|u_1 - u_2|^{p-1} S(u_2)\, d\sigma \leq C_2, \tag{23}$$

which conclude the proof of Lemma 3.1.

Theorem 3.1. *Assuming that (7) – (9) hold true. Then, there exists at least one renormalized solution $u = (u_1, u_2)$ for the degenerated quasilinear elliptic problem (6).*

4. Proof of Theorem 3.1.

The proof of Theorem 3.1 relies on approximate methods combined with a priori estimation techniques, then we pass to the limit.

Step 1: Approximate problems.

For $n \in I\!N^*$, we set $f_n(x) = T_n(f(x))$ and we consider the approximate problem of the degenerated quasilinear elliptic problem (3.1), giving by

$$\begin{cases} -\mathrm{div}(a(x, T_n(u_{1,n}), \nabla u_{1,n})) = f_n(x) + \dfrac{|T_n(u_{1,n})|^{q-2} T_n(u_{1,n})}{(|x| + \frac{1}{n})^q} & \text{in} \quad \Omega_1, \\ -\mathrm{div}(a(x, T_n(u_{2,n}), \nabla u_{2,n})) = f_n(x) + \dfrac{|T_n(u_{2,n})|^{q-2} T_n(u_{2,n})}{(|x| + \frac{1}{n})^q} & \text{in} \quad \Omega_2, \\ u_n = 0 & \text{on} \quad \partial\Omega, \\ a(x, T_n(u_{1,n}), \nabla u_{1,n}) \cdot \nu_1 = a(x, T_n(u_{2,n}), \nabla u_{2,n}) \cdot \nu_1 & \text{on} \quad \Gamma, \\ a(x, T_n(u_{1,n}), \nabla u_{1,n}) \cdot \nu_1 = -g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{1,n} - u_{2,n}) & \text{on} \quad \Gamma. \end{cases} \tag{24}$$

We define the operator A_n acting from V into its dual V' as follows:

$$\begin{aligned} \langle A_n u, v \rangle \ &= \sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_i), \nabla u_i) \cdot \nabla v_i \, dx - \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-2} T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} v_i \, dx \\ &\quad + \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)\, (v_1 - v_2)\, d\sigma, \end{aligned} \tag{25}$$

for any $u, v \in V$.

Lemma 4.1. *The operator A_n acting from V into its dual V' is bounded and pseudo-monotone, Moreover, the operator A_n is coercive in the following sense:*

$$\frac{\langle A_n v, v \rangle}{\|v\|_V} \longrightarrow \infty \qquad \textit{as} \quad \|v\|_V \longrightarrow \infty \quad \textit{for any} \quad v \in V. \tag{26}$$

Proof of Lemma 4.1.

In view of Hölder's inequality and the growth condition (7), we have : for any $u, v \in V$

$$
\begin{aligned}
|\langle A_n u, v\rangle| &\leq \sum_{i=1}^{2} \int_{\Omega_i} |a(x, T_n(u_i), \nabla u_i)|\, |\nabla v_i|\, dx + \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\, |v_i|\, dx \\
&\quad + \int_{\Gamma} g(x)|u_1 - u_2|^{p-1}\, |v_1 - v_2|\, d\sigma \\
&\leq \alpha \sum_{i=1}^{2} \int_{\Omega_i} (R(x) + |T_n(u_i)|^{p-1} + |\nabla u_i|^{p-1})\, |\nabla v_i|\, dx + n^{2q-1} \sum_{i=1}^{2} \int_{\Omega_i} |v_i|\, dx \\
&\quad + \|g(x)\|_{L^\infty(\Gamma)} \|u_1 - u_2\|_{L^p(\Gamma)}^{p-1} \|v_1 - v_2\|_{L^p(\Gamma)} \\
&\leq \alpha \sum_{i=1}^{2} \Big(\|R(x)\|_{L^{p'}(\Omega_i)} + n^{p-1}\text{meas}(\Omega_i) + \|\nabla u_i\|_{L^p(\Omega_i)}^{p-1} \Big) \|\nabla v_i\|_{L^p(\Omega_i)} \\
&\quad + nC_0 \sum_{i=1}^{2} \|v_i\|_{L^p(\Omega_i)} + \|g(x)\|_{L^\infty(\Gamma)} (\|u\|_V^{p-1} + 1) \|v\|_V \\
&\leq C_1 (1 + \|u\|_V^{p-1}) \|v\|_V.
\end{aligned} \tag{27}
$$

Thus, the operator A_n is bounded.

Concerning the coercivity, thanks to (9) and Young's inequality we have: for any $u = (u_1, u_2) \in V$

$$
\begin{aligned}
\langle A_n u, u\rangle &\geq \sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_i), \nabla u_i) \cdot \nabla u_i\, dx - \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\, |u_i|\, dx \\
&\quad + \int_{\Gamma} g(x)|u_1 - u_2|^p\, d\sigma \\
&\geq \frac{b_0}{(1+n)^\lambda} \sum_{i=1}^{2} \int_{\Omega_i} |\nabla u_i|^p\, dx + \|g(x)\|_{L^\infty(\Gamma)} \|u_1 - u_2\|_{L^p(\Gamma)}^p - C_2 n^{2q-1} \|u\|_V \\
&\geq C_3 \|u\|_V^p - C_2 n^{2q-1} \|u\|_V.
\end{aligned} \tag{28}
$$

with $C_3 = \min\left\{ \dfrac{b_0}{(1+n)^\lambda}, \|g(x)\|_{L^\infty(\Gamma)} \right\}$, it follows that

$$
\frac{\langle A_n u, u\rangle}{\|u\|_V} \geq \frac{C_3 \|u\|_V^p - C_2 n^{2q-1} \|u\|_V}{\|u\|_V} \longrightarrow \infty \quad \text{as} \quad \|u\|_V \to \infty. \tag{29}
$$

It remains to show that A_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in V such that

$$
\begin{cases}
u_k \rightharpoonup u & \text{weakly in } V, \\
A_n u_k \rightharpoonup \chi_n & \text{weakly in } V', \\
\limsup\limits_{k\to\infty} \langle A_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle .
\end{cases} \tag{30}
$$

We will prove that

$$
\chi_n = A_n u \quad \text{and} \quad \langle A_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \qquad \text{as} \quad k \to \infty.
$$

We have $u_k \rightharpoonup u$ weakly in V, in view of the compact embedding $V \hookrightarrow\hookrightarrow L^1(\Omega_i)$ we obtain $u_{i,k} \to u_i$ strongly in $L^1(\Omega_i)$ for a subsequence still denoted $(u_{i,k})_k$ for $i = 1, 2$.

We have $(u_k)_k$ is a bounded sequence in V, then the sequence $(a(x, T_n(u_{i,k}), \nabla u_{i,k})_k$ is uniformly bounded in $(L^{p'}(\Omega_i))^N$, therefore there exists a measurable function $\varphi_{i,n} \in (L^{p'}(\Omega_i))^N$ such that

$$a(x, T_n(u_{i,k}), \nabla u_{i,k}) \rightharpoonup \varphi_{i,n} \quad \text{weakly in } (L^{p'}(\Omega_i))^N \text{ as } k \to \infty. \tag{31}$$

Moreover, we have $\left|\frac{|T_n(u_{i,k})|^{q-2}T_n(u_{i,k})}{(|x|+\frac{1}{n})^q}\right| \leq n^{2q-1}$, and since $\dfrac{|T_n(u_{i,k})|^{q-2}T_n(u_{i,k})}{(|x|+\frac{1}{n})^q} \to \dfrac{|T_n(u_i)|^{q-2}T_n(u_i)}{(|x|+\frac{1}{n})^q}$ a.e. in Ω_i, in view of Lebesgue's dominated convergence theorem, we obtain

$$\frac{|T_n(u_{i,k})|^{q-2}T_n(u_{i,k})}{(|x|+\frac{1}{n})^q} \to \frac{|T_n(u_i)|^{q-2}T_n(u_i)}{(|x|+\frac{1}{n})^q} \qquad \text{strongly in} \quad L^{p'}(\Omega_i) \quad \text{as} \quad k \to \infty. \tag{32}$$

