A NOVEL COMPUTATIONAL ALGORITHM FOR EFFICIENTLY SOLVING MULTIDIMENSIONAL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper introduces a novel computational algorithm for constructing approximate analytical solutions to both homogeneous and nonhomogeneous nonlinear fractional differential equations in the Caputo sense. The proposed formulation combines the iterative framework of the Hussein–Jassim method with a telescopic nonlinear decomposition technique to handle nonlinear terms efficiently. The convergence behavior of the resulting approximations is analyzed, and several benchmark problems are presented to illustrate the effectiveness of the method through numerical and graphical results. The findings show that the proposed scheme improves convergence, reduces the algebraic complexity associated with nonlinear decomposition, and yields approximations that are in excellent agreement with the exact solutions.

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1. Introduction

Fractional calculus has a long history, dating back to Leibniz's letter in 1695, which contained the first idea of a derivative of non-integer order. For nearly three centuries, the theory of fractional calculus remained within the scope of pure mathematics. Owing to its wide applicability in physics, chemistry, ecology, biology, and engineering, fractional differential equations have attracted considerable attention from researchers [1–24]. It has been shown that many physical phenomena, including rheological behavior, damping laws, and diffusion processes, can be described more accurately by derivatives of fractional order. These developments have stimulated growing interest in fractional calculus across a broad range of scientific and engineering disciplines.

The Hussein–Jassim Method (HJM) has recently been introduced as an iterative framework based on Taylor–Maclaurin expansions and has been extended to treat a wider class of problems [25, 26].

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At the same time, telescoping decomposition methods have been proposed to simplify the treatment of nonlinear operators by replacing complicated polynomial expansions with incremental correction terms [27,28]. Motivated by these developments, the present paper proposes a computational algorithm that combines the Hussein–Jassim iterative structure with the telescopic nonlinear decomposition technique for solving multidimensional nonlinear fractional differential equations in the Caputo sense. The main objective is to obtain rapidly convergent recursive series solutions while reducing the algebraic complexity usually associated with nonlinear decomposition procedures.

2. Preliminaries

Definition 1. [1,2,5] If $a > 0$, $a < t < b$, and $\vartheta(t) \in C([a,b])$, $\alpha > 0$, then the Riemann–Liouville fractional integral of order α is defined as follows:

$$I_t^{\alpha}\vartheta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\vartheta(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau.$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. [1,2,4,5] Let $\vartheta(t) \in H^1(a,b)$, $a < b$. The Caputo fractional derivative (CDF) of order α is defined as follows:

$${}_a^cD_t^{\alpha}\vartheta(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1}\vartheta^{(n)}(\tau)\, d\tau,$$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$.

Definition 3. [25, 26] The Hussein–Jassim Method (HJM) is defined by employing Maclaurin's expansion with respect to t as

$$\vartheta(t) = \lambda + \sum_{i=1}^{\infty} \frac{{}^cD_t^{(i-1)\alpha}\big(g(t) - N(\vartheta(t))\big)\Big|_{t=0}}{\Gamma(i\alpha+1)} t^{i\alpha}.$$

with initial condition $\vartheta(0) = \lambda$, where $\vartheta(t)$ constitutes an analytical function, ${}^cD_t^{i\alpha}$ is the Caputo operator, N is a nonlinear operator, and $g(t)$ is a known function.

Definition 4. [27,28] The telescopic nonlinear decomposition expresses the nonlinear operator $N(\vartheta)$ as a telescoping series of incremental corrections. Each correction measures the variation in the nonlinear term produced by the addition of a new component of the iterative solution. If $\vartheta = \sum_{i=0}^{\infty}\vartheta_i$, then the telescopic decomposition is given by

$$N\left(\sum_{i=0}^{\infty}\vartheta_i\right) = N(\vartheta_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{k=0}^{i}\vartheta_k\right) - N\left(\sum_{k=0}^{i-1}\vartheta_k\right)\right] \tag{2.1}$$

Here, ϑ is assumed to be an analytic function and N is a nonlinear operator.

3. Analysis of the Method

Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded domain, and let T be a positive constant with $0 < T \leq \infty$. To illustrate the idea of the proposed Falah–Jassim Method (FJM) technique, for any $(\vec{\mathfrak{e}}, t) \in \Omega \times (0, T]$. Let the nonlinear fractional differential equation be given in the Caputo sense as

$$ {}^{c}D_t^{\alpha}\vartheta(\vec{\mathfrak{e}}, t) + L\big(\vartheta(\vec{\mathfrak{e}}, t)\big) + N\big(\vartheta(\vec{\mathfrak{e}}, t)\big) = g(\vec{\mathfrak{e}}, t), \qquad n-1 < \alpha < n \tag{3.1} $$

The problem is supplemented with the initial conditions

$$ \vartheta^{(\mathfrak{g})}(\vec{\mathfrak{e}}, 0) = \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}), \qquad \mathfrak{g} = 0, 1, 2, \dots, n-1,\ \vec{\mathfrak{e}} \in \bar{\Omega} \tag{3.2} $$

where $\vec{\mathfrak{e}} = (\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_n) \in \Omega$, $\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})$ are given functions, and $\vartheta(\vec{\mathfrak{e}}, t)$ denotes the analytic function for $t \geq 0$, ${}^{c}D_t^{\alpha}$ is Caputo operator, L and N represent the linear and nonlinear operators, respectively, $g(\vec{\mathfrak{e}}, t)$ is a given function.

