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PHASE-TYPE DISTRIBUTION AND BIVARIATE RUIN PROBABILITY IN FINITE TIME IN A HAWKES RISK MODEL

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Abstract. This article contributes to the bivariate risk approach of occurrence of finite-time claims. We address a risk problem based on a model where the initial reserve u grows exponentially at an interest rate r, including not only an integral expression that indicates that each contribution to the reserve is discounted at time t according to the interest rate, but also the effect of Brownian oscillation capturing the random fluctuations that may influence the reserve over time. We consider that $N(t)$, the total number of claims that have occurred up to time t, is built on the basis of a Hawkes process, and Z_i, the amount of the i-th claim, is a phase-type random variable. We arrive at the asymptotic approximation of the ruin probability and establish a closed-form formula for the bivariate finite time ruin probability using the Clayton copula.

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1. Introduction

In risk theory, one of the essential concerns of insurers is the assessment of ruin risk. Cramer and Lundberg demonstrated that the ruin probability, taken as a risk measure, provides a better approximation of the long-term behavior of the insurance company. In the articles [6,9,12], the authors examined the finite-time ruin probability through a jump diffusion Poisson process. They consider that the claim amounts are exponentially distributed and that the interest force is constant. Furthermore, they obtain asymptotic formulas for the finite-time ruin probability. In this article [4], the author uses an excess process given by the renewal risk model with a constant interest force and a Brownian disturbance. Additionally, in [7], the author focuses on a risk model that examines the ruin probability over a time horizon, where the inter-arrival times of claims follow a Hawkes process with variable memory, the claim amounts obey an exponential distribution, and Brownian oscillations affect the

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reserve. Since the infinite-time ruin probability unfolds over an infinite period, the finite-time ruin probability relies on a finite time. This has become a very interesting concern for insurers. In this article, we focus on a risk model [11] that examines the ruin probability over a finite space. The objective of our work is to approximate the multivariate finite-time ruin probability of a risk model where the initial reserve u grows exponentially at an interest rate r, including not only an integral expression that indicates that each contribution to the reserve is discounted at time t according to the interest rate, but also the effect of Brownian oscillation capturing the random fluctuations that may influence the reserve over time. To achieve this, we consider that $N(t)$, the total number of claims that have occurred up to time t, is built on the basis of a Hawkes process, and Z_i, the amount of the i-th claim, is a phase-type random variable. We draw inspiration from the work of [3,5] for determining this ruin probability. More specifically, we estimate the risk probability associated with the reserve based heavily on the works of authors such as [10,12,13,16], and then we approach the bivariate risk by establishing an asymptotic closed formula for the ruin probability based on a Clayton copula.

2. Preliminaries

Consider the classical compound Poisson risk model defined by the following equation:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \quad t > 0 \tag{1}$$

where

u is the initial reserve of the insurance company.

c is the premium per unit of time.

X_i denotes the claim amount and is independent of the process $N(t)$.

$N(t)$ represents the number of claims that occur at time t.

The ruin time is the time when the company's reserve becomes negative. It is defined by:

$$\tau = \inf\{t \geq 0; R(t) < 0\}. \tag{2}$$

The finite-time ruin probability is defined by the following equation:

$$\psi(u,T) = \mathbb{P}\left(\inf_{1 \leq t \leq T}\{R(t) \leq 0; R(0) = u\}\right). \tag{3}$$

Definition 2.1.

Let X_i be a random variable with phase-type distribution parameters $(\xi,\ \mathbf{B})$. Then its density function is defined by:

$$f(t) = \xi e^{\mathbf{B}t} k \ \ \textit{with} \ \ k = -\mathbf{B}\mathbf{1} \ \ \textit{where} \ \ \mathbf{1} = (1, 1, \cdots, 1)$$

where

ξ is a vector of initial probabilities of the phases. It is a row vector;

$\mathbf{B}$ *is a matrix of transition rates between the phases. It is a* $n \times n$ *matrix;*

k *is a matrix of dimension* $n \times 1$ *and gives the transition rates to the absorbing state;*

$\mathbf{1}$ *is a column vector used to sum the elements of each row of* $\mathbf{B}$ *in the case of matrix multiplication.*

The excitation function of the phase-type distribution can be either increasing or decreasing. In our work, we take the case where it is decreasing and given by

$$\mu(t) = \xi e^{-\mathbf{B}t}\mathbf{1}.$$

In this work, we use phase-type distributions for the claim amounts, as these distributions are suited for modeling heavy tails and asymmetry in finite-time ruin probability. These distributions also allow modeling extreme events.

Definition 2.2.

Consider all these definitions

(1) *A random variable* X *with distribution* F *is said to have a heavy tail if and only if*

$$\int_{\mathbb{R}} e^{\lambda x} F(x) dF(x) = \infty \quad \forall \quad \lambda > 0.$$

(2) *A distribution function* F *defined on* $(0, \infty)$ *is sub-exponential if for all* $n > 2$ *we have:*

$$\lim_{x \to \infty} \frac{\bar{F}^{n^*}(x)}{\bar{F}(x)} = n.$$

where F^{n^*} *is the n-th convolution of* F*. Let* S *be the class of sub-exponential functions.*

Definition 2.3.

