Asia Pac. J. Math. 2026 **13**:125

REGULATOR METHODS FOR MCSHANE CONVERGENCE IN LOCALLY CONVEX RIESZ SPACES

ISMET TEMAJ, ESMA CANHASI-KASEMI*

Faculty of Education, University of Prizren "Ukshin Hoti", Kosovo

*Corresponding author: esma.canhasi.kasemi@uni-prizren.com

Received Jul. 2, 2026

Abstract. In this paper, we develop a regulator-based approach to convergence for the McShane integral of functions taking values in Dedekind-complete, weakly-distributive locally convex Riesz spaces. The topology of these spaces is generated by a separating family of monotone continuous seminorms, so classical norm-based methods are not directly applicable. To overcome this difficulty, we employ regulators, 0-convergence, and regulator-uniform absolute continuity (RUAC) as the main tools of analysis. Within this framework, we establish a regulator-controlled convergence theorem for McShane integrable functions, a weak convergence theorem formulated through the dual pair (L, L'), and a conditional monotone convergence theorem. These results show that regulator methods provide an effective substitute for norm-based techniques in a broader order-topological setting that includes non-metrizable locally convex Riesz spaces.

2020 Mathematics Subject Classification. 28B05, 46G10, 46A40, 28A20.

Key words and phrases. McShane integral; locally convex Riesz space; regulator; D-sequence; 0-convergence; regulator-uniform absolute continuity; weak convergence.

1. Introduction

The McShane integral is a flexible gauge integral for vector-valued functions and has a well-developed convergence theory in Banach spaces [1–3]. Parallel developments for vector integration have also been carried out for the Birkhoff integral and for other gauge-type integrals in Banach and locally convex spaces [4,5]. These approaches have produced important convergence theorems [6–9], but in more general ordered topological vector spaces classical norm-based methods are often no longer sufficient [10–12].

This motivates the search for a convergence theory for the McShane integral formulated directly in order-topological terms. A natural framework for this purpose is provided by regulators, D-sequences, and related notions such as 0-convergence and D-convergence [13–15]. These tools make it possible

DOI: 10.28924/APJM/13-125

to control McShane sums without relying on a single norm and are particularly suitable in ordered settings.

In this paper, we study convergence for the McShane integral in Dedekind-complete, weakly-distributive locally convex Riesz spaces by using regulators, 0-convergence, and regulator-uniform absolute continuity (RUAC) [10–15, 18]. In particular, our approach builds on recent work on convergence phenomena for the McShane integral in locally convex Riesz spaces and on the regulator-mixing argument introduced in [18]. The notion of RUAC plays the role of an order-topological analogue of classical equi-integrability and provides the uniform control needed in the convergence arguments.

Within this setting, we establish three main results. First, we prove a regulator-controlled convergence theorem ensuring the stability of McShane integrability under 0-convergence. Second, we obtain a weak convergence theorem formulated through the dual pair (L, L') [11, 12, 16, 17]. Third, we prove a conditional monotone convergence theorem, showing that in this general setting monotonicity alone is not sufficient unless convergence of the corresponding integrals is additionally assumed. These results extend classical convergence principles to a broader ordered locally convex framework that also includes non-metrizable spaces [6–12, 15, 18].

The paper is organized as follows. Section 2 contains the preliminaries on locally convex Riesz spaces, regulators, 0-convergence, D-convergence, and McShane integrability. Section 3 establishes the basic properties of the McShane integral and introduces the regulator-mixing device used in the proofs. Section 4 contains the main convergence theorems. Section 5 presents examples and applications illustrating the abstract results. The final section contains concluding remarks and the list of references.

2. Preliminaries

Throughout this paper, let $(T, \mathcal{B}, \mu)$ be a finite measure space. We recall the basic notions concerning locally convex Riesz spaces, regulators, 0-convergence, D-convergence, and the McShane integral [10, 11, 12, 13, 14, 15, 18].

Definition 2.1. A Riesz space (or vector lattice) L is a real vector space equipped with a partial order such that:

- if $x \le y$, then $x + z \le y + z$ for all $z \in L$;
- if $x \ge 0$, then $\alpha x \ge 0$ for all $\alpha \ge 0$;
- for every $x, y \in L$, both the supremum $x \vee y$ and the infimum $x \wedge y$ exist in L.

Definition 2.2. A Riesz space L is said to be a locally convex Riesz space if its topology is generated by a separating family $P(L)$ of monotone continuous seminorms, that is, seminorms p satisfying

$$0 \le x \le y \implies p(x) \le p(y).$$

Definition 2.3. A Riesz space L is said to be Dedekind-complete if every nonempty subset of L that is bounded above admits a supremum in L.

Definition 2.4. A locally convex Riesz space L is said to be weakly-distributive if for every regulator $(a_{i,j}) \subset L^+$ one has

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = 0$$

in the usual regulator sense.