By using the Riemann–Liouville fractional integral operator of order α to both sides of (3.1), the problem can be reformulated in an equivalent integral representation as follows:

$$ \vartheta(\vec{\mathfrak{e}}, t) = \sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha + 1)} + I_t^{\alpha}\big(g(\vec{\mathfrak{e}}, t) - L(\vartheta(\vec{\mathfrak{e}}, t))\big) - I_t^{\alpha}\big(N(\vartheta(\vec{\mathfrak{e}}, t))\big). \tag{3.3} $$

We will now reformulate (3.3) by employing the Hussein–Jassim Method (HJM) to incorporate the initial conditions, the linear operator, and the source term $g(\vec{\mathfrak{e}}, t)$, obtaining

$$ \begin{aligned} \vartheta(\vec{\mathfrak{e}}, t) = {} & \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}\, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha + 1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}}, t) - L\big(\vartheta(\vec{\mathfrak{e}}, t)\big)\right) \right)\Bigg|_{t=0} \\ & - I_t^{\alpha}\big(N(\vartheta(\vec{\mathfrak{e}}, t))\big). \end{aligned} \tag{3.4} $$

Where ${}^{c}D_t^{i\alpha}$ denotes the Caputo fractional derivative of order $i\alpha$.

Let the solution be expressed in the series form

$$ \vartheta(\vec{\mathfrak{e}}, t) = \sum_{i=0}^{\infty} \vartheta_i, \tag{3.5} $$

Substituting (3.5) into (3.4) yields the result that

$$ \begin{aligned} \sum_{i=0}^{\infty} \vartheta_i = {} & \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}\, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha + 1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}}, t) - L\left(\sum_{i=0}^{\infty} \vartheta_i\right)\right) \right)\Bigg|_{t=0} \\ & - I_t^{\alpha}\left(N\left(\sum_{i=0}^{\infty} \vartheta_i\right)\right). \end{aligned} \tag{3.6} $$

The nonlinear operator in (3.6) is decomposed according to the following telescopic expansion:

$$ N\left(\sum_{i=0}^{\infty} \vartheta_i\right) = N(\vartheta_0) + \sum_{i=1}^{\infty} \left(N\left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}}\right) - N\left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}}\right) \right). \tag{3.7} $$

Substituting the decomposition (3.7) into (3.6), we obtain

$$\sum_{i=0}^{\infty}\vartheta_i=\sum_{i=0}^{\infty}\left(\begin{array}{c}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\,{}^cD_t^{i\alpha}\left(\sum_{\mathfrak{g}=0}^{n-1}\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})\frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)}\right.\\ \left.\left.+I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t)-L\left(\sum_{i=0}^{\infty}\vartheta_i\right)\right)\right)\right|_{t=0}\\ -I_t^{\alpha}\left(N\big(\vartheta_0(\vec{\mathfrak{e}},t)\big)+N\left(\sum_{\mathfrak{g}=0}^{i}\vartheta_{\mathfrak{g}}\right)-N\left(\sum_{\mathfrak{g}=0}^{i-1}\vartheta_{\mathfrak{g}}\right)\right)\end{array}\right). \tag{3.8}$$

By comparing both sides of (3.8), a recursive sequence for the components $\vartheta_i(\vec{\mathfrak{e}},t)$ is obtained:

$$\begin{cases}\vartheta_0=\vartheta(\vec{\mathfrak{e}},0)=\lambda_0(\vec{\mathfrak{e}}),\\[2mm] \vartheta_1=\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\lambda_1(\vec{\mathfrak{e}})+D_t^{1-\alpha}\big(g(\vec{\mathfrak{e}},t)-L(\vartheta_0)\big)\big|_{t=0}\right)-I_t^{\alpha}\big(N(\vartheta_0)\big),\\[2mm] \vartheta_2=\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(\lambda_2(\vec{\mathfrak{e}})+D_t^{2-\alpha}\big(g(\vec{\mathfrak{e}},t)-L(\vartheta_0+\vartheta_1)\big)\big|_{t=0}\right)-I_t^{\alpha}\big(N(\vartheta_0+\vartheta_1)-N(\vartheta_0)\big),\\[2mm] \vartheta_3=\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left(\lambda_3(\vec{\mathfrak{e}})+D_t^{3-\alpha}\big(g(\vec{\mathfrak{e}},t)-L(\vartheta_0+\vartheta_1+\vartheta_2)\big)\big|_{t=0}\right)\\ \qquad -I_t^{\alpha}\big(N(\vartheta_0+\vartheta_1+\vartheta_2)-N(\vartheta_0+\vartheta_1)\big),\\ \quad\vdots\\ \vartheta_{i+1}=\dfrac{t^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}D_t^{(i+1)\alpha}\left.\left(\sum_{\mathfrak{g}=0}^{n-1}\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})\frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)}+I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t)-L\left(\sum_{j=0}^{i}\vartheta_j\right)\right)\right)\right|_{t=0}\\ \qquad -I_t^{\alpha}\left(N(\vartheta_0+\cdots+\vartheta_i)-N(\vartheta_0+\cdots+\vartheta_{i-1})\right).\end{cases}$$

Therefore, the subsequent iterative relations are formulated to ascertain the consecutive terms of the series solution:

$$\vartheta(\vec{\mathfrak{e}},t)=\vartheta_0+\vartheta_1+\vartheta_2+\cdots=\sum_{i=0}^{\infty}\vartheta_i.$$

