Asia Pac. J. Math. 2026 **13**:123

CHARACTERIZATIONS OF REAL HPERSURFACES WITH η-PARALLEL RICCI OPERATORS IN A 2-DIMENSIONAL COMPLEX SPACE FORM

DONG HO LIM[1], JUNG HYUN KIM[2,*]

[1]Department of Mathematics Education, Sehan University, Republic of Korea

[2]Department of Mathematics Education Graduate School of Education, Ajou University, Republic of Korea

[*]Corresponding author: mirmath@ajou.ac.kr

Received Apr. 24, 2026

Abstract. We prove that a real hypersurface M in a nonflat complex space form $M_2(c)$, whose Ricci operator is η-parallel and commute with Reeb vector field on the holomorphic distribution of M

2020 Mathematics Subject Classification. 53C40; 53C15.

Key words and phrases. real hypersurface; model space of type A or B; Ricci operator; η-parallel.

1. Introduction

A complex n-dimensional Kähler manifold with constant holomorphic sectional curvature c is referred to as a complex space form, denoted as $M_n(c)$. It is widely recognized that a complete and simply connected complex space form is complex analytically isometric to either complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, depending on whether $c > 0$, $c = 0$ or $c < 0$, respectively.

Takagi [10] completely classified all such hypersurfaces into six model spaces: A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero [5] and so on. Berndt [2] catagorized all homogeneous Hopf hypersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B. If a real hypersurface M is of A_1 or A_2 in $P_n\mathbf{C}$ or of A_0, A_1, A_2 in $H_n\mathbf{C}$, then M is said to be *type* A for simplicity.

We denote the induced almost contact metric structure of the real hypersurface M in $M_n(c)$ by $(\phi, g, \xi.\eta)$. The Ricci operator of M is denoted by S, and the shape operator (or second fundmental tensor field) of M is denoted by A. The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) | g(X, \xi)_P = 0\} \tag{1}$$

DOI: 10.28924/APJM/13-123

where $T_p(M)$ is the tangent space of M at $p \in M$. A (1,1) type tensor field S of M is said to be η-parallel Ricci operator if $g((\nabla_X S)Y, Z) = 0$ for any vector fields X, Y and Z in T_0. Many authors have investigated the geometric properties of real hypersurfaces with η-parallel Ricci operators (see [1], [5], [8], [9] and [11]). D. H. Lim, W. H. Sohn and H. J. Song ([6]) studied real hypersurfaces in $M_n(c)$ under certain conditions related to the η-parallel Ricci operator. Moreover, Kimura and Maeda ([5]) and Suh ([9]) established the following results concerning real hypersurfaces with η-parallel Ricci operator.

Theorem B ([5], [9]) *Let M be a Hopf hypersurface in a complex space form of $M_n(c)$, $c \neq 0$. Then the Ricci operator of M is an η-parallel Ricci operator and structure tensor field ξ is a principal if and only if M is locally congruent to one of the model space of type A and B.*

Kim, Park and Sohn ([4]) provided a characterization of real hypersurfaces with a special Ricci operator for $n \geq 3$, that is, they proved the corresponding classification result.

Theorem C ([4]) *Let M be a real hypersurface in a complex space form of $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$g((S\phi - \phi S)X, Y)) = 0, \tag{2}$$

$$(\nabla_X S)Y = \mu g(\phi X, Y)\xi \tag{3}$$

for any X and Y in T_0, where μ is a scalar function on M, Then M is a locally congruent to one of the model space of type A and B.

Theorem D ([4]) *Let M be a real hypersurface in a complex space form of $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies (2) and*

$$(\nabla_X S)Y = \nu g(\phi AX, Y)\xi \tag{4}$$

for any X and Y in T_0, where ν is a scalar function on M, Then M is a locally congruent to one of the model space of type A and B.

The aim of this paper is to investigate, based on the results obtained by Kim([4]) et al., how the conditions holding for $n \geq 3$ influence the differential geometric structure of real hypersurfaces in complex two-dimensional space. Through this approach, we seek to gain a more precise understanding of the resulting structural properties under these conditions. In addition, this study aims to systematically analyze the geometric structure of three-dimensional real hypersurfaces and to examine their distinctive geometric properties in low-dimensional complex space forms. The main result of this study is established as follows.