Also, we have $u_{i,k} \rightharpoonup u_i$ weakly in $L^p(\Gamma)$ for $i = 1, 2$, then

$$g(x)|u_{1,k} - u_{2,k}|^{p-2}(u_{1,k} - u_{2,k}) \rightharpoonup g(x)|u_1 - u_2|^{p-2}(u_1 - u_2) \qquad \text{weakly in} \quad L^{p'}(\Gamma). \tag{33}$$

On the one hand, for all $v \in V$ we have

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle A_n u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_{i,k}), \nabla u_{i,k}) \cdot \nabla v_i \, dx - \lim_{k\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,k})|^{q-2}T_n(u_{i,k})}{(|x|+\frac{1}{n})^q} v_i \, dx \\
&\quad + \lim_{k\to\infty} \int_\Gamma g(x)|u_{1,k} - u_{2,k}|^{p-2}(u_{1,k} - u_2)(v_1 - v_2) \, dx \\
&= \sum_{i=1}^{2} \int_{\Omega_i} \varphi_{i,n} \cdot \nabla v_i \, dx - \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_k)|^{q-2}T_n(u_k)}{(|x|+\frac{1}{n})^q} v_i \, dx \\
&\quad + \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)(v_1 - v_2) \, dx.
\end{aligned} \tag{34}$$

By using (30) and (34), we obtain

(35)

$$\begin{aligned}
\limsup_{k\to\infty} \langle A_n u_k, u_k\rangle &= \limsup_{k\to\infty} \Big(\sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_{i,k}), \nabla u_{i,k}) \cdot \nabla u_{i,k} \, dx \\
&\quad - \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,k})|^{q-1}}{(|x|+\frac{1}{n})^q} |u_{i,k}| \, dx + \int_\Gamma g(x)|u_{1,k} - u_{2,k}|^p \, dx \Big) \\
&\leq \sum_{i=1}^{2} \int_{\Omega_i} \varphi_{i,n} \cdot \nabla u_i \, dx - \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_i)|^{q-1}}{(|x|+\frac{1}{n})^q} |u_i| \, dx + \int_\Gamma g(x)|u_1 - u_2|^p \, dx.
\end{aligned}$$

Thanks to (32), and the fact that $u_{i,k} \to u_i$ strongly in $L^p(\Omega_i)$ then

$$\lim_{k\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,k})|^{q-1}}{(|x|+\frac{1}{n})^q} |u_{i,k}| \, dx = \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_i)|^{q-1}}{(|x|+\frac{1}{n})^q} |u_i| \, dx. \tag{36}$$

It follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \Big(\sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_{i,k}), \nabla u_{i,k}) \cdot \nabla u_{i,k} \, dx + \int_\Gamma g(x)|u_{1,k} - u_{2,k}|^p \, dx \Big) \\
&\leq \sum_{i=1}^{2} \int_{\Omega_i} \varphi_{i,n} \cdot \nabla u_i \, dx + \int_\Gamma g(x)|u_1 - u_2|^p \, dx.
\end{aligned} \tag{37}$$

On the other hand, in view of (8) we have

$$\begin{aligned}(38)\qquad &\sum_{i=1}^{2}\int_{\Omega_i}(a(x,T_n(u_{i,k}),\nabla u_{i,k})-a(x,T_n(u_{i,k}),\nabla u_i))\cdot(\nabla u_{i,k}-\nabla u_i)\,dx\\ &+\int_{\Gamma}g(x)(|u_{1,k}-u_{2,k}|^{p-2}(u_{1,k}-u_{2,k})-|u_1-u_2|^{p-2}(u_1-u_2))\\ &\qquad\times((u_{1,k}-u_{2,k})-(u_1-u_2))\,dx\geq 0,\end{aligned}$$

it follows that

$$\begin{aligned}(39)\qquad &\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,k}),\nabla u_{i,k})\cdot\nabla u_{i,k}\,dx+\int_{\Gamma}g(x)|u_{1,k}-u_{2,k}|^{p}\,dx\\ &\geq\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,k}),\nabla u_i))\cdot(\nabla u_{i,k}-\nabla u_i)\,dx\\ &+\int_{\Gamma}g(x)|u_1-u_2|^{p-2}(u_1-u_2)((u_{1,k}-u_{2,k})-(u_1-u_2))\,dx\\ &+\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,k}),\nabla u_{i,k})\cdot\nabla u_i dx+\int_{\Gamma}g(x)|u_{1,k}-u_{2,k}|^{p-2}(u_{1,k}-u_{2,k})(u_1-u_2)dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_{i,k})\to T_n(u_i)$ strongly in $L^p(\Omega)$, then

$$a(x,T_n(u_{i,k}),\nabla u_i)\to a(x,T_n(u_i),\nabla u_i)\qquad\text{strongly in}\quad (L^{p'}(\Omega_i))^N,$$

and since $|u_{1,k}-u_{2,k}|^{p-2}(u_{1,k}-u_{2,k})\rightharpoonup|u_1-u_2|^{p-2}(u_1-u_2)$ weakly in $L^{p'}(\Gamma)$, we conclude that

$$\begin{aligned}(40)\qquad &\liminf_{k\to\infty}\Big(\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,k}),\nabla u_{i,k})\cdot\nabla u_{i,k}\,dx+\int_{\Gamma}g(x)|u_{1,k}-u_{2,k}|^{p}\,dx\Big)\\ &\geq\sum_{i=1}^{2}\int_{\Omega_i}\varphi_{i,n}\cdot\nabla u_i\,dx+\int_{\Gamma}g(x)|u_1-u_2|^{p}\,dx.\end{aligned}$$

Using (37), we conclude that

$$\begin{aligned}(41)\qquad &\lim_{k\to\infty}\Big(\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,k}),\nabla u_{i,k})\cdot\nabla u_{i,k}\,dx+\int_{\Gamma}g(x)|u_{1,k}-u_{2,k}|^{p}\,dx\Big)\\ &=\sum_{i=1}^{2}\int_{\Omega_i}\varphi_{i,n}\cdot\nabla u_i\,dx+\int_{\Gamma}g(x)|u_1-u_2|^{p}\,dx.\end{aligned}$$

By combining (34), (36) and (41), we deduce that

$$\langle A_n u_k,u_k\rangle\to\langle\chi_n,u\rangle\qquad\text{as}\quad k\to\infty.$$

Moreover, in view of (41) we have

$$\begin{aligned}(42)\qquad &\sum_{i=1}^{2}\int_{\Omega_i}(a(x,T_n(u_{i,k}),\nabla u_{i,k})-a(x,T_n(u_{i,k}),\nabla u_i))\cdot(\nabla u_{i,k}-\nabla u_i)\,dx\\ &+\int_{\Gamma}g(x)(|u_{1,k}-u_{2,k}|^{p-2}(u_{1,k}-u_{2,k})-|u_1-u_2|^{p-2}(u_1-u_2))\\ &\qquad\times((u_{1,k}-u_{2,k})-(u_1-u_2))\,dx\longrightarrow 0\quad\text{as}\quad k\to\infty,\end{aligned}$$

it follows that

$$\sum_{i=1}^{2}\int_{\Omega_i}(a(x,T_n(u_{i,k}),\nabla u_{i,k})-a(x,T_n(u_{i,k}),\nabla u_i))\cdot(\nabla u_{i,k}-\nabla u_i)\,dx\longrightarrow 0\quad\text{as}\quad k\to\infty, \tag{43}$$

Using Lemma 2.1, we get $u_k\to u$ strongly in V and $\nabla u_{i,k}\to\nabla u_i$ almost everywhere in Ω_i, then

$$a(x,T_n(u_{i,k}),\nabla u_{i,k})\rightharpoonup a(x,T_n(u_i),\nabla u_i)\qquad\text{weakly in}\quad (L^{p'}(\Omega_i))^N\quad\text{for}\quad i=1,2.$$

Thanks to (32) and (33), we deduce that $\chi_n=A_n u$ which conclude the proof of Lemma 4.1.