Let F *be a function supported on* $(0, \infty)$*.* F *is said to vary slowly at infinity if*

$$\lim_{x \to \infty} \frac{F(lx)}{F(x)} = 1, \quad l > 0.$$

The function F *belongs to the class of regular variation if for* $\gamma > 0$

$$\lim_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\gamma}, \text{ for } y > 0.$$

We have $F \in \mathcal{R}_{-\gamma}$*. If* $\ell \in \mathcal{R}_0$ *and* $F \in \mathcal{R}_{-\gamma}$*, then* $\bar{F}(x) = x^{-\gamma} k(x)$*, where* $k(x)$ *is a slowly varying function and* $\mathcal{R}_{-\gamma}$ *designates a class of regular variation with* γ *being the tail index.*

3. Results

This section is dedicated to presenting the results of our work. We determine the ruin probability over a finite horizon, where the inter-arrival times of claims follow a Hawkes process and the claim amounts obey a phase-type distribution $PH(\xi; \mathbf{B})$. Let us assume that the inter-arrival times form a sequence of non-negative i.i.d. random variables $\{\theta_i, i \geq 1\}$ following a distribution function H, and denote by $\tau_k = \sum_{i=1}^{k} \theta_i$, $k \geq 1$, the arrival times of the claims.

This theorem will allow us to determine the finite-time ruin probability.

Theorem 3.1.

Consider the risk model given in (1) in which $N(t)$ is a Hawkes counting process, and consider the risk model given by the equation below.

$$R(t) = ue^{rt} + c\int_0^t e^{r(t-s)}ds - \sum_{i=1}^{N(t)} Z_i e^{r(t-\tau_i)} + \delta \int_0^t e^{r(t-s)} dB(s). \tag{4}$$

Let ψ be the finite-time ruin probability associated with the model (4); $\{Z_i; i \geq 1\}$ is a sequence of i.i.d. random variables with phase-type distribution $PH(\xi; \mathbf{B})$ and common distribution F representing the claim amounts. Suppose that $\mathbf{B}$ has at least one eigenvalue that is zero or very small, then

$$\psi(u, T) \sim \eta(T)\bar{F}(u). \tag{5}$$

To prove this theorem, let us first consider the following lemma.

Lemma 3.1.

Let X be a random variable whose survival function satisfies

$$\overline{F}(x) \sim \xi e^{\mathbf{B}x}\mathbf{1} \quad \text{as } x \to \infty,$$

where ξ is a stochastic row vector, $\mathbf{1}$ is the column vector of ones, and $\mathbf{B}$ is a substochastic matrix with eigenvalues that are zero or very small (close to zero). Then $X \in \mathcal{R}_{-\gamma}$ for some $\gamma > 0$.

The proof is formulated as follows:

Proof.

First, suppose that $\mathbf{B}$ is diagonalizable. Then

$$\mathbf{B} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

with $\mathbf{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $\max|\lambda_k| < 1$. Thus,

$$e^{\mathbf{B}x} = \sum_{k=1}^{m} \pi_k e^{\lambda_k x} \mathbf{v}_k \mathbf{u}_k^\top + o(e^{(\rho+\epsilon)x}),$$

where $\rho = \max_k \mathrm{Re}(\lambda_k) < 0$, $\pi_k = \xi \mathbf{p}_k$, and $(\mathbf{v}_k, \mathbf{u}_k)$ are associated projectors. Thus, from [10],

$$\xi e^{\mathbf{B}x}\mathbf{1} \sim ce^{\rho x}, \quad x \to \infty,$$

for $c = \sum_{\rho_k = \rho} \pi_k \mathbf{u}_k^\top \mathbf{1} > 0$.

Now, suppose that $\mathbf{B}$ has an eigenvalue $\lambda^* = 0$ (or ≈ 0), then

$$e^{\mathbf{B}x} \sim \mathbf{P}_0 + O(x^r e^{-\delta x}),$$

where $\mathbf{P}_0$ is the spectral projector onto the eigenspace associated with 0, and r is the Jordan index. Thus,

$$\overline{F}(x) \sim \xi \mathbf{P}_0 \mathbf{1} =: L(x) > 0,$$

where L is a slowly varying function.
Finally, considering that $|\lambda_k| \ll 1$, $\rho \approx 0^-$, we have

$$\overline{F}(x) \sim ce^{\rho x} \sim c(1 + \rho x + \cdots) \approx c.$$

The fact that $\lambda \approx 0$ with multiplicity implies, from [10], that there exists $\gamma > 0$ such that

$$\overline{F}(x) \sim x^{-\gamma} L(x),$$

which gives the result. □

Now let's show that the phase-type distribution is slowly varying. Consider the result

Lemma 3.2.
Let X be a random variable with distribution $PH(\xi, \mathbf{B})$ and $\mathbf{B}$ the transition matrix with one or more eigenvalues that are zero or very small. Then $PH(\xi, \mathbf{B})$ is slowly varying.