Theorem 1. *Let M be a real hypersurface of $M_2(c)$ with $c \neq 0$. If M satisfies (2) and (3) for any $X, Y, Z \in T_0$, then M is a locally congruent to one of typ A or type B.*

Theorem 2. *Let M be a real hypersurface of $M_2(c)$, $c \neq 0$. If M satisfies* (2) *and* (4) *for any $X, Y, Z \in T_0$, then M is a locally congruent to one of the model spaces of type A or type B.*

All manifolds are assumed to be connected and of class C^∞. And real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

respectively, where X and Y are any vector fields tangent to M, g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. And we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{5}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify the followings from the Gauss and Weingarten formulas:

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{6}$$

As the ambient space has holomorphic sectional curvature c, the equations of Gauss and Codazzi are given, respectively, by:

$$\begin{aligned} R(X,Y)Z = \ &\tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &-2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{8}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M. We shall denote the Ricci operator of M by S. Then it follows from (3) that

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2 X, \tag{9}$$

where $m = traceA$ is the mean curvature of M, and the coveriant derivative of (9) is given by

$$(\nabla_X S)Y = -\frac{3c}{4}\big(g(\phi AX, Y)\xi + \eta(Y)\phi AX\big) + (Xm)AY$$
$$+ m(\nabla_X A)Y - (\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y. \tag{10}$$

Let U be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let β be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish. It is not possible to define U without specifying that $\beta \neq 0$. Then it is easily seen from the first equation (6) that

$$A\xi = \alpha\xi + \beta U. \tag{11}$$

where $\alpha = \eta(A\xi)$. We notice that U is orthogonal to ξ.
We put

$$\Omega = \{p \in M \mid \beta(p) \neq 0\}. \tag{12}$$

Then Ω is an open subset of M.

3. η-parallel Ricci operators

Let M be a real hypersurface in $M_2(c)$, $c \neq 0$, satisfying condition (2), and suppose that the Ricci operator S is η-parallel, that is,

$$g((\nabla_X S)Y, Z) = 0 \tag{13}$$

for any vector fields X, Y and Z in T_0. Assume further that Ω is not empty. Then there exist smooth functions γ, δ, ϵ and a unit vector field U and ϕU orthogonal to ξ such that

$$AU = \beta\xi + \gamma U + \delta\phi U, \qquad A\phi U = \delta U + \epsilon\phi U. \tag{14}$$

Moreover, the trace of the shape operator A is given by

$$m = traceA = \alpha + \gamma + \epsilon \tag{15}$$

in $M_2(c)$.

We shall prove the following lemmmas

Lemma 3.1 *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If M satisfies (2), then we have*

$$m = \alpha + \gamma \quad and \quad g(AU, \phi U) = 0. \tag{16}$$

Proof. If differentiate (2) coveriantly and take account of (2), (5), (9) and (11), then we have

$$(m-\alpha)\big\{g(U,Y)g(AX,Z) + g(U,Z)g(AX,Y)$$
$$+ g(\phi U, Y)g(AX, \phi Z) + g(\phi U, Z)g(AX, \phi Y)\big\}$$

$$\begin{aligned}&-g(AU,Y)g(AX,Z)-g(AU,Z)g(AX,Y)\\&+g(AU,\phi Y)(AX,\phi Z)+g(AU,\phi Z)g(AX,\phi Y)=0\end{aligned}\tag{17}$$

for any X, Y and Z in T_0, where we have used the equation

$$\begin{aligned}\nabla_X Y&=g(\nabla_X Y,\xi)\xi+(\nabla_X Y)_0\\&=-g(\phi AX,Y)\xi+(\nabla_X Y)_0\qquad(\nabla_X Y)_0\in T_0.\end{aligned}\tag{18}$$

If we put $Y=Z=U$ into equation (17), then we obtain

$$(m-\alpha-\gamma)AU+g(AU,\phi U)\,A\phi U=\beta(m-\alpha-\gamma)\,\xi.\tag{19}$$

Next, letting $Y=U$ and $Z=\phi U$ in (17), we obtain

$$g(AU,\phi U)AU-(m-\alpha-\gamma)A\phi U=\beta g(AU,\phi U)\xi.\tag{20}$$

Putting $X=Y=U$ into (17), we have

$$\begin{aligned}&(m-\alpha)\{g(AU,Z)+g(U,Z)g(AU,U)+g(\phi U,Z)g(AU,\phi U)\}\\&-2g(AU,U)g(AU,Z)+2g(AU,\phi U)g(AU,\phi Z)=0.\end{aligned}\tag{21}$$

Finally, setting $X=\phi U$ and $Y=U$ in (17), we obtain

$$\begin{aligned}&(m-\alpha)\{g(A\phi U,Z)+g(U,Z)g(A\phi U,U)+g(\phi U,Z)g(A\phi U,\phi U)\}\\&-g((AU,U)g(A\phi U,Z)-g(AU,Z)g(A\phi U,U)\\&+g(AU,\phi U)g(A\phi U,\phi Z)+g(AU,\phi Z)(A\phi U,\phi U)=0.\end{aligned}\tag{22}$$