In view of Lemma 4.1, there exists at least one weak solution $u_n=(u_{1,n},u_{2,n})$ in V for the approximate problem (24). i.e.

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega_i}a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot\nabla v_i\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})\,(v_1-v_2)\,d\sigma\\&=\sum_{i=1}^{2}\int_{\Omega_i}f_n(x,u_{i,n})v_i\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}v_i\,dx,\end{aligned}\tag{44}$$

for any $v=(v_1,v_2)\in V$.

Step 2: Weak convergence of truncations.

Taking $\theta>1$ small enough such that $\theta<p-\lambda-q+1$, and $q<\dfrac{N(p-\theta-\lambda+1)}{N+p-\lambda-\theta}$. We set $B(s)=\displaystyle\int_0^s\frac{d\tau}{(1+|\tau|)^\theta}$.

By taking $B(u_n)=(B(u_{1,n}),B(u_{2,n}))\in V$ as a test function for the approximate problem (44), we obtain

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega_i}\frac{a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot\nabla u_{i,n}}{(1+|u_{i,n}|)^\theta}\,dx\\&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})\,(B(u_{1,n})-B(u_{2,n}))\,d\sigma\\&=\sum_{i=1}^{2}\int_{\Omega_i}f_n(x)B(u_{i,n})\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}B(u_{i,n})\,dx.\end{aligned}\tag{45}$$

We have $B(s)$ is an increasing function and $|B(s)|\le\dfrac{1}{\theta-1}$. In view of (9) and Young's inequality, we conclude that

$$\begin{aligned}&b_0\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}|B(u_{1,n})-B(u_{2,n})|\,d\sigma\\&\le\sum_{i=1}^{2}\int_{\Omega_i}\frac{b(|T_n(u_{i,n})|)|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^\theta}\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|B(u_{1,n})-B(u_{2,n})|\,d\sigma\\&\le\sum_{i=1}^{2}\int_{\Omega_i}|f_n(x)|\,|B(u_{i,n})|\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}|B(u_{i,n})|\,dx\\&\le\frac{1}{\theta-1}\|f(x)\|_{L^1(\Omega\backslash\Gamma)}+\frac{\varepsilon}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}|T_n(u_{i,n})|^{p-\lambda-\theta}\,dx+\frac{C_0}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}\frac{1}{|x|^{\frac{q(p-\lambda-\theta)}{p-\lambda-\theta-q+1}}}\,dx.\end{aligned}\tag{46}$$

with $\varepsilon > 0$ is a positive constant. We set $G(r) = \displaystyle\int_0^r \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p}}}$. On the one hand we have

$$\begin{aligned}
b_0 \sum_{i=1}^{2} \int_{\Omega_i} \frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\, dx &\geq b_0 \sum_{i=1}^{2} \int_{\Omega_i} \Big| \frac{|\nabla u_{i,n}|}{(1+|u_{i,n}|)^{\frac{\lambda+\theta}{p}}} \Big|^p dx \\
&= b_0 \sum_{i=1}^{2} \int_{\Omega_i} \Big| \nabla \int_0^{u_{i,n}} \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p}}} \Big|^p dx \\
&= b_0 \sum_{i=1}^{2} \int_{\Omega_i} \Big| \nabla G(u_{i,n}) \Big|^p dx.
\end{aligned} \tag{47}$$

In view of Hölder's inequality we have

$$\begin{aligned}
\int_\Gamma |G(u_{2,n}) - G(u_{1,n})|^p \, d\sigma &= \int_\Gamma \Big| \int_{u_{1,n}}^{u_{2,n}} \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p}}} \Big|^p d\sigma \\
&\leq \int_\Gamma \Big| \Big(\int_{u_{1,n}}^{u_{2,n}} ds \Big)^{\frac{1}{p'}} \Big(\int_{u_{1,n}}^{u_{2,n}} \frac{ds}{(1+|s|)^{\frac{(\lambda+\theta)p}{p}}} \Big)^{\frac{1}{p}} \Big|^p d\sigma \\
&= \int_\Gamma |u_{2,n} - u_{1,n}|^{p-1} \Big| \int_{u_{1,n}}^{u_{2,n}} \frac{ds}{(1+|s|)^{\frac{(\lambda+\theta)p}{p}}} \Big| \, d\sigma \\
&\leq \int_\Gamma |u_{2,n} - u_{1,n}|^{p-1} \Big| \int_{u_{1,n}}^{u_{2,n}} \frac{ds}{(1+|s|)^{\theta}} \Big| \, d\sigma \\
&\leq \frac{1}{g_0} \int_\Gamma g(x) |u_{2,n} - u_{1,n}|^{p-1} |B(u_{2,n}) - B(u_{1,n})| \, d\sigma.
\end{aligned} \tag{48}$$

By combining (47) – (48), and using Remark 2.3 we conclude that

$$\begin{aligned}
& b_0 \sum_{i=1}^{2} \int_{\Omega_i} \frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\, dx + \int_\Gamma g(x) |u_{1,n} - u_{2,n}|^{p-1} |B(u_{1,n}) - B(u_{2,n})| \, d\sigma \\
&\geq b_0 \sum_{i=1}^{2} \int_{\Omega_i} \Big| \nabla G(u_{i,n}) \Big|^p dx + g_0 \int_\Gamma |G(u_{2,n}) - G(u_{1,n})|^p \, d\sigma \\
&= b_0 \sum_{i=1}^{2} \big\| \nabla G(u_{i,n}) \big\|^p_{L^p(\Omega_i)} + g_0 \big\| |G(u_{2,n}) - G(u_{1,n})| \big\|^p_{L^p(\Gamma)} \\
&\geq C_1 \big\| G(u_n) \big\|_V^p \\
&\geq C_2 \sum_{i=1}^{2} \big\| G(u_{i,n}) \big\|^p_{L^p(\Omega_i)} \\
&= C_2 \sum_{i=1}^{2} \Big\| \int_0^{u_{i,n}} \frac{ds}{(1+|s|)^{\frac{\lambda+\theta}{p}}} \Big\|^p_{L^p(\Omega_i)} \\
&\geq C_3 \sum_{i=1}^{2} \int_{\Omega_i} \frac{|u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\, dx \\
&\geq C_4 \sum_{i=1}^{2} \int_{\Omega_i} |u_{i,n}|^{p-\lambda-\theta} \, dx - C_5.
\end{aligned} \tag{49}$$

Having in mind (46), and since $1 < q < \dfrac{N(p-\theta-\lambda+1)}{N+p-\lambda-\theta}$ then $\dfrac{q(p-\lambda-\theta)}{p-\lambda-\theta-q+1} < N$, we conclude that

$$
\begin{aligned}
&C_4\sum_{i=1}^{2}\int_{\Omega_i}|u_{i,n}|^{p-\lambda-\theta}\,dx-C_5\\
&\le b_0\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}|B(u_{1,n})-B(u_{2,n})|\,d\sigma\\
&\le \frac{1}{\theta-1}\|f(x)\|_{L^1(\Omega\setminus\Gamma)}+\frac{\varepsilon}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}|T_n(u_{i,n})|^{p-\lambda-\theta}\,dx+\frac{C_0}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}\frac{1}{|x|^{\frac{q(p-\lambda-\theta)}{p-\lambda-\theta-q+1}}}\,dx\\
&\le C_6+\frac{\varepsilon}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}|u_{i,n}|^{p-\lambda-\theta}\,dx.
\end{aligned}
\tag{50}
$$

By choosing $\varepsilon>0$ small enough (for example $\varepsilon\le\dfrac{C_5}{2(\theta-1)}$), we conclude that

$$
\sum_{i=1}^{2}\int_{\Omega_i}|u_{i,n}|^{p-\lambda-\theta}\,dx\le C_7,
\tag{51}
$$

with C_7 is a positive constant that does not depend on n and k. Moreover, thanks to (50) we conclude that

$$
\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}|B(u_{1,n})-B(u_{2,n})|\,d\sigma\le C_8.
\tag{52}
$$