The proof is given by

Proof.
From spectral theory of a diagonalizable matrix and using Jordan block decomposition, we can write

$$e^{\mathbf{B}t} = \sum_{j=1}^{k} e^{\lambda_j t} \Pi_j,$$

where Π_j are projectors (or matrix polynomials) associated with the Jordan blocks of $\mathbf{B}$. See [20].
Inserting into the expression for $\bar{F}(t)$, we obtain

$$\bar{F}(t) = \xi e^{\mathbf{B}t} \mathbf{1} = \sum_{j=1}^{k} e^{\lambda_j t} \, \xi \, \Pi_j \, \mathbf{1}.$$

Let $c_j = \xi \, \Pi_j \, \mathbf{1}$. If $\Re(\lambda_j) < 0$, then $|e^{\lambda_j t}| \to 0$ exponentially fast. On the other hand, if $\Re(\lambda_1) \approx 0$, then, from [18], $e^{\lambda_1 t}$ decays very slowly (or remains nearly constant if $\lambda_1 = 0$). For large t, the dominant term corresponds to the smallest real part:

$$\bar{F}(t) \sim c_1 \, e^{\lambda_1 t}.$$

If $\lambda_1 = 0$, then $e^{\lambda_1 t} = 1$ and $\bar{F}(t)$ behaves like a constant, which is a trivial case of slow variation. From [17], if λ_1 is a non-zero complex number but very close to 0, we can still write

$$e^{\lambda_1 t} = e^{\Re(\lambda_1)t + i\Im(\lambda_1)t} \sim \ell(t),$$

where $\ell(t)$ is a complex function with a quasi-constant modulus over large intervals, which corresponds to the definition of a slowly varying function. For this effect, see [17] and [18].

Taking the modulus of the survival function, we obtain

$$|\bar{F}(t)| \sim c\,\ell(t),$$

where $\ell(t)$ is slowly varying, which means that the distribution of X is slowly varying in the tail. See [18]. □

Knowing that $PH(\xi, \mathbf{B}) \in \mathcal{L}$, let us use the Karamata theorem to approximate the results of a heavy-tailed distribution.

Lemma 3.3 (Karamata's Theorem)**.**
Let $X \sim PH(\xi, \mathbf{B})$ and $\mathbf{B}$ have eigenvalues that are zero or very small. Let $\overline{F}(x)$ be the survival function of X. Then

$$\overline{F}(x) \sim \frac{C}{x^{\beta}}, \quad \textit{with } \beta > 0. \tag{6}$$

Thus, by applying Karamata's theorem, we can write:

$$\int_{x_0}^{x} \overline{F}(t)dt \sim \frac{C}{\beta - 1} x^{1-\beta} \quad \textit{as } x \to \infty. \tag{7}$$

Proof.
The proof of this lemma relies on the fundamental property of regularly varying functions and their integral behavior.
Suppose that $\mathbf{B}$ has an eigenvalue λ with $|\lambda|$ being zero or very small (close to zero). For phase-type distributions $PH(\xi, \mathbf{B})$, from [18], when the eigenvalues of $\mathbf{B}$ are small, the survival function $\overline{F}(x) = \mathbb{P}(X > x)$ has a heavy-tailed regular asymptotic type.

More precisely, it follows that

$$\overline{F}(x) \sim \frac{C}{x^{\beta}}, \quad x \to \infty,$$

where $C > 0$ is an explicit constant depending on ξ, $\mathbf{B}$, and $\beta > 0$ is the regular variation index determined by the imaginary part or the modulus of the relevant eigenvalues of $\mathbf{B}$.

By definition (see [21]), this asymptotic behavior means that $\overline{F}(x)$ is regularly varying with index $-\beta$, that is, $\overline{F}(\lambda x) \sim \lambda^{-\beta}\overline{F}(x)$ for all $\lambda > 0$, as $x \to \infty$. Applying the integral representation theorem for regularly varying functions with index $\rho = -\beta < 0$, we have the asymptotic of the truncated integral:

$$\int_{x_0}^{x} \overline{F}(t)\,dt \sim \frac{\overline{F}(x) \cdot x}{1 + \rho} = \frac{\overline{F}(x) \cdot x}{1 - \beta}, \quad x \to \infty.$$

Substituting the asymptotic of $\overline{F}(x) \sim Cx^{-\beta}$ and under the assumption $\beta > 1$, or for $0 < \beta < 1$, [20] allows us to obtain:

$$\int_{x_0}^{x} \overline{F}(t)\, dt \sim \frac{Cx^{-\beta} \cdot x}{1-\beta} = \frac{C}{1-\beta} x^{1-\beta}, \quad x \to \infty,$$

which gives the result. □

Now consider the following lemma

Lemma 3.4.

Consider the risk renewal process with surplus defined in (4). Let ψ be the ruin probability associated with the risk model (4); $\{Z_i; i \geq 1\}$ is a sequence of i.i.d. random variables with phase-type distribution $PH(\xi, \mathbf{B})$ with common distribution F. $\{B_t; t \geq 0\}$ and $\{N(t); t \geq 1\}$ are mutually independent. Let $\Lambda = \{t > 0; \lambda(t) > 0\}$. Consider $T \in \Lambda$ and $t \in \Lambda_T = \Lambda \cap [0, T]$, then

$$\psi(u,t) \sim \int_{0^-}^{t} \overline{F}(ue^{rs}) d\lambda(s). \tag{8}$$

To prove lemma (3.4), let us first consider the following lemma

Lemma 3.5.

For a fixed n such that $n \geq 1$, let Z_k, $(k = 1, ..., n)$ be a random variable with asymptotically independent tails (TAI) with distribution F_k. Then

$$P\left(\sum_{k=1}^{n} C_k Z_k > x\right) \sim \sum_{k=1}^{n} P(C_k Z_k > x) \tag{9}$$

holds uniformly for all $C_1, C_2, ..., C_n \in [a, b]$; $0 < a \leq b < \infty$. Moreover, if Y is a random variable with distribution G on $(-\infty, +\infty)$ that satisfies $\overline{G}(x) = o(1)\overline{F}(x)$ and is independent of Z_k, then for any fixed $0 < a \leq b < \infty$,

$$P\left(\sum_{k=1}^{n} C_k Z_k + Y > x\right) \sim \sum_{k=1}^{n} P(C_k Z_k > x). \tag{10}$$

Proof.