From equations (19) and (20), we obtain the following result.

$$\begin{aligned}&\{(m-\alpha-\gamma)^2+g(AU,\phi U)^2\}(AU-\beta\xi)=0,\\&\{(m-\alpha-\gamma)^2+g(AU,\phi U)^2\}A\phi U=0.\end{aligned}$$

Assume that there exists a point p of M such that $(m-\alpha-\gamma)^2+g(AU,\phi U)^2\neq 0$ at p. Then, on an open neighborhood of p, it follows that

$$AU=\beta\xi,\qquad A\phi U=0.\tag{23}$$

These relations imply that

$$\gamma=g(AU,U)=0,\quad \epsilon=g(A\phi U,\phi U)=0\quad \textit{and}\quad m=\alpha.\tag{24}$$

by virtue of equation (15). However, substituting equation (23) and (24) into the given assumption makes the resulting equation vanishes which is a contradiction. Therefore, Theorem 3.1 holds.□

Using relation (14) and (15) together with the result of Lemma 3.1, we can easily obtain the following lemma.

Lemma 3.2 *Under the assumptions as in Lemma 3.1, then we have*

$$AU = \beta\xi + \gamma U. \qquad A\phi U = 0 \tag{25}$$

Lemma 3.3 *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. It M satisfies (2), then we obtain*

$$\beta^2 = \alpha\gamma. \tag{26}$$

Proof. *The condition of the given assumption can be expressed in terms of the following relation by using equation (9),*

$$\begin{aligned}(A^2\phi - \phi A^2)X - m(A\phi X - \phi AX)\\ = \beta\, g((\alpha - m)U + AU, \phi X)\,\xi.\end{aligned} \tag{27}$$

for any X in T_0. If we put $X = U$ into (27) and using (25) in Lemma 3.2, Then we have

$$m\gamma = \beta^2 + \gamma^2. \tag{28}$$

If we substitute the first equation (16) into (28), we obtain the equation (26).□

By use of (11), (25) and (26), the relation (9) gives rise to

$$S\xi = \frac{c}{2}\xi \tag{29}$$

Differentiting (29) covariantly along X in T_0 and using (6) and(9), we get

$$(\nabla_X S)\xi = -\frac{3c}{4}\phi AX - mA\phi AX + A^2\phi AX \tag{30}$$

for any X in T_0. On the other hand, we see from (5) and (29) that $S\phi - \phi S = 0$, which toghter with (2) implies that $S\phi - \phi S = 0$ on M, or equivalently

$$(A^2\phi - \phi A^2)X - m(A\phi - \phi A)X = 0. \tag{31}$$

By substituting Equation (31) into Equation (30) and rearranging, equation (30) can be expressed as

$$(\nabla_X S)\xi = -\frac{3c}{4}\phi AX - (\alpha + \gamma)\phi A^2 X + \phi A^3 X. \tag{32}$$

If we take inner product of Y in T_0, equation (32) is rewritten to

$$(\nabla_X S)Y = -\{g(A^3X - (\alpha + \gamma)A^2X - \frac{3c}{4}AX, \phi Y)\}\xi. \tag{33}$$

From the given relation (11) and Lemma 3.2, it follows (26) that

$$A^2\xi - mA\xi = 0, \qquad A^2U - mAU = 0. \tag{34}$$

4. Proof of Theorems

This section is devoted to proofs of Theorem 1 and 2. Let M be a real hpersurface in a complex space form $M_2(c)$, $c \neq 0$.

Proof of Theorem 1 *Assume that* $\Omega = \{p \in M \mid \beta(p) \neq 0\}$. *Since equation* (3) *shows that the Ricci operator* S *is* η*-parallel, it follows from equation* (3) *and* (33) *that*

$$g(A^3X - (\alpha+\gamma)A^2X - \frac{3c}{4}AX - \mu X, Y) = 0 \tag{35}$$

for any X *and* Y *in* T_0. *Substituting* $X = Y = U$ *into* (35) *and using the second equation* (34), *we have*

$$\mu + \frac{3c}{4}\gamma = 0. \tag{36}$$

If we put $X = Y = \phi U$ *and using the second equation* (25) *in Lemma 3.2, then we have*

$$\mu = 0. \tag{37}$$

Substituting equation (37) *into equation* (36) *and applying equation* (26) *in Lemma 3.3, it follows that* $\beta = 0$ *on* Ω*, which yields a contradiction.*