Having in mind that $|B(r)-B(s)|\ge\dfrac{|r-s|}{(1+|s|+|r|)^\theta}$ for any $r,s\in I\!R$, we obtain

$$
\int_\Gamma|u_{1,n}-u_{2,n}|^{p-1}|B(u_{1,n})-B(u_{2,n})|\,d\sigma\ge\int_\Gamma\frac{|u_{1,n}-u_{2,n}|^p}{(1+|u_{1,n}|+|u_{2,n}|)^\theta}\,d\sigma.
\tag{53}
$$

It follows that

$$
\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla u_{i,n}|^p}{(1+|u_{i,n}|)^{\lambda+\theta}}\,dx+\int_\Gamma\frac{|u_{1,n}-u_{2,n}|^p}{(1+|u_{1,n}|+|u_{2,n}|)^\theta}\,d\sigma\le C_9.
\tag{54}
$$

On the other hand, by taking $B(T_k(u_n))=(B(T_k(u_{1,n})),B(T_k(u_{2,n})))\in V$ as a test function for the approximate problem (44), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i}\frac{a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot\nabla T_k(u_{i,n})}{(1+|T_k(u_{i,n})|)^\theta}\,dx\\
&\quad+\int_\Gamma g|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})\left(B(T_k(u_{1,n}))-B(T_k(u_{2,n}))\right)d\sigma\\
&=\sum_{i=1}^{2}\int_{\Omega_i}f_n(x)B(T_k(u_{i,n}))\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}B(T_k(u_{i,n}))\,dx.
\end{aligned}
\tag{55}
$$

Thanks to (9) and Young's inequality, and since $|B(T_k(u_{i,n}))|\le\dfrac{1}{\theta-1}$ we obtain

$$
\begin{aligned}
&b_0\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla T_k(u_{i,n})|^p}{(1+|T_k(u_{i,n})|)^{\lambda+\theta}}\,dx+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}|B(T_k(u_{1,n}))-B(T_k(u_{2,n}))|\,d\sigma\\
&\le\sum_{i=1}^{2}\int_{\Omega_i}|f(x)|\,|B(T_k(u_{i,n}))|\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}\,|B(T_k(u_{i,n}))|\,dx
\end{aligned}
$$

$$\leq \frac{1}{\theta-1}\|f\|_{L^1(\Omega\setminus\Gamma)} + \frac{1}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}|u_{i,n}|^{p-\lambda-\theta}\,dx + \frac{1}{\theta-1}\sum_{i=1}^{2}\int_{\Omega_i}\frac{1}{|x|^{\frac{q(p-\lambda-\theta)}{p-\lambda-\theta-q+1}}}\,dx. \tag{56}$$

According to (51), we obtain

$$\begin{aligned} &b_0\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla T_k(u_{i,n})|^p}{(1+|T_k(u_{i,n})|)^{\lambda+\theta}}\,dx \\ &\quad+\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}|B(T_k(u_{1,n}))-B(T_k(u_{2,n}))|\,d\sigma \leq C_{10}. \end{aligned} \tag{57}$$

Having in mind $|B(T_k(r))-B(T_k(s))| \geq \dfrac{|r-s|}{(1+k)^{\theta}}$ for any $r,s\in I\!R$, we conclude that

$$\begin{aligned} &\frac{b_0}{(1+k)^{\lambda+\theta}}\sum_{i=1}^{2}\int_{\Omega_i}|\nabla T_k(u_{i,n})|^p\,dx \\ &\quad+\frac{g_0}{(1+k)^{\theta}}\int_{\Gamma}|u_{1,n}-u_{2,n}|^{p-1}|T_k(u_{1,n})-T_k(u_{2,n})|\,d\sigma \leq C_{10}. \end{aligned} \tag{58}$$

Thus, we conclude that

$$\sum_{i=1}^{2}\int_{\Omega_i}|\nabla T_k(u_{,n})|^p\,dx + \int_{\Gamma}|T_k(u_{1,n})-T_k(u_{2,n})|^p\,d\sigma \leq C_{11}k^{\lambda+\theta} \qquad \text{for any} \quad k\geq 1. \tag{59}$$

Hence, we get

$$\|T_k(u_{i,n})\|_V = \Big(\sum_{i=1}^{N}\|\nabla T_k(u_{i,n})\|^p_{L^p(\Omega_i)} + \|T_k(u_{1,n})-T_k(u_{2,n})\|^p_{L^p(\Gamma)}\Big)^{\frac{1}{p}} \leq C_{12}k^{\frac{\lambda+\theta}{p}}. \tag{60}$$

with C_{12} is a positive constant that does not depend on n and k. Therefore, the sequence $(T_k(u_n) := (T_k(u_{1,n}),T_k(u_{2,n})))_n$ is uniformly bounded in V for every $k\geq 1$. Then there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $v_k := (v_{k,1},v_{k,2})\in V$ such that

$$T_k(u_n) \rightharpoonup v_k \quad \text{weakly in} \quad V. \tag{61}$$

Recalling that the embedding $V \hookrightarrow\hookrightarrow L^p(\Omega_1)\times L^p(\Omega_2)$ is compact, and the trace operator $\gamma_i : W^{1,p}(\Omega_i)\to L^p(\Gamma)$ is continuous, then

$$\begin{cases} T_k(u_{i,n}) \to v_{k,i} & \text{strongly in} \quad L^p(\Omega_i) \quad \text{and} \quad \text{a.e in} \quad \Omega_i, \\ \gamma_i(T_k(u_{i,n})) \rightharpoonup \gamma_i(v_{k,i}) & \text{weakly in} \quad L^p(\Gamma) \quad \text{and} \quad \text{a.e on} \quad \Gamma. \end{cases} \tag{62}$$

Thanks to Remark 2.3 and (60) we have

$$\begin{aligned} k\text{meas}\{|u_n|>k\} &= \int_{\{|u_{1,n}|>k\}} k\,dx + \int_{\{|u_{2,n}|>k\}} k\,dx \\ &\leq \int_{\Omega_1}|T_k(u_{1,n})|\,dx + \int_{\Omega_2}|T_k(u_{2,n})|\,dx \\ &\leq C_{13}(\|T_k(u_{1,n})\|_{L^p(\Omega_1)} + \|T_k(u_{2,n})\|_{L^p(\Omega_2)}) \\ &\leq C_{14}\|T_k(u_n)\|_V \\ &\leq C_{15}k^{\frac{\lambda+\theta}{p}}, \end{aligned} \tag{63}$$

where C_{15} is a positive constant that does not depend on k and n, we deduce that

$$\text{meas}\{|u_n| > k\} \leq \frac{C_{15} k^{\frac{\lambda+\theta}{p}}}{k} \longrightarrow 0 \qquad \text{as} \quad k \to \infty. \tag{64}$$

Moreover, for the term $\gamma_1(u_{1,n})$, in view of the trace theorem we obtain

$$\begin{aligned}
k \ \text{meas}_\Gamma\{|\gamma_1(u_{1,n})| > k\} &= \int_{\{|\gamma_1(u_{1,n})|>k\}} k \, d\sigma \\
&\leq \int_\Gamma |\gamma_1(T_k(u_{1,n})| \, d\sigma \\
&\leq C_{16} \|\gamma_1(T_k(u_{1,n}))\|_{L^p(\Gamma)} \\
&\leq C_{17} \|T_k(u_n)\|_V \\
&\leq C_{18} k^{\frac{\lambda+\theta}{p}}.
\end{aligned}$$

Hence

$$\text{meas}_\Gamma\{|\gamma_1(u_{1,n})| > k\} \leq \frac{C_{18} k^{\frac{\lambda+\theta}{p}}}{k} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{65}$$

Similarly for $\gamma_2(u_{2,n})$, it's clear that

$$\begin{aligned}
k\text{meas}_\Gamma\{|\gamma_2(u_{2,n})| > k\} &= \int_{\{|\gamma_2(u_{2,n})|>k\}} k \, d\sigma \\
&\leq \|\gamma_2(T_k(u_{2,n}))\|_{L^p(\Gamma)} \\
&\leq C_{19} \|T_k(u_n)\|_V \\
&\leq C_{20} k^{\frac{\lambda+\theta}{p}}.
\end{aligned}$$

Consequently

$$\text{meas}_\Gamma\{|\gamma_2(u_{2,n})| > k\} \leq \frac{C_{20} k^{\frac{\lambda+\theta}{p}}}{k} \longrightarrow 0 \qquad \text{as} \quad k \to \infty. \tag{66}$$

Now, we will show that $(u_{i,n})_n$ is Cauchy sequence in measure.

