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GENERALIZED UNCERTAINTY PRINCIPLE FOR MODIFIED OFFSET FRACTIONAL STOCKWELL TRANSFORM AND APPLICATION TO LFM SIGNAL

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Abstract. In this paper, the one-dimensional modified offset fractional Stockwell transform (MOFST) is introduced and investigated. Several fundamental properties of the proposed transform, including orthogonality relations and inversion formulas, are established. Furthermore, the relationship between the one-dimensional MOFST and the Fourier transform is derived. Based on these results, various uncertainty inequalities associated with the proposed transform are obtained, including Heisenberg-type uncertainty inequalities and logarithmic uncertainty inequalities. Finally, an application of the proposed transform to linear frequency modulated (LFM) signals is presented to illustrate its time-frequency localization capability.

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1. Introduction

Time-frequency analysis plays an important role in the study of non-stationary signals. Among various techniques, the Stockwell transform, introduced in [1], combines features of the windowed Fourier transform and the wavelet transform, providing frequency-dependent resolution while preserving phase information. Its mathematical properties, including spectral characterization and inversion formulas, have been extensively studied [2–5].

Due to its effectiveness in signal representation, the Stockwell transform has found applications in signal processing, power quality analysis, biomedical signal analysis, and ground penetrating radar (GPR) data interpretation [6–10]. Several variants have also been developed, including the β-Stockwell transform, modified Stockwell transform, fractional Stockwell transform, and generalized Stockwell

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transforms [11–17], extending its applicability in time-frequency analysis. In addition, the offset fractional Fourier transform (OFrFT) has attracted increasing attention due to its rich mathematical structure and uncertainty properties [18,19].

Uncertainty principles are fundamental tools in harmonic analysis and signal processing, describing the limitations of simultaneous localization in the time and frequency domains. In recent years, uncertainty principles associated with Stockwell-type transforms have attracted considerable attention. For example, uncertainty inequalities for the non-isotropic angular Stockwell transform were established in [20,21]. More recently, uncertainty principles for a generalized Stockwell transform in the offset fractional Fourier transform setting were investigated in [22]. Although several theoretical properties of the modified Stockwell transform have been studied [12–14], uncertainty principles associated with modified offset fractional Stockwell-type transforms remain relatively unexplored.

Motivated by these observations, this paper introduces the one-dimensional modified offset fractional Stockwell transform (MOFST) and investigates its fundamental properties. In particular, orthogonality relations, inversion formulas, and several uncertainty inequalities, including Heisenberg-type and logarithmic uncertainty principles, are established. An application to linear frequency modulated (LFM) signals is also presented to illustrate the localization capability of the proposed transform.

The remainder of this paper is organized as follows. Section 2 presents the basic definitions and preliminary results. Section 3 introduces the one-dimensional modified offset fractional Stockwell transform (MOFST). The fundamental properties of the proposed transform are established in Section 4. Section 5 is devoted to uncertainty principles associated with the one-dimensional modified offset fractional Stockwell transform. An application of the proposed transform to linear frequency modulated (LFM) signals is presented in Section 6. Finally, concluding remarks are given in Section 7.

2. Preliminaries

This part introduces the fractional Fourier transform (FrFT), a generalization of the classical Fourier transform, together with some of its fundamental properties [23–25]. We also establish a direct connection between the FrFT and the classical Fourier transform.

Definition 1. *The FT for the signal $f \in L^2(\mathbb{R})$ is given by*

$$\mathcal{F}\{f\}(\varphi) = \int_{\mathbb{R}} f(z) e^{-i\varphi t}\, dz. \tag{1}$$

Definition 2. *The FrFT for the function $f \in L^2(\mathbb{R})$ with the order β is defined by*

$$\mathcal{F}^{\beta}\{f\}(\varphi) = \int_{\mathbb{R}} f(z) C^{\beta}(z,\varphi)\, dz, \tag{2}$$

where

$$C^{\beta}(z,\varphi) = A_{\beta} e^{i\frac{(z^2+\varphi^2)}{2}\cot\beta - iz\varphi\csc\beta}, \qquad \beta \neq n\pi,$$

$$A_\beta = \frac{e^{i\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}}{\sqrt{2\pi\sin\beta}} = \sqrt{\frac{1-i\cot\beta}{2\pi}}.$$

Remark 1. *By selecting $\beta = \frac{\pi}{2}$, equation (2) reduces to equation (1).*

Definition 3. *The inverse transform of the FrFT is computed by the formula*

$$f(z) = \int_{\mathbb{R}} \mathcal{F}^{\beta}\{f\}(\varphi)\,\overline{C^{\beta}(z,\varphi)}\,d\varphi. \tag{3}$$

where "overline" denotes the conjugation.

In what follows, we introduce the definition of the OFrFT. We recall its useful properties that will be needed in the later parts of this paper.

Definition 4. *For f in $L^1(\mathbb{R})$. The OFrFT of function f is defined as*

$$\mathcal{F}^{(\beta,a,b)}\{f\}\{\varphi\} = \int_{\mathbb{R}} f(z)C_{(\beta,a,b)}(z,\varphi)\,dz, \tag{4}$$

where $C_{(\beta,a,b)}(z,\varphi)$ is

$$C_{(\beta,a,b)}(z,\varphi) = A_\beta e^{\frac{i}{2}\left((\varphi^2+z^2+a^2)\cot\beta+2z(a-\varphi)\csc\beta+2\varphi(b-a\cot\beta)\right)}. \tag{5}$$

Here, (β,a,b) are real numbers and $\beta \neq n\pi$, $n\in\mathbb{Z}$.

It is easy to verify the basic relation between the FT and the OFrFT for $f\in L^2(\mathbb{R})$ is

$$\mathcal{F}\{\hat{f}_\beta\}(\varphi\csc\beta) = \frac{1}{A_\beta}\mathcal{F}^{(\beta,a,b)}\{f\}(\varphi)e^{\frac{i}{2}\left((\varphi^2+a^2)\beta+2\varphi(b-a\cot\beta)\right)}. \tag{6}$$

where

$$f_\beta(z) = f(z)e^{\frac{i}{2}\left(z^2\cot\beta+2za\csc\beta\right)}. \tag{7}$$

The $f(z)$ can obtained using the inverse of the OFrFT as follows

Theorem 2.1. *For every f, $\mathcal{F}^{(\beta,a,b)}\{f\}$ belong to $L^2(\mathbb{R})$ the inverse of the OFrFT is described through*

$$f(z) = \int_{\mathbb{R}} \mathcal{F}^{(\beta,a,b)}\{f\}(\varphi)\,\overline{C_{(\beta,a,b)}(z,\varphi)}\,d\varphi. \tag{8}$$

Lemma 2.2. *For every $g,h\in L^2(\mathbb{R})$, then*

$$\overline{\langle g,h\rangle} = \langle \mathcal{F}^{(\beta,a,b)}\{g\},\mathcal{F}^{(\beta,a,b)}\{h\}\rangle. \tag{9}$$

3. One-Dimensional Modified offset fractional Stockwell Transform and Properties

In this section, we initiate by giving the definition of the one-dimensional modified offset fractional Stockwell transform and provide the essential properties.We also make a basic relation between the one-dimensional modified Stockwell transform and the Fourier transform.