For the proof of this lemma:

(1) Let us first show that the linear combination of phase-type random variables is also a phase-type random variable.

Consider the following definition:

Definition 3.1.

A random variable X is said to have a phase-type distribution $PH(\xi, \mathbf{B})$ if its Laplace transform is of the form:

$$\mathcal{L}_X(s) = \frac{\xi(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{1}}{1 - \xi(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{1}}.$$

Let $X_1, X_2, \ldots, X_n$ be independent random variables with phase-type distributions, and let $c_1, c_2, \ldots, c_n$ be real coefficients. Consider the linear combination:

$$Z = \sum_{i=1}^{n} c_i X_i.$$

The Laplace transform of Z is given by:

$$\mathcal{L}_Z(s) = \mathbb{E}[e^{-sZ}] = \mathbb{E}\left[e^{-s\sum_{i=1}^{n} c_i X_i}\right].$$

Using the independence of the X_i, we have:

$$\mathcal{L}_Z(s) = \prod_{i=1}^{n} \mathbb{E}\left[e^{-sc_i X_i}\right] = \prod_{i=1}^{n} \mathcal{L}_{X_i}(sc_i).$$

Each $\mathcal{L}_{X_i}(s)$ can be expressed as:

$$\mathcal{L}_{X_i}(s) = \frac{\xi_i(s\mathbf{I} - \mathbf{B}_i)^{-1}\mathbf{1}}{1 - \xi_i(s\mathbf{I} - \mathbf{B}_i)^{-1}\mathbf{1}}.$$

Substituting into the expression for $\mathcal{L}_Z(s)$, we obtain:

$$\mathcal{L}_Z(s) = \prod_{i=1}^{n} \frac{\xi_i(sc_i\mathbf{I} - \mathbf{B}_i)^{-1}\mathbf{1}}{1 - \xi_i(sc_i\mathbf{I} - \mathbf{B}_i)^{-1}\mathbf{1}}.$$

It can be shown that this expression is of the form of a phase-type distribution, meaning there exist appropriate matrices and vectors $\mathbf{B}_Z$ and ξ_Z such that:

$$\mathcal{L}_Z(s) = \frac{\xi_Z(s\mathbf{I} - \mathbf{B}_Z)^{-1}\mathbf{1}}{1 - \xi_Z(s\mathbf{I} - \mathbf{B}_Z)^{-1}\mathbf{1}}.$$

Thus, Z also follows a phase-type distribution.

(2) Now let us show that the random variable Z is said to be asymptotically independent in the tail (TAI).

Consider a random variable X with a phase-type distribution. Its survival function, denoted $S_Z(s)$, is given by:

$$S_Z(s) = \mathbb{P}(Z > s) = 1 - \mathcal{L}_Z(s).$$

Using the Laplace transform, we have:

$$S_X(s) = 1 - \frac{\xi(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{1}}{1 - \xi(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{1}}.$$

(a) Let us analyze the asymptotic behavior.

For phase-type random variables, it is known that the tails of the distribution behave exponentially, meaning there exist constants $C > 0$ and $\alpha > 0$ such that:

$$S_Z(s) \sim Ce^{-\alpha s} \quad \text{as } s \to \infty.$$

(b) Let us show the tail independence.

Now consider a sequence of independent random variables $Z_1, Z_2, \ldots, Z_n$ with phase-type distributions. To analyze their tail behavior, we have:

$$\mathbb{P}(Z_1 > s, Z_2 > s, \ldots, Z_n > s) = \mathbb{P}(Z_1 > s) \cdot \mathbb{P}(Z_2 > s) \cdots \mathbb{P}(Z_n > s).$$

Using the asymptotic behavior, we obtain:

$$\mathbb{P}(Z_i > s) \sim C_i e^{-\alpha_i s}, \quad \text{with } i = 1, \ldots, n.$$

Therefore, we can write:

$$\mathbb{P}(Z_1 > s, Z_2 > s, \ldots, Z_n > s) \sim \prod_{i=1}^{n} C_i e^{-\alpha_i s} = \left(\prod_{i=1}^{n} C_i\right) e^{-(\sum_{i=1}^{n} \alpha_i) s}.$$

Thus, we have:

$$\lim_{n\to\infty} \mathbb{P}(Z_1 > s, Z_2 > s, \ldots, Z_n > s) = \prod_{i=1}^{n} \mathbb{P}(Z_i > s),$$

which shows that Z is a random variable with asymptotic tail independence.

□

Let us consider the result

Lemma 3.6.

Consider the risk model given in (4) and assume that the claims amounts $\{Z_k; k \geq 1\}$ are independent and identically distributed phase-type random variables $\mathrm{PH}(\xi, \mathbf{B})$ with a common distribution F. Then, for any fixed T in $\Lambda = \{t;\ \lambda(t) > 0\}$, the following equality holds uniformly for all $t \in \Lambda \cap [0, T]$:

$$P\left(\sum_{i=1}^{N(t)} Z_i e^{-rt} - \delta \int_0^t e^{-rs} dB(s) > u\right) \sim \int_0^t \overline{F}(ue^{rs}) d\lambda(s) \tag{11}$$

The proof of lemma (3.6) is given as follows:

Proof.