For every $\delta > 0$,

$$\begin{aligned}
meas\,\{|u_{i,n} - u_{i,m}| > \delta\} \ &\leq \text{meas}\,\{|u_{i,n}| > k\} + meas\,\{|u_{i,m}| > k\} \\
&\quad + meas\,\{|T_k(u_{i,n}) - T_k(u_{i,m})| > \delta\}.
\end{aligned} \tag{67}$$

Let $\varepsilon > 0$, in view of (64) there exists $k = k(\varepsilon)$ large enough such that

$$\text{meas}\,\{|u_{i,n}| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\,\{|u_{i,m}| > k\} \leq \frac{\varepsilon}{3} \qquad \text{for any} \quad k \geq k(\varepsilon). \tag{68}$$

Moreover, in view of (62) we have $(T_k(u_{i,n}))_n$ is a Cauchy sequence in measure in Ω_i, then for all $k > 0$ and $\delta, \varepsilon > 0$ there exists $n_0 = n_0(k, \varepsilon, \delta)$ such that

$$\text{meas}\{|T_k(u_{i,n}) - T_k(u_{i,m})| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{69}$$

Thanks to (67) and (68) $-$ (69), we deduce that: for any $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$meas\,\{|u_{i,n} - u_{i,m}| > \delta\} \leq \varepsilon \qquad \forall n, m \geq n_0(\delta, \varepsilon),$$

Thus, the sequence $(u_{i,n})_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u_i. Therefore

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad V. \tag{70}$$

Using similar arguments, we can deduce that $\gamma_i(u_{i,n})$ converges almost everywhere on Γ, for a subsequence, to some measurable function w_i, i.e.

$$\gamma_i(u_{i,n}) \to w_i \quad \text{a.e. on} \quad \Gamma \qquad \text{for} \quad i = 1, 2. \tag{71}$$

Now, we will show that $\gamma(u_i) = w_i$, in view of (60) we have

$$\|T_k(u)\|_V \leq C_{12} k^{\frac{\lambda+\theta}{p}} \quad \text{for any} \quad k \geq 1. \tag{72}$$

Thanks to the Proposition 2.2, we have $\gamma_i(u_i)$ is well defined. By combining (62), (71) – (72), we obtain that

$$\text{for any} \quad k \geq 1 \qquad \text{we have} \qquad T_k(w_i) = \gamma_i(v_{k,i}) = \gamma_i(T_k(u_i)) = T_k(\gamma_i(u_i)) \quad \text{a.e. on } \Gamma.$$

Thus, we have $w_i = \gamma_i(u_i)$.

Consequently, we conclude that

$$\begin{cases} u_{i,n} \to u_i & \text{a.e. in} \quad \Omega_i, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad V, \\ T_k(u_{i,n}) \to T_k(u_i) & \text{strongly in} \quad L^p(\Omega_i) \quad \text{and} \quad \text{a.e. in} \quad \Omega_i, \\ \gamma_i(u_{i,n}) \to \gamma_i(u_i) & \text{a.e. on} \quad \Gamma, \\ \gamma_i(T_k(u_{i,n})) \to \gamma_i(T_k(u_i)) & \text{strongly in} \quad L^p(\Gamma) \quad \text{and} \quad \text{a.e. on} \quad \Gamma. \end{cases} \tag{73}$$

Moreover, in view of Fatou's Lemma and (58), we conclude that

$$\begin{aligned} &\int_\Gamma g(x)|u_1 - u_2|^{p-1}|T_k(u_1) - T_k(u_2)|\, d\sigma \\ &\leq \liminf_{n\to\infty} \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-1}|T_k(u_{1,n}) - T_k(u_{2,n})|\, d\sigma \\ &\leq C_6 k^\theta \qquad\qquad \text{for any} \quad k \geq 1. \end{aligned} \tag{74}$$

Step 3: Equi-integrability of the sequence $\left(\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q}\right)_n$**.**

In this part, we will show that:

$$\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} \longrightarrow \frac{|u_i|^{q-2}u_i}{|x|^q} \quad \text{strongly in} \quad L^1(\Omega_i) \quad \text{for} \quad i = 1, 2. \tag{75}$$

We have $\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q}$ converges to $\dfrac{|u_i|^{q-2}u_i}{|x|^q}$ almost everywhere in Ω_i for $i = 1, 2$. By using the Vitali's theorem, it is sufficient to prove that the sequence $\left(\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q}\right)_n$ is uniformly equi-integrable for $i = 1, 2$.

Let $h > 0$, for any measurable subset $E_i \subset \Omega_i$, we have

$$\sum_{i=1}^{2} \int_{E_i} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx \le \sum_{i=1}^{2} \int_{E_i} \frac{|T_h(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx + \sum_{i=1}^{2} \int_{\{h<|u_{i,n}|\}} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx. \tag{76}$$

On the one hand, we have $\frac{|T_h(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q} \in L^1(\Omega_i)$. Then, for any $\varepsilon > 0$ there exists $\beta(\varepsilon, h)$ small enough such that: for any subset $E_i \subset \Omega_i$ with $meas(E_i) \le \beta(\varepsilon, h)$, we have

$$\sum_{i=1}^{2} \int_{E_i} \frac{|T_h(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx \le \frac{\varepsilon}{2} \qquad \text{with} \quad \text{meas}(E_i) \le \beta(\varepsilon, h). \tag{77}$$

On the other hand, then there exists a positive constant $\sigma > 0$ such that $0 < \sigma < p - \lambda - \theta - q + 1$, and in view of (51) we obtain

$$\begin{aligned} h^\sigma \sum_{i=1}^{2} \int_{\{h<|u_{i,n}|\}} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx &\le \sum_{i=1}^{2} \int_{\{h<|u_{i,n}|\}} \frac{|u_{i,n}|^{q-1+\sigma}}{(|x| + \frac{1}{n})^q}\,dx \\ &\le \sum_{i=1}^{2} \int_{\Omega_i} |u_{i,n}|^{p-\lambda-\theta}\,dx + \sum_{i=1}^{2} \int_{\Omega_i} \frac{1}{|x|^{\frac{q(p-\lambda-\theta)}{p-\lambda-\theta-q+1-\sigma}}}\,dx \\ &\le C_{21}. \end{aligned} \tag{78}$$

It follows that

$$\sum_{i=1}^{2} \int_{\{h<|u_{i,n}|\}} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx \le \frac{C_8}{h^\sigma} \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{79}$$

Thus, we conclude that: for all $\varepsilon > 0$, there exists a positive constant $h_0(\varepsilon) > 0$ such that

$$\sum_{i=1}^{2} \int_{\{h<|u_{i,n}|\}} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx \le \frac{\varepsilon}{2} \qquad \text{for any} \quad h \ge h_0(\varepsilon). \tag{80}$$

By combining (76), (77) and (80), we conclude that: for any $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{2} \int_{E_i} \frac{|T_n(u_{i,n})|^{q-1}}{(|x| + \frac{1}{n})^q}\,dx \le \varepsilon \qquad \text{with} \quad \text{meas}(E_i) \le \beta(\varepsilon). \tag{81}$$

Consequently, the sequence $\left(\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q}\right)_n$ is uniformly equi-integrable in Ω_i for $i = 1, 2$, and in view of Vitali's theorem we deduce that

$$\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} \longrightarrow \frac{|u_i|^{q-2}u_i}{|x|^q} \qquad \text{strongly in } L^1(\Omega_i) \quad \text{for} \quad i = 1, 2. \tag{82}$$

Step 4: Some regularity results.