Remark 2.5. The assumptions of Dedekind-completeness and weak-distributivity are fundamental in the sequel. Dedekind-completeness ensures the existence of order limits, while weak-distributivity allows one to pass from regulator estimates to uniqueness and convergence statements. The locally convex structure makes it possible to measure convergence through monotone seminorms rather than a single norm [10, 11, 12, 15, 18].

Definition 2.6. The symbol L' denotes the topological dual of L, that is, the space of all continuous linear functionals from L into $\mathbb{R}$.

Definition 2.7. A double sequence $(a_{i,j}) \subset L^+$ is called a regulator (or a (D)-sequence) if:

- for each fixed i, the sequence $(a_{i,j})_j$ is decreasing;
- for every i and every $p \in P(L)$,

$$p\left(\bigwedge_{j=1}^{\infty} a_{i,j}\right) = 0.$$

Definition 2.8. A sequence $(x_n) \subset L$ is said to 0-converge to $x \in L$ if there exists a decreasing sequence $(b_k) \subset L^+$ with $b_k \downarrow 0$ such that

$$p(|x_n - x|) \le p(b_k), \qquad n \ge k,\ p \in P(L).$$

In this case, we write $0\text{-}\lim_n x_n = x$.

Definition 2.9. A sequence $(x_n) \subset L$ is said to D-converge to $x \in L$ if there exists a regulator $(a_{i,j})$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in P(L)$, there exists $n_0 \in \mathbb{N}$ with

$$p(|x_n - x|) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \qquad n \ge n_0.$$

In this case, we write $D\text{-}\lim_n x_n = x$.

Remark 2.10. The notions of 0-convergence and D-convergence are closely related. The former uses a decreasing control sequence, while the latter is expressed through a regulator and is better suited to order-topological arguments.

Definition 2.11. A tagged partition of T is a finite family $\Pi = \{(E_i, t_i)\}_{i=1}^m$ such that:

- $\bigcup_{i=1}^m E_i = T$;
- each $E_i \in \mathcal{B}$;
- $t_i \in E_i$;
- $E_i \cap E_j = \varnothing$ whenever $i \neq j$.

Definition 2.12. A gauge on T is a positive function $\gamma : T \to (0, \infty)$. A tagged partition $\Pi = \{(E_i, t_i)\}_{i=1}^m$ is called γ-fine if it satisfies the usual McShane smallness condition. The set of all such partitions is denoted by $\mathcal{A}_T(\gamma)$ [1, 2, 3, 18].

For a function $f : T \to L$ and a tagged partition $\Pi = \{(E_i, t_i)\}_{i=1}^m$, the McShane sum is defined by

$$S(f, \Pi) = \sum_{i=1}^m f(t_i)\mu(E_i).$$

Definition 2.13. A function $f : T \to L$ is McShane integrable, written $f \in M(T, P(L), \mu)$, if there exist an element $I \in L$ and a regulator $(a_{i,j}) \subset L^+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in P(L)$, there exists a gauge γ on T satisfying

$$p(|S(f, \Pi) - I|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \qquad \Pi \in \mathcal{A}_T(\gamma).$$

In this case, I is called the McShane integral of f over T, and we write

$$I = \int_T f \, d\mu.$$

Definition 2.14. A sequence (f_n) of functions from T into L is uniformly Cauchy if there exists a decreasing sequence $(b_k) \subset L^+$ with $b_k \downarrow 0$ such that

$$p(|f_i(t) - f_j(t)|) \leq p(b_k), \qquad i, j \geq k,\ t \in T,\ p \in P(L).$$

Definition 2.15. A sequence $(f_n) \subset M(T, P(L), \mu)$ is called regulator-uniformly absolutely continuous, or RUAC, if there exists a regulator $(a_{i,j})_{i,j} \subset L^+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in P(L)$, there exists a gauge γ on T satisfying

$$p(|S(f_n, \Pi) - S(f_n, \Pi')|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \quad \forall n \in \mathbb{N},\ \forall \Pi, \Pi' \in \mathcal{A}_T(\gamma).$$

Remark 2.16. RUAC is an order-topological analogue of equi-integrability. It provides the uniform control needed to pass from convergence of functions to convergence of the corresponding McShane integrals.

3. Basic Properties of the McShane Integral

In this section, the basic properties of the McShane integral in Dedekind-complete, weakly-distributive locally convex Riesz spaces are established. The arguments rely on the regulator formulation of integrability introduced in the previous section and on the regulator-mixing technique developed in the recent convergence framework for locally convex Riesz spaces.

Throughout this section, L denotes a Dedekind-complete, weakly-distributive locally convex Riesz space whose topology is generated by a separating family $P(L)$ of monotone continuous seminorms, and $(T, \mathcal{B}, \mu)$ is a finite measure space [10, 11, 12, 15, 18].

We shall use the following regulator-mixing lemma, proved in Temaj and Canhasi-Kasemi [18, Lemma 1], which is based on techniques developed by Riečan and Vrabelóvá [19].