Definition 5. *Let f be a signal in $L^2(\mathbb{R})$ and let ψ be a non-zero window function in $L^2(\mathbb{R})$. For $s \in [1, \infty)$, the modified offset fractional Stockwell transform $\mathcal{G}_{\psi}^{\beta,a,b}$ of the signal f with respect to the window ψ is given by*

$$\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta, \mu) = \frac{1}{\sqrt{2\pi}} \mu^{\frac{1}{s}} \int_{\mathbb{R}} f(z) \overline{\psi(\mu(z-\vartheta))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz. \tag{10}$$

for all $\vartheta \in \mathbb{R}$ and $s \in \mathbb{R} - \{0\}$.

Definition 6. [26] *The function f is l-Hölder continuous at $t_0 \in \mathbb{R}$ and denoted by $f \in C^l(t_0)$. If there exists a constant C such that*

$$|f(z) - f(t_0)| \leq C|z - t_0|^l, \qquad l \in (0, 1) \tag{11}$$

for all z in a neighborhood of t_0.

Based on Definition 5, we get the following result.

Theorem 3.1. *Assume that*

$$\int_{\mathbb{R}} |\psi(z)| \, dz = 0, \tag{12}$$

and

$$\int_{\mathbb{R}} |\psi(z)|(1 + |z|) \, dz < \infty. \tag{13}$$

For any $f \in C^l(t_0)$ with $l \in (0, 1)$, it holds

$$\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta, \mu)\right| \leq \frac{\sin\beta}{2\pi} C_1 |\mu|^{\frac{1}{s}-l-1}. \tag{14}$$

With

$$C_1 = C \int_{\mathbb{R}} |z|^l |\psi(z)| dz. \tag{15}$$

Proof. A direct calculation yields

$$\begin{aligned}
\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\varphi, \mu)\right| &\leq \left|\frac{A_\beta}{\sqrt{2\pi}} |\mu|^{\frac{1}{s}} \int_{\mathbb{R}} f(z) \overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz\right| \\
&= \frac{\sin\beta}{2\pi} |\mu|^{\frac{1}{s}} \left|\int_{\mathbb{R}} (f(z) - f(t_0) + f(t_0)) \overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz\right| \\
&\leq \frac{\sin\beta}{2\pi} |\mu|^{\frac{1}{s}} \left|\int_{\mathbb{R}} (f(z) - f(t_0)) \overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz\right| \\
&\quad + \frac{\sin\beta}{2\pi} |\mu|^{\frac{1}{s}} \left|\int_{\mathbb{R}} f(t_0) \overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz\right|.
\end{aligned} \tag{16}$$

On the other hand, we have

$$\left|\int_{\mathbb{R}} f(t_0) \overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \, dz\right| \leq \frac{|f(t_0)|}{\mu} \int_{\mathbb{R}} |\overline{\psi(u)}| \, du = 0. \tag{17}$$

Now equation (17) will lead to

$$\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\right| \leq \frac{C_1}{2\pi}|\mu|^{\frac{1}{s}} \int_{\mathbb{R}} \left| f(z) - f(t_0)\overline{\psi(\mu(z-t_0))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \right| dz$$
$$= \frac{C_1}{2\pi}|\mu|^{\frac{1}{s}} \int_{\mathbb{R}} |(f(z)-f(t_0))| \left|\overline{\psi(\mu(z-t_0))}\right| dz \tag{18}$$

Hence,

$$\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\right| \leq \frac{C_1}{2\pi}|\mu|^{\frac{1}{s}} C \int_{\mathbb{R}} |z-t_0|^l \left|\overline{\psi(\mu(z-t_0))}\right| dz$$
$$\leq \frac{C_1}{2\pi}|\mu|^{\frac{1}{s}} \frac{C}{|\mu|^l} \int_{\mathbb{R}} |\mu(z-t_0)|^l |\psi(\mu(z-t_0))| dz. \tag{19}$$

Choosing

$$C_1 = C\int_{\mathbb{R}} |z|^l |\psi(z)| dz,$$

the above equation becomes

$$\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\right| \leq \frac{C_1}{2\pi}|\mu|^{\frac{1}{s}-l-1}, \tag{20}$$

which completes the proof. □

4. Main Properties

The next results are helpful in deriving the inversion formula associated with the onedimensional modified offset fractional Stockwell transform.

From (10) we find that

$$\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu) = \frac{1}{\sqrt{2\pi}}\mu^{\frac{1}{s}} \int_{\mathbb{R}} f(z)\overline{\psi(\mu(z-\vartheta))} e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} dz$$
$$= \int_{\mathbb{R}} f(z)\overline{\psi_{\beta,a,b}^{s,\vartheta,\mu}} dz$$
$$= \langle f, \psi_{\beta,a,b}^{s,\vartheta,\mu}\rangle. \tag{21}$$

Where

$$\psi_{\beta,a,b}^{s,\vartheta,\mu} = \frac{\mu^{\frac{1}{s}}}{\sqrt{2\pi}}\psi(\mu(z-\vartheta))e^{-\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta}. \tag{22}$$

Lemma 4.1. *For all $\varphi \in \mathbb{R}$ and $\psi \in L^1(\mathbb{R})$, one has*

$$\mathcal{F}^{\beta,a,b}\{\psi_{\beta,a,b}^{s,\vartheta,\mu}\}(\varphi) = \frac{A_\beta \mu^{\frac{1}{s}-1}}{\sqrt{2\pi}} e^{i\vartheta(a-\varphi)\csc\beta} e^{\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}$$
$$\times e^{\frac{i}{2}\left((\mu^2+a^2-(\frac{a}{\mu})^2-(\frac{\varphi}{\mu})^2)\cot\beta+2\varphi(b-a\cot\beta)-\frac{2\varphi}{\mu}(b-a\cot\beta)\right)}$$
$$\times \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right). \tag{23}$$

Proof. Due to equation (10), we have

$$\begin{aligned}\mathcal{F}^{\beta,a,b}\{\psi^{s,\vartheta,\mu}_{\beta,a,b}\}(\varphi) &= \int_{\mathbb{R}} \psi^{s,\vartheta,\mu}_{\beta,a,b}(z) K_{(\beta,a,b)}(z,\varphi)\,dz \\ &= A_\beta \int_{\mathbb{R}} \frac{\mu^{\frac{1}{s}}}{\sqrt{2\pi}} \psi(\mu(z-\vartheta)) e^{-\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta} \\ &\quad \times e^{\frac{i}{2}(a^2+\varphi^2+z^2)\cot\beta + 2z(a-\varphi)\csc\beta + 2\varphi(b-a\cot\beta)}\,dz. \end{aligned} \tag{24}$$