Let $h \ge 1$, by taking $\frac{T_h(u_n)}{h} = (\frac{T_h(u_{1,n})}{h}, \frac{T_h(u_{2,n})}{h}) \in V$ as a test function for the approximate problem (44), we obtain

$$
(83)\qquad \begin{aligned}
&\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_h(u_{i,n}),\nabla T_h(u_{i,n}))\cdot\nabla T_h(u_{i,n})\,dx\\
&\quad+\frac{1}{h}\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})\,(T_h(u_{1,n})-T_h(u_{2,n}))\,d\sigma\\
&=\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i} f_n(x)T_h(u_{i,n})\,dx+\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}T_h(u_{i,n})\,dx.
\end{aligned}
$$

In view of (9) and Young's inequality, we get

$$
(84)\qquad \begin{aligned}
&\frac{1}{2h}\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_h(u_{i,n}),\nabla T_h(u_{i,n}))\cdot\nabla T_h(u_{i,n})\,dx+\frac{b_0}{2h}\sum_{i=1}^{2}\int_{\Omega_i}\frac{|\nabla T_h(u_{i,n})|^p}{(1+|T_h(u_{i,n})|)^{\lambda}}\,dx\\
&\quad+\frac{1}{h}\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}|T_h(u_{1,n})-T_k(u_{2,n})|\,d\sigma\\
&\le\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}|f_n(x)|\,|T_h(u_{i,n})|\,dx+\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}\,|T_h(u_{i,n})|\,dx.
\end{aligned}
$$

For the first term on the right-hand side of (84), in view of (82) we have $|f_n(x)|$ tends to $|f(x)|$ strongly in $L^1(\Omega\backslash\Gamma)$ as n tends to infinity, and since $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega\backslash\Gamma)$ as $h\to\infty$, we obtain

$$
(85)\qquad \lim_{n\to\infty}\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}|f_n(x)|\,|T_h(u_{i,n})|\,dx=\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}|f(x)|\,|T_h(u_i)|\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty.
$$

Similarly, thanks to (82) we have $\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}$ converges to $\dfrac{|u_i|^{q-2}u_i}{|x|^q}$ strongly in $L^1(\Omega_i)$ for $i=1,2$, then

$$
(86)\qquad \lim_{n\to\infty}\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}\,|T_k(u_{i,n})|\,dx=\frac{1}{h}\sum_{i=1}^{2}\int_{\Omega_i}\frac{|u_i|^{q-1}}{|x|^q}\,|T_h(u_i)|\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty.
$$

It follows that

$$
(87)\qquad \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{2h}\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_h(u_{i,n}),\nabla T_h(u_{i,n}))\cdot\nabla T_h(u_{i,n})\,dx=0\qquad\text{for}\quad i=1,2.
$$

$$
(88)\qquad \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\Omega_i}\frac{|\nabla T_h(u_{i,n})|^p}{(1+|T_h(u_{i,n})|)^{\lambda}}\,dx=0\qquad\text{for}\quad i=1,2,
$$

and

$$
(89)\qquad \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}|T_h(u_{1,n})-T_k(u_{2,n})|\,d\sigma=0.
$$

Step 5: Convergence of the gradient.

In the sequel, we denote by $\varepsilon_i(n)$ for $i=1,2,\dots$ some various functions of real variables that converges to 0 as n tends to infinity, respectively, we define $\varepsilon_i(h)$ and $\varepsilon_i(n,h)$.

Let $h>k>0$, and we set $S_h(r)=1-\dfrac{|T_{2h}(r)-T_h(r)|}{h}$. By taking $v=(T_k(u_n)-T_k(u))S_h(u_n)\in V$ as a test function for the approximate problem (44), we have

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))S_h(u_{i,n})\,dx\\
&\qquad-\sum_{i=1}^{2}\frac{1}{h}\int_{\Omega_i\cap\{h<|u_{u_{i,n}}|\le 2h\}} a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot\nabla u_{i,n}(T_k(u_{i,n})-T_k(u_i))\,dx\\
&\qquad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})(T_k(u_{1,n})-T_k(u_1))S_h(u_{1,n})\,d\sigma\\
&\qquad-\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-2}(u_{1,n}-u_{2,n})(T_k(u_{2,n})-T_k(u_2))S_h(u_{2,n})\,d\sigma\\
&=\sum_{i=1}^{2}\int_{\Omega_i} f_n(x)(T_k(u_{i,n})-T_k(u_i))S_h(u_{i,n})\,dx\\
&\qquad+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}(T_k(u_{i,n})-T_k(u_i))S_h(u_{i,n})\,dx.
\end{aligned}
\tag{90}
$$

It follows that

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))S_h(u_{i,n})\,dx\\
&\le\sum_{i=1}^{2}\int_{\Omega_i}|f_n(x,u_{i,n})|\,|T_k(u_{i,n})-T_k(u_i)|\,dx+\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}\,|T_k(u_{i,n})-T_k(u_i)|\,dx\\
&\quad+\sum_{i=1}^{2}\frac{2k}{h}\int_{\{h<|u_{u_{i,n}}|\le 2h\}} a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n}))\cdot\nabla T_{2h}(u_{i,n})\,dx\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_h(u_{1,n})\,d\sigma\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{2,n})-T_k(u_2)|\,S_h(u_{2,n})|\,d\sigma.
\end{aligned}
\tag{91}
$$

It is clear that $T_k(u_{i,n})\rightharpoonup T_k(u_i)$ weak$-*$ in $L^\infty(\Omega_i)$, and since $|f_n(x)|\to|f(x)|$ strongly in $L^1(\Omega\backslash\Gamma)$, it follows that

$$
\varepsilon_1(n)=\sum_{i=1}^{2}\int_{\Omega_i}|f_n(x)|\,|T_k(u_{i,n})-T_k(u_i)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.
\tag{92}
$$

Similarly, thanks to (82) we have $\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x|+\frac{1}{n})^q}$ tends to $\dfrac{|u_i|^{q-2}u_i}{|x|^q}$ strongly in $L^1(\Omega_i)$ for $i=1,2$, we obtain

$$
\varepsilon_2(n)=\sum_{i=1}^{2}\int_{\Omega_i}\frac{|T_n(u_{i,n})|^{q-1}}{(|x|+\frac{1}{n})^q}\,|T_k(u_{i,n})-T_k(u_i)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.
\tag{93}
$$

Concerning the third term on the right-hand side of (91), in view of (87) we have

$$
\varepsilon_3(h)=\sum_{i=1}^{2}\frac{2k}{h}\int_{\{h<|u_{u_{i,n}}|\le 2h\}} a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n}))\cdot\nabla T_{2h}(u_{i,n})\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty.
\tag{94}
$$

By combining (91) and (92) – (94), we deduce that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))S_h(u_{i,n})\,dx\\
&\le \int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_h(u_{1,n})\,d\sigma.\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{2,n})-T_k(u_2)|\,S_h(u_{2,n})|\,d\sigma+\varepsilon_4(n,h).
\end{aligned}\tag{95}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_i}(a(x,T_k(u_{i,n}),\nabla T_k(u_{i,n}))-a(x,T_k(u_{i,n}),\nabla T_k(u_i)))\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))\,dx\\
&\le -\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_k(u_{i,n}),\nabla T_k(u_i))\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))\,dx\\
&\quad+\sum_{i=1}^{2}\int_{\{k<|u_{i,n}|\le 2h\}} a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n}))\cdot\nabla T_k(u_i)\;S_h(u_{i,n})\,dx\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_h(u_{1,n})\,d\sigma\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{2,n})-T_k(u_2)|\,S_h(u_{2,n})\,d\sigma+\varepsilon_4(n,h).
\end{aligned}\tag{96}$$