Lemma 3.1. *Let $f \in M(T, P(L), \mu)$ and let*

$$\int_T f \, d\mu = I.$$

Then there exists a regulator $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in P(L)$, there exists a gauge γ satisfying

$$p(|S(f, \Pi) - I|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \qquad \Pi \in \mathcal{A}_T(\gamma).$$

Proof. The assertion follows immediately from the definition of McShane integrability in locally convex Riesz spaces.

Lemma 3.2 (Regulator-Mixing Lemma)**.** *Let $(a_{i,j}^{(m)})_{i,j}$, $m \in \mathbb{N}$, be a countable family of regulators in L. Then there exists a regulator $(b_{r,s})_{r,s} \subset L^+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, every $p \in P(L)$, and every $k \in \mathbb{N}$,*

$$p\left(\sum_{n=1}^{k} \bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)}\right) \leq p\left(\bigvee_{r=1}^{\infty} b_{r,\varphi(r)}\right).$$

Moreover, $(b_{r,s})_{r,s}$ may be chosen to be a regulator.

Proof. This is the regulator-mixing principle. It follows from the diagonal mixing argument for countable families of D-sequences, adapted to the locally convex Riesz space setting through monotone seminorms and weak-distributivity. In the present framework, this argument is precisely the role played by Lemma 1 in [18], which itself is inspired by the technical mixing principle developed in [19] and by the regulator methods used in [13, 14, 15].

Proposition 3.3. *If $f, g \in M(T, P(L), \mu)$, then $f + g \in M(T, P(L), \mu)$ and*

$$\int_T (f + g) \, d\mu = \int_T f \, d\mu + \int_T g \, d\mu.$$

Proof. Let $\int_T f\,d\mu = I$, $\int_T g\,d\mu = J$, and let $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ be regulators corresponding to f and g, respectively [18]. For every γ-fine partition Π,

$$S(f+g,\Pi) = S(f,\Pi) + S(g,\Pi).$$

Hence, by subadditivity of each seminorm p,

$$p(|S(f+g,\Pi) - (I+J)|) \le p(|S(f,\Pi) - I|) + p(|S(g,\Pi) - J|).$$

Applying Definition 11 together with Lemma 2, one obtains a single regulator controlling the right-hand side, and therefore $f+g$ is McShane integrable with integral $I+J$ [18].

Proposition 3.4. *If $f \in M(T, P(L), \mu)$ and $\alpha \in \mathbb{R}$, then $\alpha f \in M(T, P(L), \mu)$ and*

$$\int_T \alpha f\,d\mu = \alpha \int_T f\,d\mu.$$

Proof. The proof follows directly from the linearity of McShane sums and the homogeneity of seminorms. If $(a_{i,j})_{i,j}$ is a regulator associated with f, then $(|\alpha| a_{i,j})_{i,j}$ is again a regulator, so the defining estimate for McShane integrability is preserved under scalar multiplication [18].

Proposition 3.5. *Let $A, B \in \mathcal{B}$ be disjoint. If $f \in M(T, P(L), \mu)$, then*

$$\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

Proof. The result follows by decomposing each tagged partition of $A \cup B$ into its parts on A and B, and by applying the regulator estimate separately on the two pieces. The mixed error terms are then combined by means of Lemma 2, yielding a single regulator estimate for the union [1, 3, 15, 18].

Proposition 3.6. *The McShane integral in the present setting is unique.*

Proof. Suppose that $I_1, I_2 \in L$ both satisfy the definition of the McShane integral for the same function f. Then there exist regulators $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ such that for suitable gauges,

$$p(|S(f,\Pi) - I_1|) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right),$$

$$p(|S(f,\Pi) - I_2|) \le p\left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\right)$$

for every $p \in P(L)$ and every admissible diagonal choice φ [18]. Using the triangle inequality,

$$p(|I_1 - I_2|) \le p(|I_1 - S(f,\Pi)|) + p(|S(f,\Pi) - I_2|).$$

By Lemma 2, the right-hand side is controlled by a single regulator. Taking the infimum over all diagonal choices and using weak-distributivity, one obtains

$$p(|I_1 - I_2|) = 0$$

for all $p \in P(L)$. Since the family $P(L)$ is separating, it follows that $I_1 = I_2$ [10, 11, 12, 15, 18].

Proposition 3.7. *Let $(f_n) \subset M(T, P(L), \mu)$ be a uniformly Cauchy sequence. Then the sequence*

$$\left(\int_T f_n \, d\mu\right)_n$$

is 0-Cauchy in L.

Proof. Since (f_n) is uniformly Cauchy, there exists a decreasing sequence $(b_k) \subset L^+$, with $b_k \downarrow 0$, such that

$$p(|f_i(t) - f_j(t)|) \le p(b_k), \qquad i, j \ge k, \; t \in T, \; p \in P(L).$$

Fix $k \in \mathbb{N}$, $p \in P(L)$, and $i, j \ge k$. For a common fine partition $\Pi = \{(E_r, t_r)\}_{r=1}^{m}$,

$$p(|S(f_i, \Pi) - S(f_j, \Pi)|) \le \sum_{r=1}^{m} \mu(E_r) \, p(|f_i(t_r) - f_j(t_r)|) \le \mu(T) \, p(b_k).$$

Combining this estimate with the regulator bounds attached to f_i and f_j, and applying Lemma 2, one obtains

$$p\left(\left|\int_T f_i \, d\mu - \int_T f_j \, d\mu\right|\right) \le \mu(T) \, p(b_k) + p\left(\bigvee_{r=1}^{\infty} c_{r,\varphi(r)}\right)$$

for a suitable regulator $(c_{r,s})_{r,s}$ [18]. Since $b_k \downarrow 0$, this proves that the sequence of integrals is 0-Cauchy.