Substituting $u = \mu(z-\vartheta)$, $\dfrac{u}{\mu} + \vartheta = z$ into equation (24) yields

$$\begin{aligned}\mathcal{F}^{\beta,a,b}\{\psi^{s,\vartheta,\mu}_{\beta,a,b}\}(\varphi) &= A_\beta \int_{\mathbb{R}} \frac{\mu^{\frac{1}{s}-1}}{\sqrt{2\pi}} \psi(u) e^{\frac{i}{2}(u^2+a^2+b^2)\cot\beta} \\ &\quad \times e^{\frac{i}{2}\left((a^2+\varphi^2)\cot\beta + 2\left(\frac{u}{\mu}+\vartheta\right)(a-\varphi)\csc\beta + 2\varphi(b-a\cot\beta)\right)}\,du \\ &= A_\beta e^{i\vartheta(a-\varphi)\csc\beta} e^{\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta} \\ &\quad \times \int_{\mathbb{R}} \frac{\mu^{\frac{1}{s}-1}}{\sqrt{2\pi}} \psi(u) e^{\frac{i}{2}\left((a^2+\varphi^2)\cot\beta + 2u\left(\frac{a}{\mu}-\frac{\varphi}{\mu}\right)\csc\beta + 2\varphi(b-a\cot\beta)\right)}\,du \\ &= A_\beta e^{i\vartheta(a-\varphi)\csc\beta} e^{\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta} e^{\frac{i}{2}\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta + 2\varphi(b-a\cot\beta) - \frac{2\varphi}{\mu}(b-a\cot\beta)} \\ &\quad \times \int_{\mathbb{R}} \frac{\mu^{\frac{1}{s}-1}}{\sqrt{2\pi}} \psi(u) e^{-\frac{i}{2}u^2\cot\beta} e^{\frac{i}{2}\left(\left(u^2+\left(\frac{a}{\mu}\right)^2+\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta + 2u\left(\frac{a}{\mu}-\frac{\varphi}{\mu}\right)\csc\beta + \frac{2\varphi}{\mu}(b-a\cot\beta)\right)}\,du. \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned}\mathcal{F}^{\beta,a,b}\{\psi^{s,\vartheta,\mu}_{\beta,a,b}\}(\varphi) &= \frac{A_\beta \mu^{\frac{1}{s}-1}}{\sqrt{2\pi}} e^{i\vartheta(a-\varphi)\csc\beta} e^{\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta} \\ &\quad \times e^{\frac{i}{2}\left(\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta + 2\varphi(b-a\cot\beta) - \frac{2\varphi}{\mu}(b-a\cot\beta)\right)} \\ &\quad \times \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u) e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right). \end{aligned} \tag{26}$$

Which finishes the proof. □

Proposition 1. *For any $f, \psi \in L^2(\mathbb{R})$, we have*

$$\begin{aligned}\mathcal{G}^{\beta,a,b}_{\psi}\{h\}(\vartheta,\mu) &= \left(\mathcal{F}^{\beta,a,b}\right)^{-1}\Bigg\{\mathcal{F}^{\beta,a,b}\{h\}(\varphi) e^{-\frac{i}{2}\left((\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2)\cot\beta - \frac{2\varphi}{\mu}(b-a\cot\beta)\right)} \\ &\quad \times \overline{\frac{1}{\mu^{\frac{1}{s}-1}} \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u) e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)}\Bigg\}(b). \end{aligned} \tag{27}$$

Proof. Applying equation (9) and equation (10) yields

$$\begin{aligned}\mathcal{G}^{\beta,a,b}_{\psi}\{h\}(\vartheta,\mu) &= \langle h, \psi^{s,\vartheta,\mu}_{\beta,a,b}\rangle \\ &= \left\langle \mathcal{F}^{\beta,a,b}\{h\}, \mathcal{F}^{\beta,a,b}\{\psi^{s,\vartheta,\mu}_{\beta,a,b}\}\right\rangle \end{aligned}$$

$$= \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h\}(\varphi)\overline{\mathcal{F}^{\beta,a,b}\{\psi^{s,\vartheta,\mu}_{\beta,a,b}\}(\varphi)}\,d\varphi. \tag{28}$$

Based on equation (23), the above equation (28) can be rewritten as

$$\begin{aligned}\mathcal{G}^{\beta,a,b}_{\psi}\{h\}(\vartheta,\mu) = \frac{\overline{A_\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} &\mathcal{F}^{\beta,a,b}\{h\}(\varphi)e^{-i\vartheta(a-\varphi)\csc\beta}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta} \\ &\times e^{-\frac{i}{2}\left(\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta+2\varphi(b-a\cot\beta)-\frac{2\varphi}{\mu}(b-a\cot\beta)\right)} \\ &\times \overline{\frac{1}{\mu^{\frac{1}{s}-1}}\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)}\,d\varphi.\end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}\mathcal{G}^{\beta,a,b}_{\psi}\{h\}(\vartheta,\mu) &= \overline{A_\beta}\int_{\mathbb{R}}\Bigg(\mathcal{F}^{\beta,a,b}\{h\}(\varphi)e^{-\frac{i}{2}\left(\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta-\frac{2\varphi}{\mu}(b-a\cot\beta)\right)} \\ &\quad\times \overline{\frac{1}{\mu^{\frac{1}{s}-1}}\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)}\Bigg) \\ &\quad\times e^{-\frac{i}{2}(a^2+\varphi^2+b^2)\cot\beta+2b(a-\varphi)\csc\beta+2\varphi(b-a\cot\beta)}\,d\varphi \\ &= (\mathcal{F}^{\beta,a,b})^{-1}\Bigg\{\mathcal{F}^{\beta,a,b}\{h\}(\varphi)e^{-\frac{i}{2}\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta-\frac{2\varphi}{\mu}(b-a\cot\beta)} \\ &\quad\times \overline{\frac{1}{\mu^{\frac{1}{s}-1}}\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)}\Bigg\}(b).\end{aligned} \tag{30}$$

Which finishes the proof. □

Definition 7. *The function $f \in L^2(\mathbb{R})$ is called an offset fractional wavelet if it satisfies*

$$0 < C^{\beta,a,b}_{\psi} = \int_{\mathbb{R}-\{0\}}\left|\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)\right|^2\frac{d\mu}{\mu} < \infty, \tag{31}$$

for almost every $\varphi \in \mathbb{R}$. $C^{\beta,a,b}_{\psi}$ is called the admissibility condition.