3.1. **Convergence of the Method.**

Theorem 3.1. *The proposed decomposition procedure for (3.1) can be equivalently formulated in terms of the sequence $\{\zeta_\eta\}$ defined by*

$$\zeta_\eta=\vartheta_1+\vartheta_2+\vartheta_3+\cdots+\vartheta_\eta,\qquad \zeta_0=0. \tag{3.9}$$

where ζ_η denotes the η-th partial sum of the series components generated by the proposed decomposition scheme.

$$\zeta_{\eta+1}=\sum_{i=1}^{\eta+1}\left(\begin{array}{c}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\,{}^cD_t^{i\alpha}\left(\sum_{\mathfrak{g}=0}^{n-1}\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})\frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)}\right.\\ \left.\left.+I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t)-L\left(\sum_{j=0}^{i-1}\vartheta_j\right)\right)\right)\right|_{t=0}\\ -I_t^{\alpha}\left(N\left(\sum_{\mathfrak{g}=0}^{i-1}\vartheta_{\mathfrak{g}}\right)-N\left(\sum_{\mathfrak{g}=0}^{i-2}\vartheta_{\mathfrak{g}}\right)\right)\end{array}\right). \tag{3.10}$$

Proof. For $\eta = 0$, (3.10) yields

$$\zeta_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \left(\lambda_1(\vec{\mathfrak{e}}) + D_t^{1-\alpha}\big(g(\vec{\mathfrak{e}},t) - L(\vartheta_0)\big)\big|_{t=0}\right) - I_t^{\alpha}\big(N(\vartheta_0)\big). \tag{3.11}$$

then,

$$\vartheta_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \left(\lambda_1(\vec{\mathfrak{e}}) + D_t^{1-\alpha}\big(g(\vec{\mathfrak{e}},t) - L(\vartheta_0)\big)\big|_{t=0}\right) - I_t^{\alpha}\big(N(\vartheta_0)\big). \tag{3.12}$$

The proof utilises robust mathematical induction. Assume that

$$\begin{aligned}\vartheta_{m+1} = {} & \frac{t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} D_t^{(m+1)\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t) - L\left(\sum_{j=0}^{m} \vartheta_j\right)\right)\right)\Bigg|_{t=0} \\ & - I_t^{\alpha}\big(N(\vartheta_0 + \cdots + \vartheta_m) - N(\vartheta_0 + \cdots + \vartheta_{m-1})\big),\end{aligned}$$

where $m = 1, 2, 3, \ldots, \eta - 1$, so

$$\begin{aligned}\zeta_{\eta+1} &= \sum_{i=1}^{\eta+1} \left(\begin{aligned} &\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t) - L\left(\sum_{j=0}^{i-1} \vartheta_j\right)\right)\right)\Bigg|_{t=0} \\ &- I_t^{\alpha}\left(N\left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}}\right) - N\left(\sum_{\mathfrak{g}=0}^{i-2} \vartheta_{\mathfrak{g}}\right)\right)\end{aligned}\right) \\ &= \vartheta_1 + \cdots + \vartheta_\eta + \frac{t^{(\eta+1)\alpha}}{\Gamma((\eta+1)\alpha+1)} D_t^{(\eta+1)\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)}\right. \\ &\qquad \left. + I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t) - L\left(\sum_{j=0}^{\eta} \vartheta_j\right)\right)\right)\Bigg|_{t=0} \\ &\qquad - I_t^{\alpha}\big(N(\vartheta_0 + \cdots + \vartheta_\eta) - N(\vartheta_0 + \cdots + \vartheta_{\eta-1})\big).\end{aligned}$$

According to the definition provided in (3.9), the subsequent relation can be derived.

$$\begin{aligned}\vartheta_{\eta+1} = {} & \frac{t^{(\eta+1)\alpha}}{\Gamma((\eta+1)\alpha+1)} D_t^{(\eta+1)\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t) - L\left(\sum_{j=0}^{\eta} \vartheta_j\right)\right)\right)\Bigg|_{t=0} \\ & - I_t^{\alpha}\big(N(\vartheta_0 + \cdots + \vartheta_\eta) - N(\vartheta_0 + \cdots + \vartheta_{\eta-1})\big),\end{aligned}$$

□

Theorem 3.2. *Let B be a Banach space.*

I. $\sum_{i=0}^{\infty} \vartheta_i$ *in* (3.8) *converges to* $\zeta \in B$, *if*

$$\exists\,(\,0 \le \epsilon < 1\,), \qquad \textit{such that} \qquad (\forall\,\eta \in \mathbb{N} \Rightarrow \|\vartheta_\eta\| \le \epsilon\|\vartheta_{\eta-1}\|)\,. \tag{3.13}$$

II. $\zeta = \sum_{\eta=1}^{\infty} \vartheta_\eta$ *satisfies*

$$\zeta = \sum_{i=1}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} {}^cD_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} \right. \\ \left. \left. + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{j=0}^{i} \vartheta_j \right) \right) \right) \right|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-2} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right). \tag{3.14}$$

Proof. **I.** From (3.13), we have

$$\begin{aligned} \|\zeta_{\eta+1} - \zeta_\eta\| &= \left\| \sum_{i=0}^{\eta+1} \vartheta_i - \sum_{i=0}^{\eta} \vartheta_i \right\| = \|\vartheta_{\eta+1}\| \\ &\leq \epsilon \|\vartheta_\eta\| \leq \epsilon^2 \|\vartheta_{\eta-1}\| \leq \epsilon^3 \|\vartheta_{\eta-2}\| \leq \cdots \leq \epsilon^{\eta+1} \|\vartheta_0\|. \end{aligned} \tag{3.15}$$