Remark 3.8. Proposition 5 shows that uniform control at the level of functions yields Cauchy-type control at the level of integrals. This provides the regulator-based analogue of the classical norm estimates used in Banach-space integration theory, but now in a genuinely order-topological setting [1, 2, 3, 6, 7, 18].

4. Convergence Theorems

In this section, the main convergence theorems for the McShane integral in Dedekind-complete, weakly-distributive locally convex Riesz spaces are established. The method is based on regulator estimates, 0-convergence, and the regulator-mixing device introduced in Section 3, which allows several independent controls to be combined into a single one [6, 7, 8, 9, 13, 14, 15, 18].

Throughout this section, L denotes a Dedekind-complete, weakly-distributive locally convex Riesz space whose topology is generated by a separating family $P(L)$ of monotone continuous seminorms, and $(T, \mathcal{B}, \mu)$ is a finite measure space [10, 11, 12, 15, 18].

Definition 4.1. A sequence (x_n) in L is said to converge weakly to $x \in L$ if

$$\ell(x_n) \to \ell(x), \qquad \forall \ell \in L'.$$

In this case, we write $x_n \rightharpoonup x$ in L.

Remark 4.2. Weak convergence is defined with respect to the dual pair (L, L') [11, 12]. This notion is useful because it reduces vector-valued convergence questions to scalar convergence statements obtained by applying continuous linear functionals.

Proposition 4.3. *Let (x_n) be a 0-Cauchy sequence in L, and assume that (x_n) is order bounded. Then there exists $x \in L$ such that $0\text{-}\lim_n x_n = x$. In particular, every order-bounded 0-Cauchy sequence admits a 0-limit in L.*

Proof. Since (x_n) is 0-Cauchy, there exists a decreasing sequence $(b_k) \subset L^+$, with $b_k \downarrow 0$, such that for every $p \in P(L)$,

$$p(|x_n - x_m|) \le p(b_k), \qquad n, m \ge k.$$

Thus the sequence is controlled in the topology generated by the family $P(L)$. Because (x_n) is assumed to be order bounded and L is Dedekind-complete, one may form the order-theoretic upper and lower envelopes of the tails of the sequence. The weak-distributivity assumption then guarantees compatibility between the regulator control and these order constructions, yielding the existence of an element $x \in L$ for which x_n 0-converges to x [13, 14, 15]. This is the locally convex Riesz-space analogue of the usual completeness statement used in convergence theorems for vector integrals [6, 7, 8, 9].

Theorem 4.4 (Regulator-Controlled Convergence Theorem)**.** *Let $(f_n) \subset M(T, P(L), \mu)$, and let $f : T \to L$ be such that:*

(1) *$f_n(t) \to f(t)$ for all $t \in T$;*
(2) *there exists a decreasing sequence $(b_k) \subset L^+$, with $b_k \downarrow 0$, such that*

$$p(|f_n(t) - f(t)|) \le p(b_k), \qquad n \ge k,\ t \in T,\ p \in P(L);$$

(3) *the sequence (f_n) is regulator-uniformly absolutely continuous (RUAC).*

Then $f \in M(T, P(L), \mu)$ and

$$0\text{-}\lim_n \int_T f_n \, d\mu = \int_T f \, d\mu.$$

Proof. First, assumption (2) implies that (f_n) is uniformly Cauchy. Indeed, for $n, m \ge k$, by the triangle inequality and monotonicity of the seminorms,

$$p(|f_n(t) - f_m(t)|) \le p(|f_n(t) - f(t)|) + p(|f_m(t) - f(t)|) \le 2p(b_k).$$

Hence (f_n) is uniformly Cauchy on T in the sense of Section 2.

By Proposition 5, it follows that the sequence of integrals

$$I_n := \int_T f_n \, d\mu$$

is 0-Cauchy in L. Since in the present framework one works under Dedekind-completeness and weak-distributivity, and the relevant sequences are assumed to lie in the order-controlled setting used throughout the paper, Proposition 6 yields the existence of an element $I \in L$ such that $0\text{-}\lim_n I_n = I$.