Lemma 4.2. *Suppose that C_ψ is the admissibility condition of the classical mother wavelet*

$$C_\psi = \int_{\mathbb{R}-\{0\}}\left|\mathcal{F}\{\psi\}\left(\frac{\varphi}{\mu}\csc\beta\right)\right|^2\frac{d\mu}{\mu}. \tag{32}$$

The relationship between (10) and (31) is given by

$$C_\psi = 2\pi\sin\beta\, C^{\beta,a,b}_{\psi} < \infty. \tag{33}$$

Proof. From Definition 6, we have

$$\begin{aligned}C^{\beta,a,b}_{\psi} &= \int_{\mathbb{R}-\{0\}}\left|\int_{\mathbb{R}} A_\beta\psi(u)e^{-\frac{i}{2}u^2\cot\beta}e^{\frac{i}{2}\left(\left(u^2+\left(\frac{a}{\mu}\right)^2+\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta+2u\left(\frac{a}{\mu}-\frac{\varphi}{\mu}\right)\csc\beta+\frac{2\varphi}{\mu}(b-a\cot\beta)\right)}du\right|^2\frac{d\mu}{\mu} \\ &= \int_{\mathbb{R}-\{0\}}\left|\int_{\mathbb{R}} A_\beta\psi(u)e^{-i\frac{\varphi}{\mu}u\csc\beta}\right|^2\left|e^{\frac{i}{2}\left(\left(u^2+\left(\frac{a}{\mu}\right)^2+\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta+\frac{2ua}{\mu}\csc\beta+\frac{2\varphi}{\mu}(b-a\cot\beta)\right)}du\right|^2\frac{d\mu}{\mu}\end{aligned}$$

$$= |A_\beta|^2 \int_{\mathbb{R}-\{0\}} \left| \int_{\mathbb{R}} \psi(u) e^{-i\frac{\varphi}{\mu} u \csc\beta} \, du \right|^2 \frac{d\mu}{\mu}$$
$$= \frac{1}{2\pi \sin\beta} \int_{\mathbb{R}-\{0\}} \left| \mathcal{F}\{\psi\} \left(\frac{\varphi}{\mu} \csc\beta \right) \right|^2 \frac{d\mu}{\mu}$$
$$= \frac{1}{2\pi \sin\beta} C_\psi. \tag{34}$$

This is the desired result. □

Note that the FT of equation (1) has the form

$$\mathcal{F}\{\psi_{a,b}\} \left(\frac{\varphi}{\mu} \right) = \sqrt{a}\, e^{-ib\eta}\, \mathcal{F}\{\psi\} \left(\frac{\varphi}{\mu} \right). \tag{35}$$

The next theorems are focused on the derivation of inversion theorem and orthogonality formula for modified offset fractional Stockwell transform. These results are proved by applying relation between the modified offset fractional Stockwell transform and the WT.

Theorem 4.3. *Assume that $\psi \in L^2(\mathbb{R})$ related the OFrFT satisfies equation (10). Then, for every $h \in L^2(\mathbb{R})$ we have*

$$\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta, \mu)\, \overline{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}(\vartheta, \mu)}\, \frac{d\vartheta\, d\mu}{\mu^{\frac{2}{s}-1}} = \frac{1}{2\pi \sin\beta} C_\psi \langle h_1, h_2 \rangle. \tag{36}$$

Proof. Using Parseval's formula for the Fourier transform in equation (9), we find that

$$\begin{aligned}
&\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta, \mu) \overline{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}(\vartheta, \mu)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} \\
&\quad = \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h_1\}\}(\varphi) \overline{\mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}\}(\varphi)} \frac{d\varphi d\mu}{\mu^{\frac{2}{s}-1}} \\
&\quad = \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h_1\}(\varphi) e^{-\frac{i}{2}\left((\mu^2 + a^2 - (\frac{a}{\mu})^2 - (\frac{\varphi}{\mu})^2) \cot\beta - \frac{2\varphi}{\mu}(b - a\cot\beta) \right)} \\
&\qquad \times \overline{\mu^{\frac{1}{s}-1} \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right)} \\
&\qquad \times \overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\varphi) e^{-\frac{i}{2}\left((\mu^2 + a^2 - (\frac{a}{\mu})^2 - (\frac{\varphi}{\mu})^2) \cot\beta - \frac{2\varphi}{\mu}(b - a\cot\beta) \right)}} \\
&\qquad \times \overline{\mu^{\frac{1}{s}-1} \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right)} \frac{d\varphi d\mu}{\mu^{\frac{2}{s}-1}}.
\end{aligned} \tag{37}$$

Above equation (37) can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta, \mu) \overline{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}(\vartheta, \mu)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} \\
&= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mu^{\frac{2}{s}-2} \mathcal{F}^{\beta,a,b}\{h_1\}(\varphi) \overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\varphi)} \\
&\times \overline{\mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right)} \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right) \frac{d\varphi d\mu}{\mu^{\frac{2}{s}-1}}
\end{aligned}$$

$$= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h_1\}(\varphi)\overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\varphi)} \left| \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right) \right|^2 \frac{d\varphi d\mu}{\mu}$$

$$= \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h_1\}(\varphi)\overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\varphi)} \left(\int_{\mathbb{R}-\{0\}} \left| \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right) \right|^2 \frac{d\mu}{\mu} \right) d\varphi$$

$$= C_\psi^{\beta,a,b} \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h_1\}(\varphi)\overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\varphi)}\, d\varphi$$

$$= \frac{1}{2\pi \sin\beta} C_\psi \int_{\mathbb{R}} \mathcal{F}^{\beta,a,b}\{h_1\}(\psi)\overline{\mathcal{F}^{\beta,a,b}\{h_2\}(\psi)} d\psi. \tag{38}$$

By virtue of equation (9) we have

$$\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\overline{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}(\vartheta,\mu)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} = \frac{1}{2\pi \sin\beta} C_\psi \langle h_1, h_2 \rangle. \tag{39}$$

□

Remark 2. *If $h_1 = h_2$, equation (36) above changes to*

$$\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \left| \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \right|^2 \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} = \frac{1}{2\pi \sin\beta} C_\psi \, \|h_1\|_{L^2(\mathbb{R})}. \tag{40}$$

The reconstruction of the function from the appropriate modified Stockwell transform is given by the following result.

Theorem 4.4 (Inversion formula)**.** *Let $\psi \in L^2(\mathbb{R})$ be a non-zero window function. Then for every $h \in L^2(\mathbb{R})$ we have*

$$h_1(z) = \frac{1}{C_\psi^{\beta,a,b}} \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\, \psi_{\beta,a,b}^{s,\vartheta,\mu} \overline{h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}}. \tag{41}$$

Proof. Applying Theorem 4.3 yields

$$\begin{aligned} C_\psi^{\beta,a,b} \langle h_1, h_2 \rangle &= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\overline{\mathcal{G}_\psi^{\beta,a,b}\{h_2\}(\vartheta,\mu)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} \\ &= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \overline{\int_{\mathbb{R}} h_2(z) \overline{\psi_{\beta,a,b}^{s,\vartheta,\mu}}(z)\, dz} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \psi_{\beta,a,b}^{s,\vartheta,\mu} \overline{h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}}\, dz \\ &= \left\langle \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \psi_{\beta,a,b}^{s,\vartheta,\mu} \overline{h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}},\ h_2(z) \right\rangle. \end{aligned} \tag{42}$$