For the first term on the right-hand side of (96), we have $T_k(u_{i,n})\to T_k(u_i)$ strongly in $L^{p(\cdot)}(\Omega_i)$, in view of Lebesgue dominated convergence theorem and (7) we have $a(x,T_k(u_{i,n}),\nabla T_k(u_i))\to a(x,T_k(u_i),\nabla T_k(u_i))$ strongly in $(L^{p'}(\Omega_i))^N$, and since $\nabla T_k(u_{i,n})\rightharpoonup\nabla T_k(u_i)$ weakly in $(L^p(\Omega_i))^N$, it follows that

$$\varepsilon_5(n)=\sum_{i=1}^{2}\int_{\Omega_i} a(x,T_k(u_{i,n}),\nabla T_k(u_i))\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\tag{97}$$

Concerning the second term on the right-hand side of (96), since the sequence $(a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n})))_n$ is uniformly bounded in $(L^{p'}(\Omega_i))^N$, then, there exists a measurable function $\varphi_{i,h}\in(L^{p'}(\Omega_i))^N$ such that $a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n}))\rightharpoonup\varphi_{i,h}$ weakly in $(L^{p'}(\Omega_i))^N$, and since $\nabla T_k(u_i)\;S_h(u_{i,n})$ tends to $\nabla T_k(u_i)\;S_h(u_i)$ strongly in $(L^p(\Omega_i))^N$, we obtain

$$\begin{aligned}
\varepsilon_6(n)\;&=\sum_{i=1}^{2}\int_{\{k<|u_{i,n}|\le 2h\}} a(x,T_{2h}(u_{i,n}),\nabla T_{2h}(u_{i,n}))\cdot\nabla T_k(u_i)S_h(u_{i,n})\,dx\\
&\longrightarrow\sum_{i=1}^{2}\int_{\{k<|u_i|\le 2h\}}\varphi_{i,h}\cdot\nabla T_k(u_i)S_h(u_i)\,dx=0\quad\text{as}\quad n\to\infty.
\end{aligned}\tag{98}$$

For the third term on the right-hand side of (96), it's clear that $S_h(u_{1,n})=S_h(u_{1,n})S_{2h}(u_{1,n})$, then

$$\begin{aligned}
&\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_h(u_{1,n})\,d\sigma\\
&=\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_{2h}(u_{1,n})(S_h(u_{1,n})-S_h(u_{2,n}))\,d\sigma\\
&\quad+\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_{2h}(u_{1,n})S_h(u_{2,n})\,d\sigma\\
&\le\frac{2k}{h}\int_\Gamma g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_{2h}(u_{1,n})-T_{2h}(u_{2,n})|\,d\sigma\\
&\quad+\int_\Gamma g(x)|T_{4h}(u_{1,n})-T_{2h}(u_{2,n})|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,d\sigma.
\end{aligned}\tag{99}$$

In view of (89) we have

$$\frac{2k}{h}\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_{2h}(u_{1,n})-T_{2h}(u_{2,n})|\,d\sigma \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{100}$$

Concerning the second term on the right-hand side of (99), we have $|T_{4h}(u_{1,n})-T_{2h}(u_{2,n})|^{p-1}\to |T_{4h}(u_1)-T_{2h}(u_2)|^{p-1}$ strongly in $L^{p'}(\Gamma)$, and since $T_k(u_{1,n})-T_k(u_1)\rightharpoonup 0$ weakly in $L^p(\Gamma)$, then

$$\int_{\Gamma} g(x)|T_{4h}(u_{1,n})-T_{2h}(u_{2,n})|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,d\sigma \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{101}$$

Thus, according to (99) and (100) – (101), we deduce that

$$\varepsilon_7(n,h)=\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{1,n})-T_k(u_1)|\,S_h(u_{1,n})\,d\sigma \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{102}$$

Similarly, we can show that

$$\varepsilon_8(n,h)=\int_{\Gamma} g(x)|u_{1,n}-u_{2,n}|^{p-1}\,|T_k(u_{2,n})-T_k(u_2)|\,S_h(u_{2,n})\,d\sigma \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{103}$$

By combining (96) – (98) and (102) – (103), we conclude that

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega_i}\left(a(x,T_k(u_{i,n}),\nabla T_k(u_{i,n}))-a(x,T_k(u_{i,n}),\nabla T_k(u_i))\right)\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))\,dx\\ &\le \varepsilon_9(n,h).\end{aligned} \tag{104}$$

By passing to the limit as n and h tend to infinity, we conclude that

$$\sum_{i=1}^{2}\int_{\Omega_i}\left(a(x,T_k(u_{i,n}),\nabla T_k(u_{i,n}))-a(x,T_k(u_{i,n}),\nabla T_k(u_i))\right)\cdot(\nabla T_k(u_{i,n})-\nabla T_k(u_i))\,dx \longrightarrow 0, \tag{105}$$

as $n\to\infty$. In view of Lemma 2.1, we deduce that

$$\begin{cases} T_k(u_n)\to T_k(u) & \text{strongly in} \quad V,\\ \nabla u_{i,n}\to\nabla u_i & \text{almost everywher in} \quad \Omega_i \quad \text{for} \quad i=1,2. \end{cases} \tag{106}$$

Moreover, we obtain $a(x,T_n(u_{i,n}),\nabla u_{i,n})\cdot\nabla u_{i,n}\to a(x,u_i,\nabla u_i)\cdot\nabla u_i$ a.e. in Ω_i, and in view of Fatou's Lemma and (87) we conclude that

$$\begin{aligned}&\lim_{h\to\infty}\frac{1}{h}\int_{\{|u_i|\le h\}} a(x,T_h(u_i),\nabla T_h(u_i))\cdot\nabla T_h(u_i)\,dx\\ &\le \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\int_{\{|u_{i,n}|\le h\}} a(x,T_h(u_{i,n}),\nabla T_h(u_{i,n}))\cdot\nabla T_h(u_{i,n})\,dx\\ &\le \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_{i,n}|\le h\}} a(x,T_h(u_{i,n}),\nabla T_h(u_{i,n}))\cdot\nabla T_h(u_{i,n})\,dx=0.\end{aligned} \tag{107}$$

Step 6: Passage to limit.

Let $\varphi = (\varphi_1, \varphi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$, and let $S(\cdot)$ be a smooth function in $C_0^1(I\!R)$ such that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.

By taking $v = (S(u_{1,n})\varphi_1, S(u_{2,n})\varphi_2) \in V$ as a test function for the approximate problem (44), we have

$$\begin{aligned} &\sum_{i=1}^{2} \int_{\Omega_i} a(x, T_n(u_{i,n}), \nabla u_{i,n}) \cdot (S'(u_{i,n})\varphi_i \nabla u_{i,n} + S(u_{i,n}) D\varphi_i)\, dx \\ &\quad + \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{1,n} - u_{2,n})(S(u_{1,n})\varphi_1 - S(u_{2,n})\varphi_2)\, d\sigma \\ &= \sum_{i=1}^{2} \int_{\Omega_i} f_n(x) S(u_{i,n})\varphi_i\, dx + \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-2} T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} S(u_{i,n})\varphi_i\, dx. \end{aligned} \tag{108}$$

Having in mind that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$, it follows that

$$\begin{aligned} &\sum_{i=1}^{2} \int_{\Omega_i} a(x, T_M(u_{i,n}), \nabla T_M(u_{i,n})) \cdot (S'(T_M(u_{i,n}))\varphi_i \nabla T_M(u_{i,n}) + S(T_M(u_{i,n}))\nabla\varphi_i)\, dx \\ &\quad + \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{1,n} - u_{2,n})(S(T_M(u_{1,n}))\varphi_1 - S(T_M(u_{2,n}))\varphi_2)\, d\sigma \\ &= \sum_{i=1}^{2} \int_{\Omega_i} f_n(x) S(u_{i,n})\varphi_i\, dx + \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-2} T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} S(u_{i,n})\varphi_i\, dx. \end{aligned} \tag{109}$$