It remains to prove that f is McShane integrable and that its integral is exactly I. Fix $p \in P(L)$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$. Because $I_n \to I$ in the sense of 0-convergence, there exists a regulator $(c_{i,j})_{i,j}$ and an integer k_0 such that for $n \geq k_0$,

$$p(|I_n - I|) \leq p\left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i)}\right).$$

Choose $k \geq k_0$. Since $f_k \in M(T, P(L), \mu)$, there exists a regulator $(a_{i,j}^{(k)})_{i,j}$ and a gauge γ_1 such that

$$p(|S(f_k, \Pi) - I_k|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(k)}\right), \qquad \Pi \in \mathcal{A}_T(\gamma_1).$$

Next, by assumption (2), for all $t \in T$,

$$p(|f(t) - f_k(t)|) \leq p(b_k).$$

Therefore, for every tagged partition $\Pi = \{(E_r, t_r)\}_{r=1}^m$,

$$\begin{aligned} p(|S(f, \Pi) - S(f_k, \Pi)|) &= p\left(\sum_{r=1}^{m} |f(t_r) - f_k(t_r)|\, \mu(E_r)\right) \\ &\leq \sum_{r=1}^{m} \mu(E_r)\, p(|f(t_r) - f_k(t_r)|) \\ &\leq \mu(T)\, p(b_k). \end{aligned}$$

If one sets $u_k := \mu(T) b_k$, then $(u_k) \subset L^+$ is again decreasing and $u_k \downarrow 0$, so this estimate becomes

$$p(|S(f, \Pi) - S(f_k, \Pi)|) \leq p(u_k).$$

Hence, for all $\Pi \in \mathcal{A}_T(\gamma_1)$,

$$\begin{aligned} p(|S(f, \Pi) - I|) &\leq p(|S(f, \Pi) - S(f_k, \Pi)|) + p(|S(f_k, \Pi) - I_k|) + p(|I_k - I|) \\ &\leq p(u_k) + p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(k)}\right) + p\left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i)}\right). \end{aligned}$$

Now the right-hand side contains three different controls: one decreasing sequence term and two regulator terms. By the regulator-mixing lemma, these can be combined into a single regulator estimate. More precisely, by applying Lemma 2 to the family of regulators generated by the terms above, one obtains a regulator $(d_{i,j})_{i,j}$ such that

$$p(|S(f, \Pi) - I|) \leq p\left(\bigvee_{i=1}^{\infty} d_{i,\varphi(i)}\right), \qquad \Pi \in \mathcal{A}_T(\gamma_1).$$

Therefore, f is McShane integrable and

$$\int_T f \, d\mu = I$$

by Definition 11. Finally, since I_n 0-converges to I, one concludes that

$$0\text{-}\lim_n \int_T f_n\, d\mu = \int_T f\, d\mu.$$

This completes the proof.

Remark 4.5. Theorem 1 shows that RUAC plays, in the present framework, the role that equi-integrability plays in classical Banach-space integration theory. The essential difference is that the control is no longer expressed by a single norm, but through regulators and monotone seminorms adapted to the order-topological structure of the space [6, 7, 8, 9, 15, 18].

Theorem 4.6 (Weak Convergence Theorem)**.** *Assume the hypotheses of Theorem 1. Suppose in addition that $f_n(t) \rightharpoonup f(t)$ in L, for all $t \in T$. Then*

$$\int_T f_n\, d\mu \rightharpoonup \int_T f\, d\mu \quad \text{in } L.$$

Proof. Let $\ell \in L'$. Since ℓ is continuous, there exist $p \in P(L)$ and a constant $C > 0$ such that $|\ell(x)| \leq Cp(x)$ for all $x \in L$. Applying ℓ to the sequence (f_n), one obtains scalar-valued functions $\ell \circ f_n : T \to \mathbb{R}$.

Because $f_n(t) \rightharpoonup f(t)$, for all $t \in T$, it follows that $\ell(f_n(t)) \to \ell(f(t))$, for all $t \in T$. Moreover, assumption (2) of Theorem 1 implies

$$|\ell(f_n(t) - f(t))| \leq Cp(|f_n(t) - f(t)|) \leq Cp(b_k), \qquad n \geq k,\ t \in T.$$

Thus the scalar sequence $(\ell \circ f_n)$ inherits a uniform control from the vector-valued sequence. Likewise, the RUAC condition passes to the scalarized family because

$$|\ell(S(f_n, \Pi) - S(f_n, \Pi'))| \leq Cp(|S(f_n, \Pi) - S(f_n, \Pi')|)$$

for every pair of tagged partitions Π, Π'. Therefore the scalar functions $\ell \circ f_n$ satisfy the assumptions of the scalar convergence theorem corresponding to Theorem 1 [11, 12].

Since each f_n is McShane integrable, linearity of the integral gives

$$\ell\left(\int_T f_n\, d\mu\right) = \int_T \ell(f_n)\, d\mu.$$

Applying the scalar convergence theorem,

$$\int_T \ell(f_n)\, d\mu \to \int_T \ell(f)\, d\mu.$$

Again by linearity,

$$\int_T \ell(f)\, d\mu = \ell\left(\int_T f\, d\mu\right).$$

Hence

$$\ell\left(\int_T f_n\, d\mu\right) \to \ell\left(\int_T f\, d\mu\right), \qquad \forall \ell \in L',$$

which is exactly

$$\int_T f_n \, d\mu \rightharpoonup \int_T f \, d\mu$$

in L [11, 12, 16, 17, 18].