Since $h \in L^2(\mathbb{R})$ is arbitrary, it follows that

$$C_\psi^{\beta,a,b}\, h_1(z) = \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \psi_{\beta,a,b}^{s,\vartheta,\mu} \overline{h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}}. \tag{43}$$

Or, equivalently,

$$h_1(z) = \frac{1}{C_\psi^{\beta,a,b}} \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \psi_{\beta,a,b}^{s,\vartheta,\mu} \overline{h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}}$$

$$= \frac{2\pi \sin\beta}{C_\psi} \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu) \overline{\psi_{\beta,a,b}^{s,\vartheta,\mu} h_2(z)} \frac{d\vartheta d\mu}{\mu^{\frac{2}{s}-1}}. \tag{44}$$

This is the required result. □

5. Uncertainty Principles for One-Dimensional Modified Stockwell Transform

In this section, we establish several uncertainty principles associated with the one-dimensional modified offset fractional Stockwell transform. These results describe the limitations on the simultaneous localization of a signal in the transform domain and provide theoretical insight into the time-frequency behavior of the proposed transform. In particular, we derive Heisenberg-type and logarithmic uncertainty inequalities.

5.1. **Heisenberg-Type Uncertainty Principle.** In this part, we derive an analogue of Heisenberg-type uncertainty principle for the one-dimensional modified offset fractional Stockwell transform. For this purpose, we need to introduce the following lemma.

Lemma 5.1. *For all $h, \psi \in L^2(\mathbb{R})$, one has*

$$\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi) \right|^2 d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}} = \frac{1}{2\pi \sin\beta} C_\psi \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^2 d\varphi. \tag{45}$$

Proof. In fact, we have

$$\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi) \right|^2 d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}} = \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi) \times \overline{\mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi)}\, d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{46}$$

The use of equation (3.26) gives

$$\begin{aligned}
&\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi) \right|^2 d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}} \\
&= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \mathcal{F}^{\beta,a,b}\{h\}(\varphi) e^{-\frac{i}{2}\left(\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta - \frac{2\varphi}{\mu}(b-a\cot\beta)\right)} \\
&\quad \times \overline{\mu^{\frac{1}{s}-1} \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)} \\
&\quad \times \overline{\mathcal{F}^{\beta,a,b}\{h\}(\varphi) e^{-\frac{i}{2}\left(\left(\mu^2+a^2-\left(\frac{a}{\mu}\right)^2-\left(\frac{\varphi}{\mu}\right)^2\right)\cot\beta - \frac{2\varphi}{\mu}(b-a\cot\beta)\right)}} \\
&\quad \times \overline{\mu^{\frac{1}{s}-1} \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}}\left(\frac{\varphi}{\mu}\right) d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}}.
\end{aligned} \tag{47}$$

We arrive at

$$\begin{aligned}
&\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}\}(\varphi) \right|^2 d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}} \\
&= \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^2 \left| \mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right) \right|^2 \frac{d\varphi d\mu}{\mu}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 \left(\int_{\mathbb{R}-\{0\}} \left|\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^2\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)\right|^2 \frac{d\mu}{\mu}\right) d\varphi \\
&= C_\psi^{\beta,a,b} \int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 d\varphi \\
&= \frac{1}{2\pi\sin\beta} C_\psi \int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 d\varphi. \qquad (48)
\end{aligned}
$$

This is the desired result. □

We are now in a position to derive the uncertainty principles for the one-dimensional modified Stockwell transform.

Theorem 5.2. *For any $h \in L^2(\mathbb{R})$, one has*

$$
\left\{\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 d\varphi\right\}^{\frac{1}{2}} \geq \frac{1}{2\sqrt{2}}\sqrt{\frac{\pi}{\sin\beta}}\sqrt{C_\psi}\,\|h\|_{L^2(\mathbb{R})}. \qquad (49)
$$

Proof. With the aid of Heisenberg-type uncertainty principle for the Fourier transform, we infer that

$$
\left\{\int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h_1\}\}(\varphi)\right|^2 d\varphi\right\}^{\frac{1}{2}} \geq \frac{\pi}{2}\int_{\mathbb{R}} \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta. \qquad (50)
$$

Integrating both sides of equation (50) with respect to the measure $\frac{d\mu}{\mu^{\frac{2}{s}-1}}$, we immediately obtain

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{2}} \\
&\quad\times \left\{\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{\mathcal{G}_\psi^{\beta,a,b}\{h_1\}\}(\varphi)\right|^2 d\varphi \frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{2}} \\
&\geq \frac{\pi}{2}\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}}. \qquad (51)
\end{aligned}
$$

Applying Lemma 5.1 results in

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^2 d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{2}} \times \left\{C_\psi^{\beta,a,b}\int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 d\varphi\right\}^{\frac{1}{2}} \\
&\geq \frac{\pi}{2} C_\psi^{\beta,a,b}\|h\|^2_{L^2(\mathbb{R})}. \qquad (52)
\end{aligned}
$$

By equation (33) we find that

$$
\left\{\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^2 \left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}} \varphi^2 \left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^2 d\varphi\right\}^{\frac{1}{2}}
$$

$$\geq \frac{1}{2\sqrt{2}}\sqrt{\frac{\pi}{\sin\beta}}\sqrt{C_\psi}\,\|h\|_{L^2(\mathbb{R})}. \tag{53}$$

Thus, the proof is complete. □

5.2. **Logarithmic Uncertainty Principle.** In this part, we explore an analogue of the logarithmic uncertainty principle for the one-dimensional modified offset fractional Stockwell transform. We obtain the following result.

Theorem 5.3. *For any $h \in L^2(\mathbb{R})$, it holds*

$$\left(\ln\pi + \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\frac{1}{2\pi\sin\beta}C_\psi\,\|h\|_{L^2(\mathbb{R})} \leq \frac{1}{2\pi\sin\beta}C_\psi\int_{\mathbb{R}}\ln|\varphi|\,\left|\mathcal{F}^{(\beta,m,n)}\{h\}(\varphi)\right|^2 d\varphi$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\ln|\vartheta|\,\left|\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^2 d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{54}$$

Here, $S(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$.