For all $\eta, m \in \mathbb{N}$, $\eta \geq m$,

$$\begin{aligned} \|\zeta_\eta - \zeta_m\| &= \|(\zeta_\eta - \zeta_{\eta-1}) + (\zeta_{\eta-1} - \zeta_{\eta-2}) + \cdots + (\zeta_{m+1} - \zeta_m)\| \\ &\leq \|\zeta_\eta - \zeta_{\eta-1}\| + \|\zeta_{\eta-1} - \zeta_{\eta-2}\| + \cdots + \|\zeta_{m+1} - \zeta_m\| \\ &\leq \epsilon^\eta \|\vartheta_0\| + \epsilon^{\eta-1} \|\vartheta_0\| + \epsilon^{\eta-2} \|\vartheta_0\| + \cdots + \epsilon^{m+1} \|\vartheta_0\| \\ &\leq \epsilon^{m+1} \|\vartheta_0\| \left(\epsilon^{\eta-m-1} + \epsilon^{\eta-m-2} + \cdots + 1 \right) \\ &= \frac{1-\epsilon^{\eta-m}}{1-\epsilon} \epsilon^{m+1} \|\vartheta_0\|. \end{aligned} \tag{3.16}$$

Then,

$$\lim_{\eta, m \to \infty} \|\zeta_\eta - \zeta_m\| = 0.$$

Thus,

$$\lim_{\eta \to \infty} \zeta_\eta = \sum_{\eta=1}^{\infty} \vartheta_\eta = \zeta.$$

II. From (3.10),

$$\lim_{\eta \to \infty} \zeta_{\eta+1} = \lim_{\eta \to \infty} \sum_{i=1}^{\eta+1} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} {}^cD_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} \right. \\ \left. \left. + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{j=0}^{i-1} \vartheta_j \right) \right) \right) \right|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-2} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right),$$

$$= \sum_{i=1}^{\infty} \left(\begin{array}{c} \left. \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} {}^cD_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{j=0}^{i-1} \vartheta_j \right) \right) \right) \right|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-2} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right).$$

Thus,

$$\zeta = \sum_{i=1}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} \right. \\ \left. \left. + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{i=0}^{\infty} \vartheta_i \right) \right) \right) \right|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right) = \sum_{i=1}^{\infty} \vartheta_i.$$

□

Theorem 3.3. *The subsequent* (3.14)

$$\zeta = \sum_{i=1}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{j=0}^{\infty} \vartheta_j \right) \right) \right) \Big|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right),$$

is equivalent to (3.1)*;*

$${}^{c}D_t^{\alpha} \vartheta(\vec{\mathfrak{e}}, t) + L(\vartheta(\vec{\mathfrak{e}}, t)) + N(\vartheta(\vec{\mathfrak{e}}, t)) = g(\vec{\mathfrak{e}}, t), \qquad \vartheta^{(\mathfrak{g})}(\vec{\mathfrak{e}}, 0) = \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}), \quad \mathfrak{g} = 0, 1, 2, \ldots, n-1.$$

Proof. Adding $\lambda_0(\vec{\mathfrak{e}})$ to both sides of (3.14) and using the definition of the partial sum ζ, the series representation of $\vartheta(\vec{\mathfrak{e}}, t)$ is obtained

$$\lambda_0(\vec{\mathfrak{e}}) + \zeta = \lambda_0(\vec{\mathfrak{e}}) + \sum_{i=1}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} \right. \\ \left. \left. + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{i=0}^{\infty} \vartheta_i \right) \right) \right) \right|_{t=0} \\ - I_t^{\alpha} \left(N \left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right)$$

$$= \sum_{i=0}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{i=0}^{\infty} \vartheta_i \right) \right) \right) \Big|_{t=0} \\ - I_t^{\alpha}(N(\vartheta_0)) + \left(N \left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right),$$

Noting that $\vartheta = \zeta + \lambda_0(\vec{\mathfrak{e}})$ and hence $\vartheta = \sum_{i=1}^{\infty} \vartheta_i + \vartheta_0 = \sum_{i=0}^{\infty} \vartheta_i$, we obtain

$$\sum_{i=0}^{\infty} \vartheta_i = \sum_{i=0}^{\infty} \left(\begin{array}{c} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \, {}^{c}D_t^{i\alpha} \left(\sum_{\mathfrak{g}=0}^{n-1} \lambda_{\mathfrak{g}}(\vec{\mathfrak{e}}) \frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha} \left(g(\vec{\mathfrak{e}}, t) - L \left(\sum_{i=0}^{\infty} \vartheta_i \right) \right) \right) \Big|_{t=0} \\ - I_t^{\alpha}(N(\vartheta_0)) + \left(N \left(\sum_{\mathfrak{g}=0}^{i} \vartheta_{\mathfrak{g}} \right) - N \left(\sum_{\mathfrak{g}=0}^{i-1} \vartheta_{\mathfrak{g}} \right) \right) \end{array} \right), \tag{3.17}$$

The right-hand side of (3.17) corresponds to a Maclaurin series expansion; therefore, (3.17) can be rewritten in the following form.