We have $a(x, T_M(u_{i,n}), \nabla T_M(u_{i,n})) \to a(x, T_M(u_i), \nabla T_M(u_i))$ a.e. in Ω_i. In view of (7), the sequence $(a(x, T_M(u_{i,n}), \nabla T_M(u_{i,n})))_n$ is uniformly bounded in $(L^{p'}(\Omega_i))^N$, it follows that

$$a(x, T_M(u_{i,n}), \nabla T_M(u_{i,n})) \rightharpoonup a(x, T_M(u_i), \nabla T_M(u_i)) \qquad \text{weakly in} \quad (L^{p'}(\Omega_i))^N. \tag{110}$$

On the other hand, thanks to (106), we have

$$S'(u_{i,n})\varphi_i \nabla T_M(u_{i,n}) + S(u_{i,n})\nabla\varphi_i \to S'(u_i)\varphi_i \nabla T_M(u_i) + S(u_i)\nabla\varphi_i \quad \text{strongly in} \ \ (L^p(\Omega_i))^N.$$

We conclude that

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} a(x, T_M(u_{i,n}), \nabla T_M(u_{i,n})) \cdot (S'(u_{i,n})\varphi_i \nabla T_M(u_{i,n}) + S(u_{i,n})\nabla\varphi_i)\, dx \\ &\qquad = \sum_{i=1}^{2} \int_{\Omega_i} a(x, T_M(u_i), \nabla T_M(u_i)) \cdot (S'(u_i)\varphi_i \nabla T_M(u_i) + S(u_i)\nabla\varphi_i)\, dx \\ &\qquad = \sum_{i=1}^{2} \int_{\Omega_i} a(x, u_i, \nabla u_i) \cdot (S'(u_i)\nabla u_i \varphi_i + S(u_i)\nabla\varphi_i)\, dx. \end{aligned} \tag{111}$$

Concerning the second term on the left-hand side of (109). Let $h \geq M$, we set

$$\psi_h(r) = 1 - \frac{|S_{2h}(r) - S_h(r)|}{h}.$$

We have $S(\cdot) \subset [-M, M]$, it follows that $S(u_{1,n}) = S(u_{1,n})\psi_h(u_{1,n})\psi_{2h}(u_{1,n})$ a.e. in Ω_1. It follows that

$$\begin{aligned}
&\int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{1,n})\varphi_1 \, d\sigma \\
&= \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{1,n})\psi_{2h}(u_{1,n})\psi_h(u_{1,n})\varphi_1 \, d\sigma \\
&= \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{1,n})\psi_{2h}(u_{1,n})(\psi_h(u_{1,n}) - \psi_h(u_{2,n}))\varphi_1 \, d\sigma \\
&\quad + \int_\Gamma g(x)|T_{4h}(u_{1,n}) - T_{2h}(u_{2,n})|^{p-2}(T_{4h}(u_{1,n}) - T_{2h}(u_{2,n}))S(u_{1,n})\psi_{2h}(u_{1,n})\psi_h(u_{2,n})\varphi_1 d\sigma.
\end{aligned} \tag{112}$$

For the first term on the right-hand side of (112), thanks to (89) we have

$$\begin{aligned}
&\Big| \int_\Gamma g(x)|u_{1,n} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{1,n})\psi_{2h}(u_{1,n})(\psi_h(u_{1,n}) - \psi_h(u_{2,n}))\varphi_1 \, d\sigma \Big| \\
&\leq \frac{\|S(\cdot)\|_{L^\infty(I\!R)}\|\varphi_1\|_{L^\infty(\Omega_1)}}{h} \int_\Gamma g(x)|u_{n.1} - u_{2,n}|^{p-1}|T_h(u_{1,n}) - T_h(u_{2,n})| \, d\sigma \to 0 \ \text{ as } \ n, h \to \infty.
\end{aligned} \tag{113}$$

Concerning the second term on the right-hand side of (112), thanks to Lebesgue dominated convergence theorem and according to remark 3.1, we conclude that

$$\begin{aligned}
&\int_\Gamma g(x)|T_{4h}(u_{n.1}) - T_{2h}(u_{2,n})|^{p-2}(T_{4h}(u_{n.1}) - T_{2h}(u_{2,n}))S(u_{1,n})\psi_{2h}(u_{1,n})\psi_h(u_{2,n})\varphi_1 d\sigma \\
&\longrightarrow \int_\Gamma g(x)|T_{4h}(u_1) - T_{2h}(u_2)|^{p-2}(T_{4h}(u_1) - T_{2h}(u_2))S(u_1)\psi_{2h}(u_1)\psi_h(u_2)\varphi_1 d\sigma \quad \text{as} \quad n \to \infty. \\
&= \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)S(u_1)\psi_{2h}(u_1)\psi_h(u_2)\varphi_1 \, d\sigma \\
&\longrightarrow \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)S(u_1)\varphi_1 \, d\sigma \qquad \text{as} \quad h \to \infty.
\end{aligned} \tag{114}$$

By combining (112) and (113) − (114), we obtain

$$\begin{aligned}
&\int_\Gamma g(x)|u_{n.1} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{1,n})\varphi_1 \, d\sigma \\
&\qquad \longrightarrow \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)S(u_1)\varphi_1 \, d\sigma \quad \text{as} \quad n, h \to \infty.
\end{aligned} \tag{115}$$

Similarly, we show that

$$\begin{aligned}
&\int_\Gamma g(x)|u_{n.1} - u_{2,n}|^{p-2}(u_{n.1} - u_{2,n})S(u_{2,n})\varphi_2 \, d\sigma \\
&\qquad \longrightarrow \int_\Gamma g(x)|u_1 - u_2|^{p-2}(u_1 - u_2)S(u_2)\varphi_2 \, d\sigma \quad \text{as} \quad n, h \to \infty.
\end{aligned} \tag{116}$$

Concerning the term on the right-hand side of (109), thanks to (82) we have $f_n(x) \to f(x)$ strongly in $L^1(\Omega_i)$, and since $S(u_{i,n})\varphi_i \rightharpoonup S(u_i)\varphi_i$ weak$-*$ in $L^\infty(\Omega_i)$ for $i = 1, 2$, it follows that

$$\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} f_n(x)S(u_{i,n})\varphi_i \, dx = \sum_{i=1}^{2} \int_{\Omega_i} f(x)S(u_i)\varphi_i \, dx. \tag{117}$$

Moreover, we have $\dfrac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q}$ tends to $\dfrac{|u_i|^{q-2}u_i}{|x|^q}$ strongly in $L^1(\Omega_i)$ for $i = 1, 2$, then

$$\lim_{n\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} \frac{|T_n(u_{i,n})|^{q-2}T_n(u_{i,n})}{(|x| + \frac{1}{n})^q} S(u_{i,n})\varphi_i \, dx = \sum_{i=1}^{2} \int_{\Omega_i} \frac{|u_i|^{q-2}u_i}{|x|^q} S(u_i)\varphi_i \, dx. \tag{118}$$

By passing to the limit as n tends to infinity in (109), and using the results established in (111), and (115) – (118), we deduce that

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega_i} a(x,u_i,\nabla u_i)\cdot(S'(u_i)\varphi_i\nabla u_i + S(u_i)\nabla\varphi_i)\,dx\\ &\quad+\int_{\Gamma} g(x)|u_1-u_2|^{p-2}(u_1-u_2)(S(u_1)\varphi_1 - S(u_2)\varphi_2)\,d\sigma\\ &=\sum_{i=1}^{2}\int_{\Omega_i} f(x)S(u_i)\varphi_i\,dx + \sum_{i=1}^{2}\int_{\Omega_i}\frac{|u_i|^{q-2}u_i}{|x|^q}S(u_i)\varphi_i\,dx.\end{aligned} \tag{119}$$

which complete the proof of the Theorem 3.1.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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