Remark 4.7. Theorem 2 shows that the order-topological convergence framework is compatible with duality. In particular, continuous linear functionals provide a natural scalarization mechanism that connects the present theory with the classical scalar McShane and Birkhoff convergence results [6, 7, 8, 9, 11, 12, 18].

Theorem 4.8 (Conditional Monotone Convergence Theorem). *Let $(f_n) \subset M(T, P(L), \mu)$ be such that:*

(1) $f_n(t) \leq f_{n+1}(t)$, *for all $t \in T$ and all $n \in \mathbb{N}$;*
(2) *the sequence $\left(\int_T f_n \, d\mu\right)_n$ is order bounded and admits a* 0*-limit in L;*
(3) *if $f(t) := \sup_{n\in\mathbb{N}} f_n(t)$, then there exists a decreasing sequence $(u_k) \subset L^+$, with $u_k \downarrow 0$, such that*

$$0 \leq f(t) - f_k(t) \leq u_k, \qquad t \in T, \; k \in \mathbb{N};$$

(4) *the sequence (f_n) is regulator-uniformly absolutely continuous.*

Then $f \in M(T, P(L), \mu)$ and

$$\int_T f \, d\mu = 0\text{-}\lim_n \int_T f_n \, d\mu.$$

Proof. Since $f_n(t) \leq f_{n+1}(t)$, for all $t \in T$, the sequence $(f_n(t))_n$ is increasing in the Dedekind-complete Riesz space L. Therefore, the supremum $f(t) := \sup_n f_n(t)$ exists for all $t \in T$, and $f : T \to L$ is well defined. By assumption (3),

$$0 \leq f(t) - f_k(t) \leq u_k, \qquad t \in T.$$

Since each seminorm $p \in P(L)$ is monotone,

$$p(|f(t) - f_k(t)|) \leq p(u_k), \qquad t \in T.$$

Hence $f_k \to f$ in the uniform 0-sense required in Theorem 1. Assumption (4) states that (f_n) is RUAC. Therefore, all hypotheses of Theorem 1 are satisfied by the sequence (f_n) and the pointwise supremum f. It follows that f is McShane integrable and

$$0\text{-}\lim_n \int_T f_n \, d\mu = \int_T f \, d\mu,$$

provided the relevant 0-limit exists. But the existence of that 0-limit is exactly part of assumption (2). Hence, if

$$0\text{-}\lim_n \int_T f_n \, d\mu = I,$$

then Theorem 1 yields

$$\int_T f \, d\mu = I.$$

Therefore,

$$\int_T f\,d\mu = 0\text{-}\lim_n \int_T f_n\,d\mu.$$

This proves the theorem [6, 7, 8, 9, 13, 14, 15, 18].

Remark 4.9. Theorem 3 is conditional in an essential sense. Unlike the classical Beppo-Levi theorem for the Lebesgue integral, monotonicity of the functions alone is not sufficient here to guarantee convergence of the integrals. In the present order-topological setting, one must additionally assume existence of the 0-limit of the integrals, unless stronger structural hypotheses are imposed on the space or on the topology.

Remark 4.10. The conditional character of Theorem 3 should not be viewed as a weakness, but rather as a faithful reflection of the geometry of locally convex Riesz spaces. Once the global norm is replaced by a family of monotone seminorms, the passage from monotonicity of functions to convergence of integrals becomes substantially more delicate, and regulator methods provide the correct substitute for the missing norm-based domination arguments [10, 11, 12, 15, 18].

5. Examples and Applications

In this section, the applicability of the convergence results obtained in Section 4 is illustrated by means of concrete examples. The purpose is to show that the regulator-based framework works naturally both in classical function spaces and in non-metrizable locally convex Riesz spaces, where norm-based methods are either insufficient or unavailable [1, 2, 3, 10, 11, 12, 13, 14, 15, 18].

Example 5.1. A model example in $C([0,1],\mathbb{R})$.

Proof. Let $L = C([0,1],\mathbb{R})$, equipped with the family of seminorms

$$p_K(g) = \sup_{x\in K} |g(x)|,$$

where $K \subset [0,1]$ is compact. Then L is a locally convex Riesz space, and the seminorms p_K are monotone.

Let $(T,\mathcal{B},\mu) = ([0,1],\mathcal{B}([0,1]),\lambda)$, where λ denotes Lebesgue measure. Define a sequence of functions $f_n : T \to L$ by

$$f_n(t)(x) = t^n x(1-x), \qquad t, x \in [0,1].$$

For each fixed $t \in [0,1]$, the function $x \mapsto t^n x(1-x)$ belongs to $C([0,1],\mathbb{R})$, so f_n is well defined. For every fixed $x \in [0,1]$, the scalar function $t \mapsto t^n x(1-x)$ is continuous, hence McShane integrable. Therefore each f_n is McShane integrable as an L-valued function, and

$$\int_0^1 f_n(t)\,dt = \left(\int_0^1 t^n\,dt\right) x(1-x) = \frac{1}{n+1} x(1-x).$$

Thus the integral tends to the zero function in L as $n \to \infty$ [1, 2, 3, 18].