$$\begin{aligned}\sum_{i=0}^{\infty}\vartheta_i = \sum_{\mathfrak{g}=0}^{n-1}\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})\frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha}\left(g(\vec{\mathfrak{e}},t) - L\left(\sum_{i=0}^{\infty}\vartheta_i\right)\right)\\ - I_t^{\alpha}\big(N(\vartheta_0)\big) + \left(N\left(\sum_{\mathfrak{g}=0}^{i}\vartheta_{\mathfrak{g}}\right) - N\left(\sum_{\mathfrak{g}=0}^{i-1}\vartheta_{\mathfrak{g}}\right)\right).\end{aligned} \tag{3.18}$$

Using (3.5) and (3.7), (3.18) can be rewritten in the following form.

$$\vartheta(\vec{\mathfrak{e}},t) = \sum_{\mathfrak{g}=0}^{n-1}\lambda_{\mathfrak{g}}(\vec{\mathfrak{e}})\frac{t^{\mathfrak{g}\alpha}}{\Gamma(\mathfrak{g}\alpha+1)} + I_t^{\alpha}\big(g(\vec{\mathfrak{e}},t) - L(\vartheta)\big) - I_t^{\alpha}\big(N(\vartheta)\big). \tag{3.19}$$

Applying the Caputo fractional derivative of order α to both sides of (3.19),

$${}^{c}D_t^{\alpha}\vartheta(\vec{\mathfrak{e}},t) + L\big(\vartheta(\vec{\mathfrak{e}},t)\big) + N\big(\vartheta(\vec{\mathfrak{e}},t)\big) = g(\vec{\mathfrak{e}},t).$$

The answer derived from (3.14) aligns with the solution of (3.1). □

4. Illustrative examples

Example 1. Consider the following time-fractional partial differential equation of Caputo type, accompanied by the specified initial conditions.

$${}^{c}D_t^{2\alpha}\vartheta(\vec{\mathfrak{e}},t) - \frac{1}{9}\left(\vartheta_{\mathfrak{e}_1\mathfrak{e}_1}(\vec{\mathfrak{e}},t) + \vartheta_{\mathfrak{e}_2\mathfrak{e}_2}(\vec{\mathfrak{e}},t) + \vartheta_{\mathfrak{e}_3\mathfrak{e}_3}(\vec{\mathfrak{e}},t)\right)^2 + \vartheta^2(\vec{\mathfrak{e}},t) = 0, \qquad 0 < \alpha \le 1. \tag{4.1}$$

where $\vec{\mathfrak{e}} = (\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3)$, with initial conditions, $\vartheta(\vec{\mathfrak{e}},0) = 0$, $\vartheta_t(\vec{\mathfrak{e}},0) = \cos(\mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3)$.

Applying the proposed F–J method framework, (4.1) is reformulated into the associated recursive integral representation.

$$\begin{aligned}\vartheta(\vec{\mathfrak{e}},t) = \sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D_t^{i\alpha}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\right)\Bigg|_{t=0}\\ + I_t^{2\alpha}\left(\frac{1}{9}\left(\vartheta_{\mathfrak{e}_1\mathfrak{e}_1} + \vartheta_{\mathfrak{e}_2\mathfrak{e}_2} + \vartheta_{\mathfrak{e}_3\mathfrak{e}_3}\right)^2 - \vartheta^2\right).\end{aligned} \tag{4.2}$$

Assume that the solution can be expressed in the series form

$$\begin{aligned}\vartheta(\vec{\mathfrak{e}},t) = \sum_{i=0}^{\infty}\vartheta_i,\\ \left(\vartheta_{\mathfrak{e}_1\mathfrak{e}_1} + \vartheta_{\mathfrak{e}_2\mathfrak{e}_2} + \vartheta_{\mathfrak{e}_3\mathfrak{e}_3}\right)^2 = \sum_{i=0}^{\infty}\omega_i, \qquad \vartheta^2 = \sum_{i=0}^{\infty}\nu_i.\end{aligned} \tag{4.3}$$

By substituting (4.3) into (4.2),

$$\sum_{i=0}^{\infty}\vartheta_i = \sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D_t^{i\alpha}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\right)\Bigg|_{t=0} + I_t^{2\alpha}\left(\frac{1}{9}\sum_{i=0}^{\infty}\omega_i - \sum_{i=0}^{\infty}\nu_i\right). \tag{4.4}$$

By comparing both sides of (4.4),

$$\begin{aligned}
\vartheta_0 &= 0,\\
\vartheta_1 &= \frac{t^\alpha}{\Gamma(\alpha+1)}\,{}^cD_t^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\right)\Bigg|_{t=0} + I_t^{2\alpha}\left(\frac{1}{9}\omega_0-\nu_0\right)\\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3),\\
\vartheta_2 &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,{}^cD_t^{2\alpha}\big(\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\big)\Big|_{t=0} + I_t^{2\alpha}\left(\frac{1}{9}\omega_1-\nu_1\right)\\
&= I_t^{2\alpha}\left(\frac{1}{9}\left(-\frac{3t^\alpha}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\right)^2-\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3)\right)^2\right)=0,\\
&\vdots
\end{aligned}$$

The approximate solution obtained by the proposed F-J method is given as

$$\vartheta(\vec{\mathfrak{e}},t)=\sum_{i=0}^{\infty}\vartheta_i=\vartheta_0+\vartheta_1+\vartheta_2+\vartheta_3+\cdots$$

$$\vartheta_a(\vec{\mathfrak{e}},t)=\frac{t^\alpha}{\Gamma(\alpha+1)}\cos(\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3).$$

And when $\mathfrak{s}=\mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3$, the HJM and HPM produce the same analytical expression [29], which is identical to the exact solution.