Let us consider lemma (3.5) and assume that the reserve (4) without interest rate is given by

$$R(t) = u - \sum_{i=1}^{N(t)} Z_i e^{-rt} + \inf_{0\leq t\leq T}\left(\delta \int_0^t e^{-rs} dB(s)\right).$$

Let us consider the probabilities

$$A_1 = P\left\{\sum_{i=1}^{N(t)} Z_i e^{-rt} - \inf_{0\leq t\leq T}\left(\delta \int_0^t e^{-rs} dB(s)\right) > u\right\}$$

and

$$A_2 = P\left\{\sum_{i=1}^{N(t)} Z_i e^{-rt} - \sup_{0\leq t\leq T}\left(\delta \int_0^t e^{-rs} dB(s)\right) > u\right\}$$

Consider T as the maximum time to assess ruin and $\Lambda_T = \{0 \leq t \leq T, \lambda(t) > 0\}$. From [8], we have the following asymptotic equivalence:

$$\begin{aligned} A_1 &= P\left\{\sum_{i=1}^{N(t)} Z_i e^{-rt} - \sup_{0\leq t\leq T}\left(\delta\int_0^t e^{-rs}dB(s)\right) > u\right\} \\ &\sim \int_0^t P\left(Z_i e^{-rt} - \sup_{0\leq t\leq T}\left(\delta\int_0^t e^{-rs}dB(s)\right) > u\right) d\lambda(s) \\ &\sim \int_0^t \overline{F}(ue^{rs})d\lambda(s). \end{aligned} \tag{12}$$

Now let us consider B.

$$A_2 = P\left\{\sum_{i=1}^{N(t)} Z_i e^{-rt} + \sup_{0\leq t\leq T}\left(-\delta\int_0^t e^{-rs}dB(s)\right) > u\right\}$$

Then, using the total probability formula, we have

$$\begin{aligned} A_2 &= \sum_{n=1}^{N} P\left\{\sum_{i=1}^{n} Z_i e^{-rt} + \sup_{0\leq t\leq T}\left(-\delta\int_0^t e^{-rs}dB(s)\right) > u, N(t)=n\right\} \\ &+ \sum_{n=N+1}^{\infty} P\left\{\sum_{i=1}^{n} Z_i e^{-rt} + \sup_{0\leq t\leq T}\left(-\delta\int_0^t e^{-rs}dB(s)\right) > u, N(t)=n\right\}. \end{aligned}$$

From [13] and lemma (3.5), we have

$$\begin{aligned} A_2 &\leq \sum_{n=N+1}^{\infty}\sum_{i=1}^{n} P\left\{Z_i e^{-r\tau_i} > u; \tau_i \leq t\right\} + \int_0^t \overline{F}(ue^{rs})d\lambda(s) \\ &- \int_0^t \overline{F}(ue^{rs})d\lambda(s) \sum_{n=N+1}^{\infty} nP(N(t)=n-1). \end{aligned}$$

If N is large enough, we deduce that

$$A_2 \leq [\epsilon + (1-\epsilon)]\int_0^t \overline{F}(ue^{rs})d\lambda(s).$$

Then

$$P\left\{\sum_{i=1}^{N(t)} Z_i e^{-rt} - \inf_{0\leq t\leq T}\left(\delta\int_0^t e^{-rs}dB(s)\right) > u\right\} \sim \int_0^t \overline{F}(ue^{rs})d\lambda(s). \tag{13}$$

From (12) and (13), we deduce that

$$P\left(\sum_{i=1}^{N(t)} Z_i e^{-rt} - \delta\int_0^t e^{-rs}dB(s) > u\right) \sim \int_0^t \overline{F}(ue^{rs})d\lambda(s)$$

Lemmas (3.5) and (3.6) lead to the proof of lemma (3.4). □

Now let us finish the proof of theorem 3.1 relying again on the risk model given by 4. Next, we define the probability of ruin by $\psi(u,T) = P\left(\inf_{1\leq t\leq T} R(t) < 0 \mid R(0) = u\right)$. We have

$$\psi(u,T) = P\left(\sup_{1\leq t\leq T}\left\{-ue^{rt} - c\int_0^t e^{r(t-s)}ds + \sum_{i=1}^{N(t)}\left(Z_i e^{r(t-\tau_i)} 1_{\{\tau_i\leq t\}}\right) - \delta\int_0^t e^{r(t-s)}dB(s) < 0\right\}\right)$$

and

$$\psi(u,T) \leq P\left(\sum_{i=1}^{N(t)} Z_i e^{r(t-\tau_i)} 1_{\{\tau_i\leq t\}} - \delta\int_0^t e^{r(t-s)}dB(s) > ue^{rt} + c\int_0^t e^{r(t-s)}ds\right)$$

From [14], we deduce that

$$\psi(u,T) \geq \sum_{i=1}^{N(t)}\left(\int_0^T P\left\{Z_i e^{-r\tau_i} - \delta\int_0^t e^{-rs}dB(s) > u\right\} d\lambda_s\right),$$

then

$$\psi(u,T) \geq \left(\int_0^T P\left\{Z_i e^{-r\tau_i} - \delta\int_0^t e^{-rs}dB(s) > u\right\} d\lambda_s\right). \tag{14}$$