Therefore β *vanishes identically on* M. *By equation* (11)*, Reeb vector field* ξ *is principal and the desired conclusion follows directly from Theorem B.* □

Proof of Theorem 2 *Assume that Assume that* $\Omega = \{p \in M \mid \beta(p) \neq 0\}$. *Comparing* (4) *and* (33) *and using* (34)*, we get*

$$\begin{aligned} &A^3X - (\alpha+\gamma)A^2X - (\nu + \frac{3c}{4})AX \\ &= -\beta(\nu + \frac{3c}{4})g(X,U)\xi \end{aligned} \tag{38}$$

for any X *in* T_0. *If we put* $X = U$ *into* (38) *and by use of* (34) *and Lemma 3.2, then we obtain*

$$\gamma(\nu + \frac{3c}{4}) = 0. \tag{39}$$

on Ω. *Assume that there is a point* p *in* Ω *such that* $\nu + \frac{3c}{4} \neq 0$ *at* p*. then, from equation* (39)*, it follows that* $\gamma = 0$. *Futhermore, by equation* (26) *in Lemma 3.3, we obtain* $\beta = 0$*, which leads to a contradiction. Hence, we conclude that* $\nu + \frac{3c}{4} = 0$ *holds on* Ω. *Consequently, the equation* (38) *reduces to*

$$A^3X - (\alpha+\gamma)A^2X = 0 \tag{40}$$

on Ω. *Let* X *be any vector field on* M *such that* $AX = \lambda X$. *Then any principal curvature* λ *of* M *is given by*

$$\lambda = 0, \qquad \lambda = \alpha + \gamma. \tag{41}$$

Since these vector fields ϕU is orthogonal to U, it lies in the eigenspace corresponding to $\lambda = 0$. Accordingly, the shape operator can be represented by

$$\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha+\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which yields the mean curvature $m = 2\alpha + \gamma$. On the other hand, by Lemma 3.2, the shape operator takes the form

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

giving the mean curvature $m = \alpha + \gamma$.

Comparing these expressions for the mean curvature, we obtain $\alpha = 0$. Substituting this into equation (26) of Lemma 3.3 yields $\beta = 0$, which leads to a contradiction. Therefore, ξ is principal, and Theorem 2 follows from Theorem B. □

Acknowledgements. This paper was supported by the AJOU University Research Fund in 2026.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] C. Baikoussis, A Characterization of Real Hypersurfaces in Complex Space Forms in Terms of the Ricci Tensor, Can. Math. Bull. 40 (1997), 257–265. https://doi.org/10.4153/cmb-1997-031-5.

[2] J. Berndt, Real Hypersurfaces with Constant Principal Curvatures in Complex Hyperbolic Space, J. Reine Angew. Math. 395 (1989), 132–141. https://doi.org/10.1515/crll.1989.395.132.

[3] U.H. Ki, Y.J. Suh, On Real Hypersurfaces of a Complex Space Form, Math. J. Okayama Univ. 32 (1990), 207–221.

[4] I.B. Kim, H.J. Park, W.H. Sohn, On Characterizations of Real Hypersurfaces with η-Parallel Ricci Operators in a Complex Space Form, Bull. Korean Math. Soc. 43 (2006), 235–244. https://doi.org/10.4134/bkms.2006.43.2.235.

[5] M. Kimura, S. Maeda, On Real Hypersurfaces of a Complex Projective Space, Math. Z. 202 (1989), 299–311.

[6] D.H. Lim, W.H. Sohn, H. Song, A Study of Real Hypersurfaces with Ricci Operators in 2-Dimensional Complex Space Forms, Pac. J. Math. 266 (2013), 305–311. https://doi.org/10.2140/pjm.2013.266.305.

[7] S. Montiel, A. Romero, On Some Real Hypersurfaces of a Complex Hyperbolic Space, Geom. Dedicata 20 (1986), 245–261. https://doi.org/10.1007/bf00164402.

[8] R. Niebergall, Real Hypersurfaces in Complex Space Forms, Tight and Taut Submanifolds, Cambridge University Press, 1997. https://cir.nii.ac.jp/crid/1573387449888599296.

[9] Y.J. Suh, On Real Hypersurfaces of a Complex Space Form with η-Parallel Ricci Tensor, Tsukuba J. Math. 14 (1990), 27–37.

[10] R. Takagi, On Homogeneous Real Hypersurfaces in a Complex Projective Space, Osaka. J. Math. 10 (1973), 495–506.

[11] T. Taniguchi, Characterizations of Real Hypersurfaces of Complex Hyperbolic Space in Terms of Ricci Tensor and Holomorphic Distribution, Tsukuba J. Math. 18 (1994), 469–482.