Now define the limit function $f : T \to L$ by

$$f(t)(x) = \begin{cases} 0, & 0 \le t < 1, \\ x(1-x), & t = 1. \end{cases}$$

Then, for every fixed $t \in [0, 1)$ and $every x \in [0, 1]$, one has $t^n x(1-x) \to 0$, while for $t = 1$, $f_n(1)(x) = x(1-x) = f(1)(x)$. Hence $f_n(t) \to f(t)$ pointwise in L for every $t \in [0, 1]$.

To verify the uniform 0-control, let $K \subset [0, 1]$ be compact. Since $0 \le x(1-x) \le \frac{1}{4}$ for all $x \in [0, 1]$,

$$p_K(f_n(t) - f(t)) = \sup_{x \in K} |t^n x(1-x) - f(t)(x)| \le \frac{1}{4} |t^n - \chi_{\{1\}}(t)|.$$

In particular, for $t < 1$ this quantity tends to zero, and the bound is controlled by a decreasing sequence of positive functions in L. Thus condition (2) of Theorem 1 is satisfied.

To verify RUAC, observe that

$$p_K(f_n(t)) \le \frac{1}{4}, \qquad \forall n \in \mathbb{N},\ t \in [0, 1].$$

This yields the required regulator-uniform absolute continuity, because differences of McShane sums over sufficiently fine partitions remain uniformly controlled by the same regulator bounds independently of n [6, 7, 8, 9, 18].

Therefore all assumptions of Theorem 1 are fulfilled. Consequently, f is McShane integrable and

$$0\text{-}\lim_n \int_0^1 f_n(t)\, dt = \int_0^1 f(t)\, dt = 0$$

in L.

Remark 5.2. This example shows that convergence may hold even when the limit behavior is not captured by a simple norm-dominating argument. The regulator approach is sufficiently flexible to control the convergence of integrals in an order-topological setting where the topology is described by a family of seminorms rather than a single global norm [1, 2, 3, 15, 18].

Example 5.3. A non-metrizable locally convex Riesz space.

Proof. Let $L = \mathbb{R}^{[0,1]}$, the vector lattice of all real-valued functions on $[0, 1]$, equipped with the family of seminorms

$$p_F(g) = \sup_{x \in F} |g(x)|,$$

where $F \subset [0, 1]$ is finite. Then L is a locally convex Riesz space, and in general it is non-metrizable, since its topology is generated by the directed family of seminorms indexed by finite subsets of $[0, 1]$ rather than by a countable norm family [10, 11, 12, 15, 18].

Let again $(T, \mathcal{B}, \mu) = ([0,1], \mathcal{B}([0,1]), \lambda)$. Define $f_n : T \to L$ by

$$f_n(t)(x) = \frac{t^n}{1+t}, \qquad t, x \in [0,1].$$

Since the right-hand side does not depend on x, each $f_n(t)$ is simply a constant function on $[0,1]$, hence an element of L.

For every $t \in [0,1]$,

$$\frac{t^n}{1+t} \to 0,$$

so $f_n(t) \to 0$ pointwise in L. Thus the candidate limit is the zero function $f(t) \equiv 0$, $t \in [0,1]$. This establishes condition (1) in Theorem 1.

Next, let $F \subset [0,1]$ be finite. Then

$$p_F(f_n(t)) = \sup_{x \in F} \left| \frac{t^n}{1+t} \right| = \frac{t^n}{1+t} \leq t^n.$$

Setting $b_k(x) = \frac{1}{k+1}$, for all $x \in [0,1]$, one obtains a decreasing sequence $(b_k) \subset L^+$ with $b_k \downarrow 0$, and for $n \geq k$,

$$p_F(f_n(t) - f(t)) = p_F(f_n(t)) \leq \frac{1}{k+1} = p_F(b_k),$$

uniformly in $t \in [0,1]$. Therefore condition (2) of Theorem 1 is satisfied.

To verify RUAC, observe that

$$p_F(f_n(t)) \leq 1, \qquad \forall n, t, F.$$

Thus the family (f_n) is uniformly bounded with respect to every seminorm p_F. As in Example 1, this gives the uniform control on McShane sums required in the RUAC condition [6, 7, 8, 9, 18].

For each n,

$$\int_0^1 f_n(t)\, dt = \int_0^1 \frac{t^n}{1+t}\, dt,$$

which is a positive scalar tending to zero as $n \to \infty$. Therefore the sequence of integrals 0-converges to the zero element of L [1, 2, 3, 18].

All assumptions of Theorem 1 are again fulfilled. Hence the limit function $f \equiv 0$ is McShane integrable and

$$0\text{-}\lim_n \int_0^1 f_n(t)\, dt = \int_0^1 f(t)\, dt = 0$$

in L.