The exact solution when $\alpha=1$ is

$$\vartheta_e(\mathfrak{s},t)=t\cos(\mathfrak{s}).$$

Example 2. Consider the two-dimensional time-fractional equation of Caputo type,

$${}^cD_t^\alpha\vartheta(\mathfrak{e},y,t)-\frac{1}{2}\big(\vartheta_{\mathfrak{e}\mathfrak{e}}+\vartheta_{yy}\big)+\frac{1}{2}\vartheta\big(\vartheta_{\mathfrak{e}}+\vartheta_y\big)=0,\qquad 0<\alpha\le 1. \tag{4.5}$$

with initial conditions, $\vartheta(\mathfrak{e},y,0)=\mathfrak{e}+y$.

Applying the Riemann–Liouville fractional integral operator of order α to both sides of (4.5) results in the corresponding integral representation.

$$\vartheta(\mathfrak{e},y,t)=\mathfrak{e}+y+\frac{1}{2}I_t^\alpha\big(\vartheta_{\mathfrak{e}\mathfrak{e}}+\vartheta_{yy}\big)-\frac{1}{2}I_t^\alpha\big(\vartheta(\vartheta_{\mathfrak{e}}+\vartheta_y)\big). \tag{4.6}$$

Applying the proposed F–J method framework, (4.6) is reformulated into the associated recursive integral representation.

$$\vartheta(\mathfrak{e},y,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D_t^{i\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^\alpha(\vartheta_{\mathfrak{e}\mathfrak{e}}+\vartheta_{yy})\right)\Bigg|_{t=0}-\frac{1}{2}I_t^\alpha\big(\vartheta(\vartheta_{\mathfrak{e}}+\vartheta_y)\big). \tag{4.7}$$

Assume that the solution can be expressed in the series form

$$\vartheta(\mathfrak{e},y,t)=\sum_{i=0}^{\infty}\vartheta_i,\qquad \vartheta\vartheta_{\mathfrak{e}}=\sum_{i=0}^{\infty}\omega_i,\qquad \vartheta\vartheta_y=\sum_{i=0}^{\infty}\nu_i. \tag{4.8}$$

By substituting (4.8) into (4.7),

$$\sum_{i=0}^{\infty}\vartheta_i = \sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D_t^{i\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}\left(\sum_{j=0}^{i-1}\vartheta_{j\mathfrak{e}\mathfrak{e}}+\sum_{j=0}^{i-1}\vartheta_{jyy}\right)\right)\Bigg|_{t=0} - \frac{1}{2}I_t^{\alpha}\left(\sum_{i=0}^{\infty}\omega_i+\sum_{i=0}^{\infty}\nu_i\right). \tag{4.9}$$

By comparing both sides of (4.9),

$$\vartheta_0 = \vartheta(\mathfrak{e},y,0) = \mathfrak{e}+y,$$

$$\begin{aligned}
\vartheta_1 &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\,{}^{c}D_t^{\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}\left(\vartheta_{0\mathfrak{e}\mathfrak{e}}+\vartheta_{0yy}\right)\right)\Bigg|_{t=0} - \frac{1}{2}I_t^{\alpha}\left(\omega_0+\nu_0\right),\\
&= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\,{}^{c}D_t^{\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}(0)\right)\Bigg|_{t=0} - \frac{1}{2}I_t^{\alpha}(2(\mathfrak{e}+y)),\\
&= -(\mathfrak{e}+y)\frac{t^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
\vartheta_2 &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,{}^{c}D_t^{2\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}\left(\vartheta_{0\mathfrak{e}\mathfrak{e}}+\vartheta_{1\mathfrak{e}\mathfrak{e}}+\vartheta_{0yy}+\vartheta_{1yy}\right)\right)\Bigg|_{t=0} - \frac{1}{2}I_t^{\alpha}\left(\omega_1+\nu_1\right),\\
&= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,{}^{c}D_t^{2\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}(0)\right)\Bigg|_{t=0}\\
&\quad - \frac{1}{2}I_t^{\alpha}\left[\begin{aligned}&(\vartheta_0+\vartheta_1)(\vartheta_0+\vartheta_1)_{\mathfrak{e}}-(\vartheta_0)(\vartheta_0)_{\mathfrak{e}}\\ &\quad+(\vartheta_0+\vartheta_1)(\vartheta_0+\vartheta_1)_y-(\vartheta_0)(\vartheta_0)_y\end{aligned}\right],\\
&= (\mathfrak{e}+y)\left(\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right),
\end{aligned}$$

$$\begin{aligned}
\vartheta_3 &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\,{}^{c}D_t^{3\alpha}\left(\mathfrak{e}+y+\frac{1}{2}I_t^{\alpha}(0)\right)\Bigg|_{t=0} - \frac{1}{2}I_t^{\alpha}\left(\omega_2+\nu_2\right),\\
&= -\frac{1}{2}I_t^{\alpha}\left[2(\mathfrak{e}+y)\left(\begin{aligned}&\left(1-\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)^2\\ &\quad-\left(1-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^2\end{aligned}\right)\right],\\
&= (\mathfrak{e}+y)\Bigg(-\frac{4t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad+\left(\frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}+\frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha}\\
&\quad-\left(\frac{4\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\right)t^{5\alpha}\\
&\quad+\frac{4\Gamma(5\alpha+1)t^{6\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)}-\frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}\Bigg).
\end{aligned}$$