Proof. Based on the logarithmic uncertainty principle for the Fourier transform, we have

$$\left(\ln\pi + \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\int_{\mathbb{R}}|h(\vartheta)|^2\,d\vartheta \leq \int_{\mathbb{R}}\ln|\varphi|\,\left|\mathcal{F}^{(\beta,m,n)}\{h\}(\varphi)\right|^2 d\varphi$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}}\ln|\vartheta|\,|h(\vartheta)|^2\,d\vartheta. \tag{55}$$

Replacing $h(z)$ by $\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)$ into equation (55), we get

$$\left(\ln\pi + \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\int_{\mathbb{R}}\left|\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^2 d\vartheta \leq \int_{\mathbb{R}}\ln|\varphi|\,\left|\mathcal{F}^{(\beta,m,n)}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\}(\varphi)\right|^2 d\varphi$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}}\ln|\vartheta|\,\left|\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^2 d\vartheta. \tag{56}$$

Integrating both sides of equation (56) with respect to the measure $\dfrac{d\mu}{\mu^{\frac{2}{s}-1}}$ results in

$$\left(\ln\pi + \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$
$$\leq \int_{\mathbb{R}-\{0\}}\left\{\int_{\mathbb{R}}\ln|\varphi|\,\left|\mathcal{F}^{(\beta,m,n)}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\}(\varphi)\right|^2 d\varphi\right.$$
$$\left.+\frac{1}{2\pi}\int_{\mathbb{R}}\ln|\vartheta|\,\left|\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^2 d\vartheta\right\}\frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{57}$$

Or, equivalently

$$\left(\ln\pi + \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\left|\mathcal{G}_\psi^{\beta,a,b}\{h_1\}(\vartheta,\mu)\right|^2 d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$
$$\leq \int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\ln|\varphi|\,\left|\mathcal{F}^{(\beta,m,n)}\{\mathcal{G}_\psi^{\beta,a,b}\{h\}(\vartheta,\mu)\}(\varphi)\right|^2 d\varphi\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$

$$+\frac{1}{2\pi}\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\ln|\vartheta|\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^{2}d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{58}$$

Implementing Lemma 5.1 yields

$$\left(\ln\pi+\frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)\frac{1}{2\pi\sin\beta}C_{\psi}\|h\|_{L^{2}(\mathbb{R})}\leq\frac{1}{2\pi\sin\beta}C_{\psi}\int_{\mathbb{R}}\ln|\varphi|\left|\mathcal{F}^{(\beta,m,n)}\{h\}(\varphi)\right|^{2}d\varphi$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\ln|\vartheta|\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\right|^{2}d\vartheta\,\frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{59}$$

which complete the proof. □

The non-trivial generalization of Theorem 5.2 above is demonstrated by the following result.

Theorem 5.4. *Under the assumptions as in Theorem 5.2, one has*

$$\left\{\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\vartheta^{p}\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}(\vartheta,\mu)\right|^{p}d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^{p}d\varphi\right\}^{\frac{1}{p}}$$
$$\geq\frac{\pi}{2}\left(\mu^{\frac{2}{s}-2+p-\frac{p}{s}}\right)^{\frac{1}{p}}\sqrt{\frac{C_{\psi}}{2\pi\sin\beta}}\|h\|_{L^{2}(\mathbb{R})}. \tag{60}$$

Proof. Based on the uncertainty principles for the Fourier transform we have

$$\left\{\int_{\mathbb{R}}\vartheta^{p}\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}(\vartheta,\mu)\right|^{p}d\vartheta\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}\}(\varphi)\right|^{p}d\varphi\right\}^{\frac{1}{p}}$$
$$\geq\frac{\pi}{2}\int_{\mathbb{R}}\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}(\vartheta,\mu)\right|^{2}d\vartheta. \tag{61}$$

Integrating both sides of equation (61) with respect to the measure $\frac{d\mu}{\mu^{\frac{2}{s}-1}}$, we immediately obtain

$$\left\{\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\vartheta^{p}\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}(\vartheta,\mu)\right|^{p}d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{p}}\times\left\{\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}\}(\varphi)\right|^{p}d\varphi\frac{d\mu}{\mu^{\frac{2}{s}-1}}\right\}^{\frac{1}{p}}$$
$$\geq\frac{\pi}{2}\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\left|\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}(\vartheta,\mu)\right|^{2}d\vartheta\frac{d\mu}{\mu^{\frac{2}{s}-1}}. \tag{62}$$

By equation (27) we obtain

$$\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{\mathcal{G}_{\psi}^{\beta,a,b}\{h_{1}\}\}(\varphi)\right|^{p}d\varphi\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$
$$=\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\overline{\mu^{\frac{1}{s}-1}\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^{2}\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)}\right|^{p}d\varphi\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$
$$=\int_{\mathbb{R}-\{0\}}\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^{p}\left(|\mu|^{2\left(\frac{1}{s}-1\right)}\left|\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^{2}\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)\right|^{2}\right)^{\frac{p}{2}}d\varphi\,\frac{d\mu}{\mu^{\frac{2}{s}-1}}$$
$$=\int_{\mathbb{R}}\varphi^{p}\left|\mathcal{F}^{\beta,a,b}\{h\}(\varphi)\right|^{p}\left(\int_{\mathbb{R}-\{0\}}\left(|\mu|^{2\left(\frac{1}{s}-1\right)}\left|\mathcal{F}^{\beta,\frac{a}{\mu},b}\left\{\psi(u)e^{-\frac{i}{2}u^{2}\cot\beta}\right\}\left(\frac{\varphi}{\mu}\right)\right|^{2}\right)^{\frac{p}{2}}\frac{d\mu}{\mu^{\frac{2}{s}-1}}\right)d\varphi$$

$$= \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^p \left(\int_{\mathbb{R}-\{0\}} \left(\left| \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right) \right|^2 \right)^{\frac{p}{2}} \frac{d\mu}{\mu^{\frac{2}{s}+1+p-\frac{p}{s}}} \right) d\varphi. \tag{63}$$

Further we get

$$\begin{aligned}
&\int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{\mathcal{G}_{\psi}^{\beta,a,b}\{h_1\}\}(\varphi) \right|^p d\varphi \, \frac{d\mu}{\mu^{\frac{2}{s}-1}} \\
&\geq \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^p \left(\int_{\mathbb{R}-\{0\}} \left| \mathcal{F}^{\beta,\frac{a}{\mu},b} \left\{ \psi(u) e^{-\frac{i}{2}u^2 \cot\beta} \right\} \left(\frac{\varphi}{\mu} \right) \right|^2 \frac{d\mu}{\mu} \right)^{\frac{p}{2}} \frac{1}{\left(\mu^{\frac{2}{s}-2+p-\frac{p}{s}} \right)^{\frac{p}{2}}} d\varphi \\
&\geq \frac{\left(C_{\psi}^{\beta,a,b} \right)^{\frac{p}{2}}}{\left(\mu^{\frac{2}{s}-2+p-\frac{p}{s}} \right)^{\frac{p}{2}}} \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^p d\varphi.
\end{aligned} \tag{64}$$

Substituting equation (64) and equation (40) into equation (62) gives

$$\begin{aligned}
\left\{ \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^p \left| \mathcal{G}_{\psi}^{\beta,a,b}\{h_1\}(\vartheta,\mu) \right|^p d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}} \right\}^{\frac{1}{p}} \times \left\{ \frac{\left(C_{\psi}^{\beta,a,b} \right)^{\frac{p}{2}}}{\left(\mu^{\frac{2}{s}-2+p-\frac{p}{s}} \right)^{\frac{p}{2}}} \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^p d\varphi \right\}^{\frac{1}{p}} \\
\geq \frac{\pi}{2} C_{\psi}^{\beta,a,b} \|h\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{65}$$