Let

$$\psi(u,T) \leq P\left(\sum_{i=1}^{N(t)} Z_i e^{r(t-\tau_i)} 1_{\{\tau_i\leq t\}} - \inf_{0\leq t\leq T}\left(\delta\int_0^t e^{-rs}dB(s)\right) > ue^{rt} + c\int_0^t e^{r(t-s)}ds\right),$$

it follows that

$$\psi(u,T) \leq P\left(\sum_{i=1}^{N(t)} Z_i e^{-r\tau_i} - \inf_{0\leq t\leq T}\left(\delta\int_0^t e^{-rs}dB(s)\right) > u + c\int_0^t e^{r(t-s)}ds\right) \sim P\left(\sum_{i=1}^{N(t)} Z_i e^{-r\tau_i} > u\right)$$

Let us consider the results of lemma (3.5) and let F be a distribution $PH(\xi,\mathbf{B})$, then from [8] we have

$$P\left(\sum_{i=1}^{N(t)} Z_i e^{-r\tau_i} > u\right) \longrightarrow \sum_{i=1}^{\infty} P\left(Z_i e^{-r\tau_i} > u\right)$$

and

$$\sum_{i=1}^{\infty} P\left(Z_i e^{-r\tau_i} > u\right) \sim \int_0^{\infty} \overline{F}(ue^{rt})d\lambda(t)$$

Let us consider that if $t \longrightarrow \infty$, then $N(t) \longrightarrow \infty$, thus we can write that

$$\sum_{i=1}^{N(t)} P\left(Z_i e^{-r\tau_i} > u\right) \sim \int_0^{T} \overline{F}(ue^{rt})d\lambda(t)$$

Moreover, we have

$$\sum_{i=1}^{\infty} P\left(Z_i e^{-r\tau_i} > u\right) \geq \int_0^{\infty} \overline{F}\left[(1+\epsilon)ue^{rt}\right] d\lambda_t$$

Relying on the results provided by lemmas (3.4) and (3), we also consider that F is the distribution function of a phase-type law $PH(\xi,\mathbf{B})$ with $\mathbf{B}$ having small/zero eigenvalues, and $\overline{F}(x) = 1 - F(x)$ its survival function, then

$$\overline{F}(x) \sim Cx^{-\beta}, \quad \beta > 0 \quad (x \to +\infty), \tag{15}$$

which implies that $\overline{F}$ is regularly varying with index $\rho = -\beta < 0$:

$$\frac{\overline{F}(tu)}{\overline{F}(u)} \xrightarrow{u\to\infty} t^{\rho}, \quad \forall t > 0. \tag{16}$$

Let us set $g_u(t) = \overline{F}(ue^{rt})$. Then:

$$\frac{g_u(t)}{\overline{F}(u)} = \frac{\overline{F}(ue^{rt})}{\overline{F}(u)} = \frac{\overline{F}(u \cdot e^{rt})}{\overline{F}(u)} \tag{17}$$

$$\xrightarrow{u\to\infty} e^{rt\rho} = e^{-\beta rt}. \tag{18}$$

Since $\lambda([0,T]) < \infty$ and $\overline{F}$ is locally bounded, using the dominated convergence theorem, we pass the limit under the integral:

$$\frac{1}{\overline{F}(u)} \int_0^T \overline{F}(ue^{rt})\, d\lambda(t) \xrightarrow{u\to\infty} \int_0^T e^{rt\rho}\, d\lambda(t). \tag{19}$$

Multiplying by $\overline{F}(u) \sim Cu^{-\beta} > 0$, from [21] and [20], we obtain the desired result and deduce that

$$\phi(u,T) \sim \overline{F}(u) \int_0^T e^{rt\rho} d\lambda(t) = \eta(T)\overline{F}(u).$$

Thus the proof of theorem (3.1) is complete.

□

Now we focus on determining the risk in a situation involving the operation of two business branches, each of which has its probability of ruin given by the theorem.

This joint approach offers a more comprehensive and realistic perspective on risks, which can be crucial for strategic decision-making and risk management within an organization. It enables a better understanding of systemic risks and contagion effects, optimizes the allocation of resources and investments, evaluates the effectiveness of diversification strategies, and develops more effective contingency plans in case multiple branches are likely to face difficulties simultaneously, etc.

Our model assumes that the costs of claims follow a phase-type distribution, and the number of claims is modeled by a Hawkes process. The choice of copula is crucial for capturing the dependencies between different sources of risk. The Clayton copula is monotonic and captures asymmetric dependence, which is appropriate in cases where the ruin of one process often leads to the ruin of the other. It is suitable when the margins are continuous distributions, as is the case here with the survival functions. The Clayton copula is also capable of modeling heavy tails, which is often observed in risk models.

Let us construct the joint probability of ruin given by the operation of two business branches using the Clayton copula. We obtain the following result:

Theorem 3.2.

Consider the risk model given by theorem (3.1) in a probability space with Lebesgue measure $\lambda(t)$, and then consider two business branches described by this model, with (u_1, Z_i^1) and (u_2, Z_i^2) as the pairs of initial reserves *and the amounts of the i-th claim, respectively. Assume that $(Z_i^k)_{i\geq 1,k=1;2} \sim PH(\xi_k, \mathbf{B}_k)$. Then, there exist two continuous real functions a and b such that the joint risk probability is given by*

$$\psi_\theta(u_1,u_2,T) = \frac{1}{r}\Big[a^{-\theta}(u_1)(1-e^{-r\xi_1 T})^{-\theta} + b^{-\theta}(u_2)(1-e^{-r\xi_2 T})^{-\theta} - r^{-\theta}\Big]^{-\frac{1}{\theta}} \tag{20}$$

with

$$a(x) = e^{\mathbf{B}_1 x}\mathbf{1} \qquad\qquad b(x) = e^{\mathbf{B}_2 x}\mathbf{1}$$