The approximate solution obtained by the proposed F–J method is given as

$$\vartheta(\mathfrak{e},y,t)=(\mathfrak{e}+y)\Bigg(1-\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}-\left(4+\frac{\Gamma^2(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$+\left(\frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}+\frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha}$$
$$-\left(\frac{4\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\right)t^{5\alpha}$$
$$+\frac{4\Gamma(5\alpha+1)t^{6\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)}-\frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}\Bigg).$$

where

$$\mathfrak{s}=\mathfrak{e}+y.$$

The exact solution for $\alpha=1$ is

$$\vartheta_e(\mathfrak{s},t)=\frac{\mathfrak{s}}{1+t}.$$

TABLE 1. Numerical comparison of the approximate solution ϑ for different values of α with the exact solution.

Variables			Solutions					Absolute errors
t	$\mathfrak{e}$	y	$\vartheta_{\alpha=0.7}$	$\vartheta_{\alpha=0.8}$	$\vartheta_{\alpha=0.9}$	$\vartheta_{\alpha=1}$	ϑ_{exact}	$\lvert\vartheta_{\alpha=1}-\vartheta_e\rvert$
0.1	0.1	0.1	0.16556	0.17167	0.17709	0.18181	0.18182	0.00001
0.1	0.1	0.5	0.49667	0.51500	0.53128	0.54544	0.54545	0.00002
0.1	0.1	0.9	0.82778	0.85833	0.88547	0.90906	0.90909	0.00003
0.1	0.5	0.1	0.49667	0.51500	0.53128	0.54544	0.54545	0.00002
0.1	0.5	0.5	0.82778	0.85833	0.88547	0.90906	0.90909	0.00003
0.1	0.5	0.9	1.15890	1.20167	1.23966	1.27269	1.27273	0.00004
0.1	0.9	0.1	0.82778	0.85833	0.88547	0.90906	0.90909	0.00003
0.1	0.9	0.5	1.15890	1.20167	1.23966	1.27269	1.27273	0.00004
0.1	0.9	0.9	1.49001	1.54500	1.59385	1.63631	1.63636	0.00005
0.3	0.1	0.1	0.13678	0.14304	0.14843	0.15353	0.15385	0.00031
0.3	0.1	0.5	0.41034	0.42912	0.44528	0.46060	0.46154	0.00094
0.3	0.1	0.9	0.68390	0.71520	0.74213	0.76767	0.76923	0.00156
0.3	0.5	0.1	0.41034	0.42912	0.44528	0.46060	0.46154	0.00094
0.3	0.5	0.5	0.68390	0.71520	0.74213	0.76767	0.76923	0.00156
0.3	0.5	0.9	0.95746	1.00128	1.03899	1.07473	1.07692	0.00219
0.3	0.9	0.1	0.68390	0.71520	0.74213	0.76767	0.76923	0.00156
0.3	0.9	0.5	0.95746	1.00128	1.03899	1.07473	1.07692	0.00219
0.3	0.9	0.9	1.23103	1.28736	1.33584	1.38180	1.38462	0.00281
0.5	0.1	0.1	0.11404	0.12165	0.12713	0.13157	0.13333	0.00176
0.5	0.1	0.5	0.34213	0.36495	0.38138	0.39472	0.40000	0.00528
0.5	0.1	0.9	0.57021	0.60824	0.63563	0.65786	0.66667	0.00880
0.5	0.5	0.1	0.34213	0.36495	0.38138	0.39472	0.40000	0.00528
0.5	0.5	0.5	0.57021	0.60824	0.63563	0.65786	0.66667	0.00880
0.5	0.5	0.9	0.79830	0.85154	0.88988	0.92101	0.93333	0.01233
0.5	0.9	0.1	0.57021	0.60824	0.63563	0.65786	0.66667	0.00880
0.5	0.9	0.5	0.79830	0.85154	0.88988	0.92101	0.93333	0.01233
0.5	0.9	0.9	1.02638	1.09484	1.14413	1.18415	1.20000	0.01585