Hence,

$$\begin{aligned}
\left\{ \int_{\mathbb{R}-\{0\}} \int_{\mathbb{R}} \vartheta^p \left| \mathcal{G}_{\psi}^{\beta,a,b}\{h_1\}(\vartheta,\mu) \right|^p d\vartheta \frac{d\mu}{\mu^{\frac{2}{s}-1}} \right\}^{\frac{1}{p}} \times \left\{ \int_{\mathbb{R}} \varphi^p \left| \mathcal{F}^{\beta,a,b}\{h\}(\varphi) \right|^p d\varphi \right\}^{\frac{1}{p}} \\
\geq \frac{\pi}{2} \left(\mu^{\frac{2}{s}-2+p-\frac{p}{s}} \right)^{\frac{1}{2}} \sqrt{\frac{C_{\psi}}{2\pi \sin\beta}} \|h\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{66}$$

Thus, we obtained the desired result. □

6. Application of 1-D MOFST to linear Frequency Modulated (LFM) Signal

Motivated by the increasing interest in time-frequency analysis and the development of fractional Stockwell-type transforms [12,13,15], as well as the widespread application of linear frequency modulation (LFM) signals in radar and communication systems due to their low probability of interception, high average power, and high range resolution capabilities [23,24], we investigate the application of the modified offset fractional Stockwell transform to LFM signals. In particular, we establish a generalized uncertainty principle for the modified offset fractional Stockwell transform and demonstrate its applicability in the analysis of LFM signals.

Suppose that a single-component LFM signal with amplitude A, initial frequency μ, and frequency rate λ is given by

$$h(z) = Ae^{i(\mu_A z + \lambda z^2)}, \qquad -\frac{T}{2} \leq z \leq \frac{T}{2},$$

and

$$\psi(z) = e^{-\frac{z^2}{4}},$$

respectively.

Let us calculate the single-component LFM signal in the 1-D MOFST domain. Direct computation yields

$$\begin{aligned}
\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu) &= \frac{1}{\sqrt{2\pi}}\mu^{\frac{1}{s}}\int_{\mathbb{R}} h(z)\overline{\psi\big(\mu(z-\vartheta)\big)}e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta}\,dz \\
&= \frac{1}{\sqrt{2\pi}}\mu^{\frac{1}{s}}\int_{-\frac{T}{2}}^{\frac{T}{2}} Ae^{i(\mu_A z+\lambda z^2)}e^{-\frac{\mu^2(z-\vartheta)^2}{4}}e^{\frac{i}{2}(z^2-\mu^2-a^2-b^2)\cot\beta}\,dz \\
&= \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(\mu_A z+\lambda z^2)}e^{-\frac{\mu^2(z-\vartheta)^2}{4}}e^{\frac{i}{2}z^2\cot\beta}\,dz. \qquad (67)
\end{aligned}$$

Above equation (67) can be rewritten as

$$\begin{aligned}
\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu) &= \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)z^2+\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)z}\,dz \\
&= \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)\left(z^2-\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)}{\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}z\right)}\,dz \\
&= \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}} \\
&\quad \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)\left[\left(z-\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)^2-\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)^2}{4\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)^2}\right]}\,dz. \qquad (68)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu) &= \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}+\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)^2}{4\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}} \\
&\quad\times\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)\left(z-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)^2}\,dz.
\end{aligned}$$

By the change of variable

$$x = \left(z-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right),$$

we get

$$\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu) = \frac{A}{\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}+\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)^2}{4\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}}$$

$$\times \int_{-\frac{T}{2}-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}}^{\frac{T}{2}-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}} e^{-\left(\sqrt{\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda}\,x\right)^2} dx.$$

Hence,

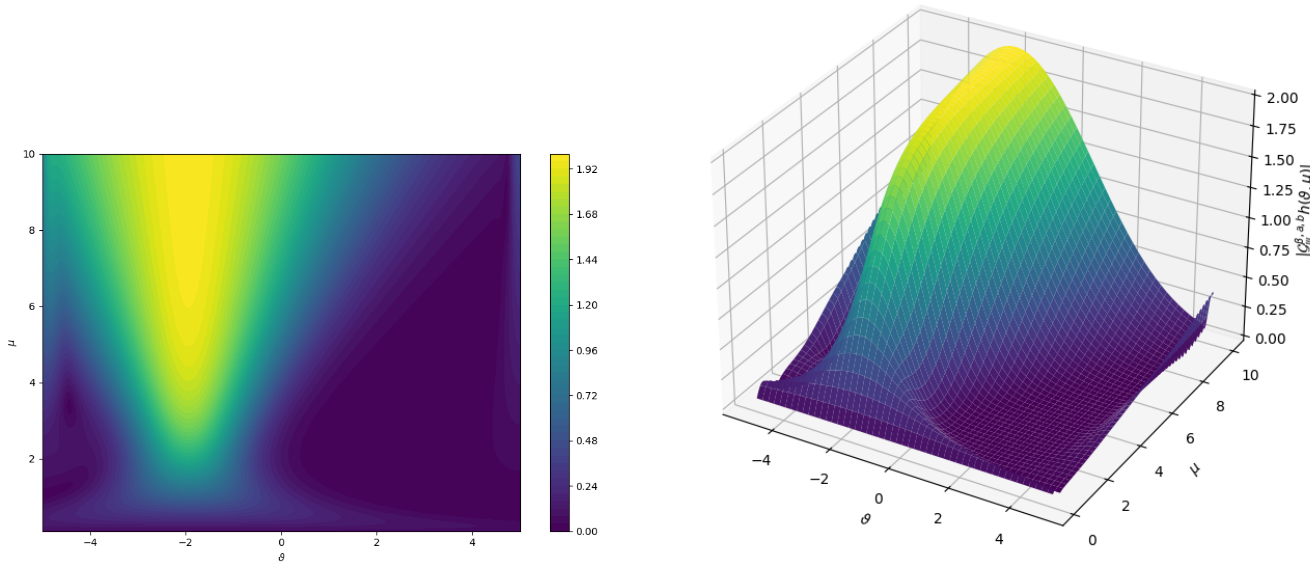
$$\begin{aligned}
&\mathcal{G}_{\psi}^{\beta,a,b}\{h\}(\vartheta,\mu)\\
&= \frac{A}{\sqrt{\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\sqrt{2\pi}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}\left(\mu^2+a^2+b^2\right)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}+\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)^2}{4\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}}\\
&\quad\times\left(\int_0^{\sqrt{\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda}\left(\frac{T}{2}-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)} e^{-u^2}du-\int_0^{-\sqrt{\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda}\left(\frac{T}{2}+\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)} e^{-u^2}du\right)\\
&= \frac{A}{2\sqrt{\left(\frac{\mu^2}{4}-i\cot\beta-i2\lambda\right)}}\mu^{\frac{1}{s}}e^{-\frac{i}{2}(\mu^2+a^2+b^2)\cot\beta}e^{-\frac{\mu^2\vartheta^2}{4}+\frac{\left(i\mu_A+\frac{\mu^2\vartheta}{2}\right)^2}{4\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}}\\
&\quad\times\left(erf\left(\sqrt{\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda}\left(\frac{T}{2}-\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)\right)\right.\\
&\quad\left.-erf\left(-\sqrt{\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda}\left(\frac{T}{2}+\frac{i\mu_A+\frac{\mu^2\vartheta}{2}}{2\left(\frac{\mu^2}{4}-\frac{i}{2}\cot\beta-i\lambda\right)}\right)\right)\right).
\end{aligned} \tag{69}$$

The magnitude distribution of the MOFST given by (69) is numerically evaluated and illustrated in Figure 1.