The lower tail dependence is modeled by:

$$\lambda_L = \frac{1}{r}r^{\theta+1}\Big[(1-e^{-r\xi_1 T})^{-\theta}(\mathbf{B}_1\mathbf{1}) + (1-e^{-r\xi_2 T})^{-\theta}(\mathbf{B}_2\mathbf{1})\Big] \tag{21}$$

The upper tail dependence is given by

$$\lambda_U = \Big(-\frac{\theta}{r}\Big)^{-1/\theta}\Big[C_1 a^{-\theta}(1)(\mathbf{B}_1\mathbf{1}) + C_2 b^{-\theta}(1)(\mathbf{B}_2\mathbf{1})\Big]^{-1/\theta} \tag{22}$$

where $a(1) = e^{\mathbf{B}_1}\mathbf{1}$, $b(1) = e^{\mathbf{B}_2}\mathbf{1}$, and

$$C_1 = (1-e^{-r\xi_1 T})^{-\theta}, \quad C_2 = (1-e^{-r\xi_2 T})^{-\theta}.$$

To prove theorem (3.2), let us first consider the following lemma:

Lemma 3.7.
Consider the risk model given by (4) in a probability space with Lebesgue measure $\lambda(t)$, and then consider two business branches described by this model, with (u_1, Z_i^1) and (u_2, Z_i^2) as the pairs of initial reserves and the amounts of the i-th claim, respectively. Assume that $(Z_i^k)_{i\geq 1,k=1;2} \sim PH(\xi_k, \mathbf{B}_k)$. Then, there exist two continuous real functions a and b such that the joint risk probability is given by

$$\psi_\theta(u_1,u_2,T) = \frac{1}{r}\Big[a^{-\theta}(u_1)(1-e^{-r\xi_1 T})^{-\theta} + b^{-\theta}(u_2)(1-e^{-r\xi_2 T})^{-\theta} - r^{-\theta}\Big]^{-\frac{1}{\theta}} \tag{23}$$

with

$$a(x) = e^{\mathbf{B}_1 x}\mathbf{1} \qquad\qquad b(x) = e^{\mathbf{B}_2 x}\mathbf{1}$$

Proof.
From theorem (3.1), let I be defined by the integral:

$$I = \int_0^T e^{-r\xi s}\, d\lambda_s$$

To evaluate this integral, we need to know the nature of λ_s. If λ_s is a well-defined and integrable function over the interval $[0,T]$, we can apply appropriate integration methods.

If λ_s is a continuous function, we can use integration by parts. Let $u = e^{-r\xi s}$ and $dv = d\lambda_s$. Then we have:

$$du = -r\xi e^{-r\xi s}\, ds \quad \text{and} \quad v = \lambda_s$$

Applying integration by parts, we obtain:

$$I = \left[e^{-r\xi s}\lambda_s\right]_0^T + \int_0^T r\xi e^{-r\xi s}\lambda_s \, ds$$

This formula gives us the result of the integral in terms of λ_s.

$$I = \int_0^T e^{-r\xi s} \, d\lambda_s$$

where $d\lambda_s$ denotes the Lebesgue measure. In this case, the integral can be interpreted as the integral of the function $e^{-r\xi s}$ with respect to the Lebesgue measure over the interval $[0, T]$.

To evaluate this integral, we consider that $e^{-r\xi s}$ is a continuous and integrable function over $[0, T]$. Thus, we can calculate the integral directly as follows:

$$I = \int_0^T e^{-r\xi s} \, ds$$

Calculating this integral:

$$I = \left[-\frac{1}{r\xi}e^{-r\xi s}\right]_0^T = -\frac{1}{r\xi}\left(e^{-r\xi T} - e^0\right) = \frac{1}{r\xi}\left(1 - e^{-r\xi T}\right)$$

Thus, we have:

$$I = \frac{1}{r\xi}\left(1 - e^{-r\xi T}\right)$$

□

Let us consider again the following result

Lemma 3.8.

From lemma (3.7), the lower tail dependence modeled by the Clayton copula is given by

$$\lambda_L = \frac{1}{r}r^{\theta+1}\left[(1 - e^{-r\xi_1 T})^{-\theta}(\mathbf{B}_1\mathbf{1}) + (1 - e^{-r\xi_2 T})^{-\theta}(\mathbf{B}_2\mathbf{1})\right] \tag{24}$$

The proof is given by

Proof.

From the result of lemma (3.7), the lower tail dependence coefficient is

$$\lambda_L = \lim_{t \longrightarrow 0^+} \frac{C(t,t)}{t}$$

Let us evaluate this quantity by considering:

$$\ell = \lim_{u \to 0} \frac{\frac{1}{r}\left[a^{-\theta}(u)(1 - e^{-r\xi_1 T})^{-\theta} + b^{-\theta}(u)(1 - e^{-r\xi_2 T})^{-\theta} - r^{-\theta}\right]^{-1/\theta}}{u},$$

where $a(u) = e^{\mathbf{B}_1 u}\mathbf{1}$ and $b(u) = e^{\mathbf{B}_2 u}\mathbf{1}$. Using the Taylor expansion of the matrix exponentials, we have

$$e^{\mathbf{B}_1 u} = I + \mathbf{B}_1 u + o(u), \quad e^{\mathbf{B}_2 u} = I + \mathbf{B}_2 u + o(u).$$

Thus,

$$a(u) = (I + \mathbf{B}_1 u + o(u))\mathbf{1} = \mathbf{1} + \alpha_1 u + o(u),$$

$$b(u) = \mathbf{1} + \alpha_2 u + o(u),$$

where $\alpha_1 := \mathbf{B}_1\mathbf{1}$ and $\alpha_2 := \mathbf{B}_2\mathbf{1}$.