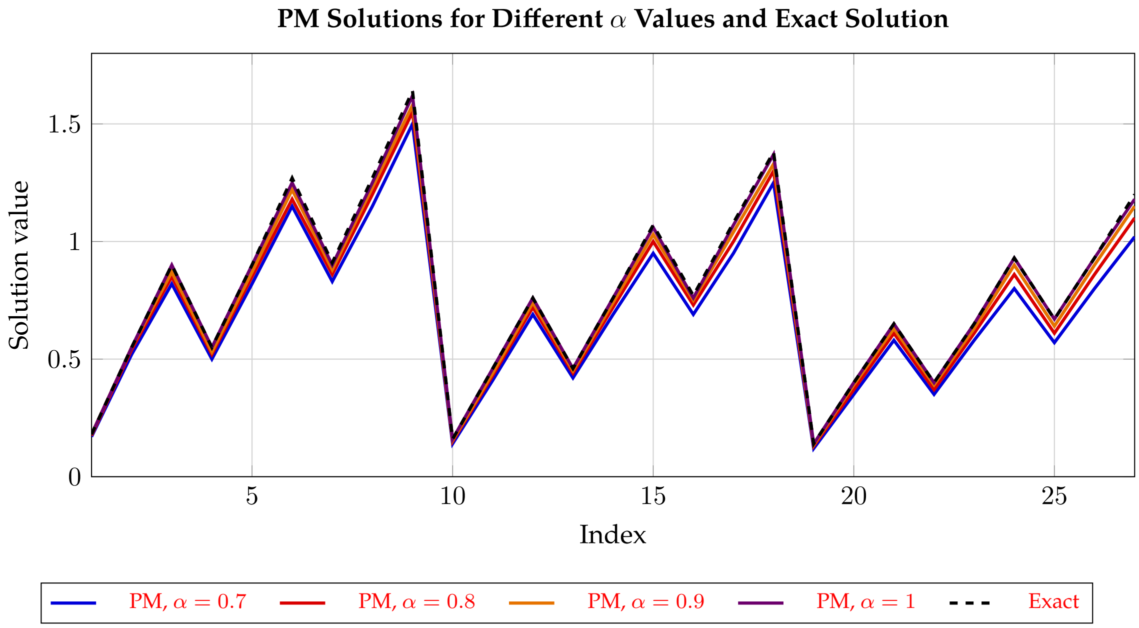

Figure 1. Graphical comparison of the Proposed Method (PM) approximate solutions for different values of α with the exact solution.

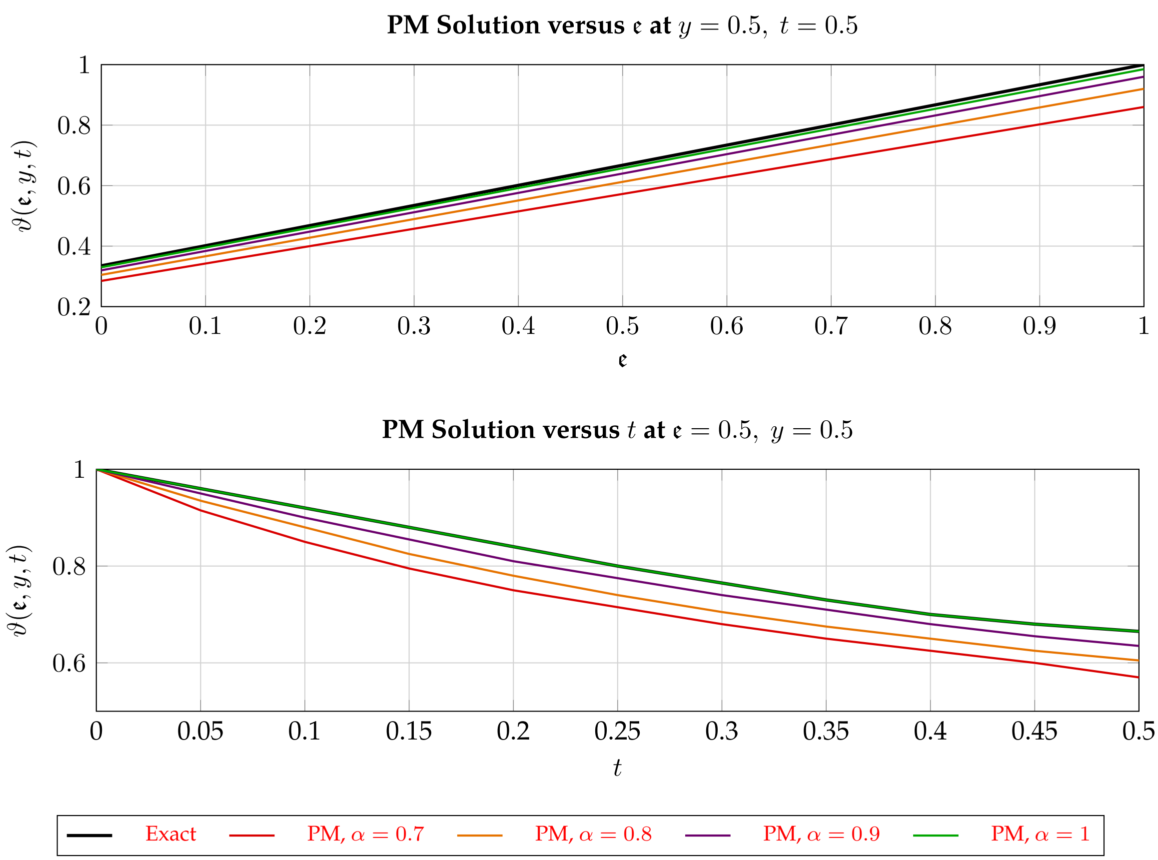

Figure 2. Graphical comparison of the Proposed Method (PM) approximate solutions with the exact solution for different values of α, versus $\mathfrak{e}$ at $y = 0.5$, $t = 0.5$, and versus t at $\mathfrak{e} = 0.5$, $y = 0.5$.

5. Conclusions

This research introduces a novel analytical method, referred to as the Falah–Jassim Method (FJM), for solving multidimensional nonlinear fractional differential equations in the Caputo sense. The proposed approach constructs approximate analytical solutions in the form of rapidly convergent recursive series. By combining the Hussein–Jassim iterative framework with the telescopic nonlinear decomposition technique, the method reduces the algebraic complexity associated with nonlinear terms and provides a reliable scheme that can be readily implemented. The illustrative examples, together with the numerical and graphical results, confirm the accuracy and applicability of the proposed method. Therefore, the FJM can be regarded as a promising addition to existing analytical and semi-analytical techniques for fractional differential equations, with potential applications in broader scientific and engineering problems.

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