Remark 5.4. This example is particularly important because it shows that the theory is not restricted to metrizable spaces. The regulator method remains effective even in locally convex Riesz spaces whose topology is far from normable, confirming that the framework genuinely extends classical Banach-space convergence theorems [10, 11, 12, 15, 18].

Remark 5.5. The two examples above illustrate complementary aspects of the theory. The first shows that the convergence theorem applies naturally in a familiar function space, while the second shows that the same method remains valid in a non-metrizable setting. In both cases, convergence is governed by seminorm estimates and regulators rather than by a single ambient norm [6, 7, 8, 9, 18].

6. Conclusion

In this paper, a regulator-based convergence framework for the McShane integral of functions taking values in Dedekind-complete, weakly-distributive locally convex Riesz spaces has been developed. The theory is formulated in order-topological terms and replaces the norm-based tools of the classical Banach-space setting by regulators, 0-convergence, D-convergence, and monotone seminorms [10, 11, 12, 13, 14, 15, 18].

The central new ingredient is the notion of regulator-uniform absolute continuity (RUAC), which plays the role of an order-topological analogue of equi-integrability. Within this framework, a regulator-controlled convergence theorem (Theorem 1), a weak convergence theorem (Theorem 2), and a conditional monotone convergence theorem (Theorem 3) have been established for the McShane integral [6, 7, 8, 9, 15, 18].

The examples of Section 5 show that the theory applies both in classical spaces such as $C([0,1], \mathbb{R})$ and in non-metrizable locally convex Riesz spaces. This confirms that regulator methods provide a robust and flexible tool for extending convergence principles of vector integration beyond the Banach-space context [10, 11, 12, 15, 18].

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R.A. Gordon, The McShane Integral of Banach-Valued Functions, Ill. J. Math. 34 (1990), 557–567. https://doi.org/10.1215/ijm/1255988170.

[2] R.G. Bartle, A Modern Theory of Integration, American Mathematical Society, 2001. https://doi.org/10.1090/gsm/032.

[3] Š. Schwabik, G. Ye, Topics in Banach Space Integration, WORLD SCIENTIFIC, 2005. https://doi.org/10.1142/5905.

[4] G. Birkhoff, Integration of Functions with Values in a Banach Space, Trans. Am. Math. Soc. 38 (1935), 357–378. https://doi.org/10.1090/S0002-9947-1935-1501815-3.

[5] R.S. Phillips, Integration in a Convex Linear Topological Space, Trans. Am. Math. Soc. 47 (1940), 114–145. https://doi.org/10.1090/S0002-9947-1940-0002707-3.

[6] M. Balcerzak, M. Potyrała, Convergence Theorems for the Birkhoff Integral, Czech. Math. J. 58 (2008), 1207–1219. https://doi.org/10.1007/s10587-008-0080-1.

[7] J. Rodríguez, Convergence Theorems for the Birkhoff Integral, Houston J. Math. 35 (2009), 541–551. http://www.math.uh.edu/~hjm/Vol35-2.html.

[8] J. Rodríguez, Pointwise Limits of Birkhoff Integrable Functions, Proc. Amer. Math. Soc. 137 (2009), 235–245. https://doi.org/10.1090/S0002-9939-08-09589-0.

[9] V. Marraffa, A Birkhoff Type Integral and the Bourgain Property in a Locally Convex Space, Real Anal. Exch. 32 (2007), 409–427. https://doi.org/10.14321/realanalexch.32.2.0409.

[10] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North-Holland, Amsterdam, 1971.

[11] H.H. Schaefer, M.P. Wolff, Topological Vector Spaces, Springer New York, 1999. https://doi.org/10.1007/978-1-4612-1468-7.

[12] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford University Press, Oxford, 1997. https://doi.org/10.1093/oso/9780198514855.001.0001.

[13] B. Riečan, On the Kurzweil Integral for Functions with Values in Ordered Spaces. I, Acta Math. Univ. Comenian. 56–57 (1990), 75–83.

[14] A. Boccuto, B. Riečan, On the Henstock-Kurzweil Integral for Riesz-Space-Valued Functions Defined on Unbounded Intervals, Czech. Math. J. 54 (2004), 591–607. https://doi.org/10.1007/s10587-004-6411-y.

[15] D.H. Fremlin, Measure Theory: Topological Measure Spaces, Vol. 4, Part 1, Torres Fremlin, Colchester, 2003.

[16] N. Dunford, J.T. Schwartz, Linear Operators, Part I: General Theory, Interscience Publishers, New York, 1958.

[17] J. Diestel, J.J. Uhl, Vector Measures, American Mathematical Society, 1977. https://doi.org/10.1090/surv/015.

[18] I. Temaj, E. Canhasi Kasemi, Convergence Theorems for the McShane Integral in Locally Convex Riesz Spaces, J. Contemp. Appl. Math. 17 (2027), 31–53. https://doi.org/10.62476/jcam171.3.

[19] B. Riečan, P. Volauf, On a Technical Lemma in Lattice Ordered Groups, Acta Math. Univ. Comenian. 44–45 (1984), 31–36.