Figure 1 illustrates the magnitude distribution of the proposed MOFST associated with the LFM signal. The contour plot in Figure 1(a) and the corresponding surface plot in Figure 1(b) show that the transform energy is concentrated in a localized region of the (ϑ,μ)-plane, with a dominant peak occurring around $\vartheta \approx -2$. This behavior demonstrates the localization capability of the proposed MOFST and indicates its effectiveness in providing a compact time-frequency representation of the LFM signal.

In addition to the magnitude representation, the phase distribution corresponding to (69) is illustrated in Figure 2.

Figure 2 presents the phase distribution of the proposed MOFST associated with the LFM signal. The phase exhibits structured variations across the (ϑ,μ)-plane, reflecting the complex-valued nature of the transform. These phase characteristics provide complementary information to the magnitude representation shown in Figure 1 and further describe the behavior of the LFM signal in the MOFST domain.



(a) (b)

FIGURE 1. Magnitude representation of the MOFST associated with the LFM signal: (a) contour plot and (b) three-dimensional surface plot.

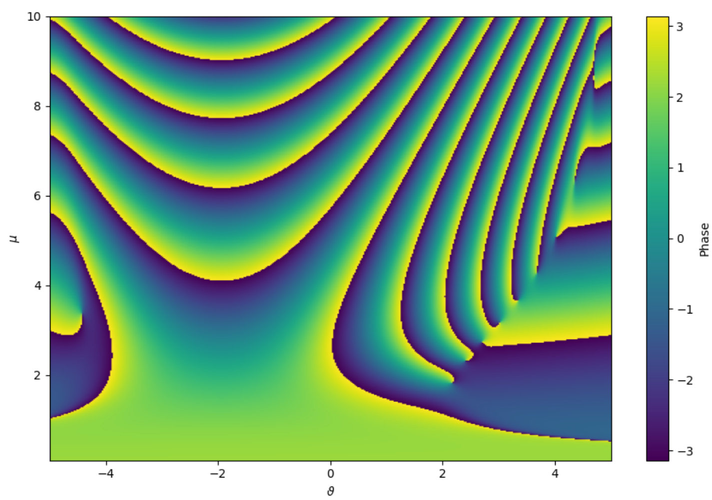
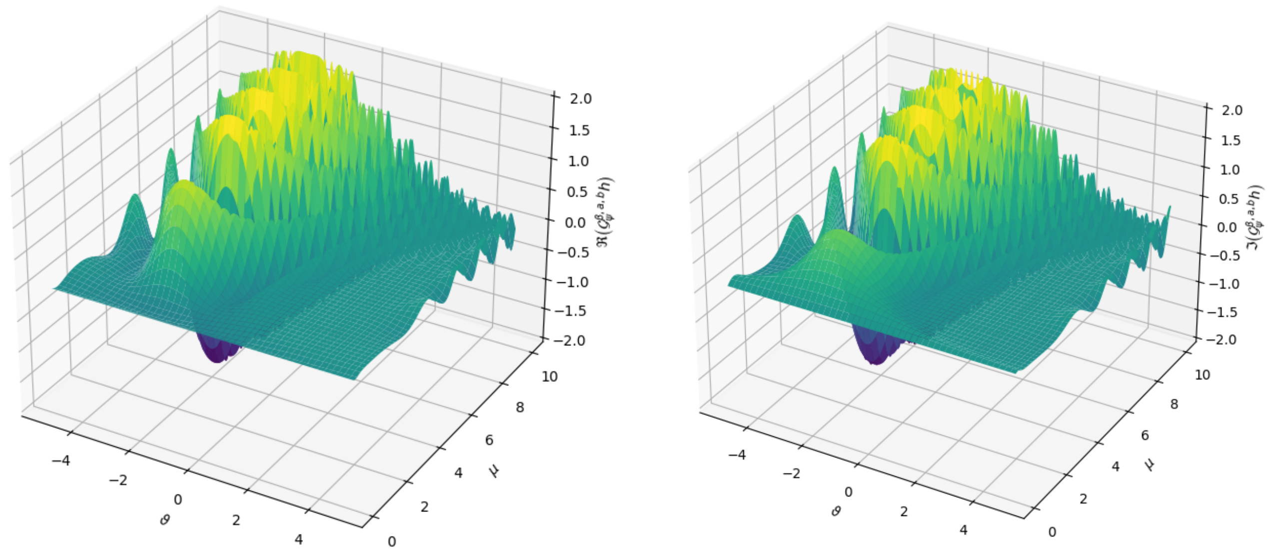


FIGURE 2. Phase distribution of the MOFST associated with the LFM signal in the (ϑ, μ)-plane.

For completeness, the real and imaginary components of the proposed MOFST are depicted in Figure 3.

Figure 3 illustrates the real and imaginary components of the MOFST associated with the LFM signal. Both components exhibit oscillatory behavior over the (ϑ, μ)-plane, reflecting the complex-valued structure of the transform. Together, they provide additional insight into the transform-domain representation beyond the magnitude and phase distributions.



(a) (b)

Figure 3. Real and imaginary components of the proposed MOFST associated with the LFM signal: (a) real part and (b) imaginary part.

The magnitude and phase representations presented in Figures 1 and 2 indicate that the proposed MOFST provides a localized time-frequency representation of LFM signals. Since LFM signals are widely used in radar and communication systems, the proposed transform may be useful for chirp signal analysis and radar signal processing [23,24]. Moreover, Stockwell-type transforms have been successfully applied in power quality analysis, ECG and EEG signal processing, and ground penetrating radar data interpretation [7–10]. Therefore, the proposed MOFST may serve as an alternative tool for analyzing non-stationary signals in these application areas.

7. Conclusion

In the current work, we have introduced one-dimensional modified Stockwell transform and investigated its properties. We have made a relation between the one-dimensional modified offset fractional Stockwell transform and Fourier transform. We developed the relation to obtain some inequalities related to the one-dimensional modified offset fractional Stockwell transform.

Authors' Contributions. All authors have read and approved the final version of the manuscript.

- Nasrullah Bachtiar: Conceptualization, supervision, and project administration.
- Andi Tenri Ajeng Nur: Literature review, mathematical formulation, and manuscript preparation.
- Putri Isnaini Cahyaning Baiti: Theoretical analysis and mathematical proofs.
- Muklas Rivai: Validation of theoretical results and manuscript review.

- Tesa Ria Winata: Numerical experiments, visualization, and application to LFM signals.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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