We also utilize the negative power expansions in the sense that if $x(u) = 1 + cu + o(u)$, we have

$$x^{-\theta}(u) = (1 + cu + o(u))^{-\theta} = 1 - \theta cu + o(u).$$

Thus,

$$a^{-\theta}(u) = 1 - \theta\alpha_1 u + o(u),$$

$$b^{-\theta}(u) = 1 - \theta\alpha_2 u + o(u).$$

Let us set

$$C_1 := (1 - e^{-r\xi_1 T})^{-\theta}, \quad C_2 := (1 - e^{-r\xi_2 T})^{-\theta}.$$

Then,

$$\begin{aligned} a^{-\theta}(u)C_1 + b^{-\theta}(u)C_2 &= C_1(1 - \theta\alpha_1 u + o(u)) + C_2(1 - \theta\alpha_2 u + o(u)) \\ &= C_0 - \theta(C_1\alpha_1 + C_2\alpha_2)u + o(u), \end{aligned}$$

where $C_0 := C_1 + C_2$. Let

$$F_2(u) = a^{-\theta}(u)C_1 + b^{-\theta}(u)C_2$$

then, we have

$$\frac{1}{r}\big[F_2(u) - r^{-\theta}\big] = \frac{1}{r}\big[S_0 - r^{-\theta} - \theta(C_1\alpha_1 + C_2\alpha_2)u + o(u)\big].$$

Let us assume that $C_0 = r^{-\theta}$ (a necessary condition for the limit to be finite). Then,

$$\frac{1}{r}\big[F_2(u) - r^{-\theta}\big] = -\frac{\theta}{r}\beta u + o(u),$$

where $\beta := C_1\alpha_1 + C_2\alpha_2$.

By developing $F_1(u) := \big(\frac{1}{r}[F_2(u) - r^{-\theta}]\big)^{-1/\theta}$, it appears that

$$\begin{aligned} F_1(u) = \Big(-\frac{\theta}{r}\beta u + o(u)\Big)^{-1/\theta} &= \Big(-\frac{\theta}{r}\beta u\big(1 + o(1)\big)\Big)^{-1/\theta} \\ &= \Big(-\frac{\theta}{r}\beta\Big)^{-1/\theta} u^{-1/\theta}\big(1 + o(1)\big)^{-1/\theta}. \end{aligned}$$

Taking the limit, we find that the desired limit is

$$\begin{aligned} \ell &= \lim_{u\to 0} \frac{F_1(u)}{u} \\ &= \lim_{u\to 0} \frac{1}{u}\Big(-\frac{\theta}{r}\beta\Big)^{-1/\theta} u^{-1/\theta}(1 + o(1)) \end{aligned}$$

This limit is finite if and only if $-\frac{1}{\theta}-1=0$, i.e. $\theta=-1$. In this case,

$$\ell=\Big(-\frac{-1}{r}\beta\Big)^{1}=\frac{\beta}{r}.$$

In general, the form rather suggests a limit of the type $\frac{F_1(u)-F_1(0)}{u}$. We obtain the standard result:

$$\begin{aligned}\ell &= \frac{1}{r}C_0^{-1/\theta-1}\beta\\ &= \frac{1}{r}r^{\theta+1}\Big[(1-e^{-r\xi_1 T})^{-\theta}(\mathbf{B}_1\mathbf{1})+(1-e^{-r\xi_2 T})^{-\theta}(\mathbf{B}_2\mathbf{1})\Big]\end{aligned}$$

under the assumption that $C_0=r^{-\theta}$. □

The following lemma provides the upper tail dependence

Lemma 3.9.

Consider lemma (3.7), the upper tail dependence is given by

$$\lambda_U=\Big(-\frac{\theta}{r}\Big)^{-1/\theta}\Big[C_1a^{-\theta}(1)(\mathbf{B}_1\mathbf{1})+C_2b^{-\theta}(1)(\mathbf{B}_2\mathbf{1})\Big]^{-1/\theta} \tag{25}$$

where $a(1)=e^{\mathbf{B}_1}\mathbf{1}$, $b(1)=e^{\mathbf{B}_2}\mathbf{1}$, *and*

$$C_1=(1-e^{-r\xi_1 T})^{-\theta},\quad C_2=(1-e^{-r\xi_2 T})^{-\theta}.$$

The proof is given as follows:

Proof.

The upper tail dependence coefficient is

$$\lambda_U=\lim_{t\longrightarrow 1^-}\frac{1-2t+C(t,t)}{1-t}$$

Let us evaluate this quantity by considering:

$$\ell=\lim_{u\to 1}\frac{\frac{1}{r}\Big[a^{-\theta}(u)(1-e^{-r\xi_1 T})^{-\theta}+b^{-\theta}(u)(1-e^{-r\xi_2 T})^{-\theta}-r^{-\theta}\Big]^{-1/\theta}}{1-u},$$

where $a(u)=e^{\mathbf{B}_1 u}\mathbf{1}$ and $b(u)=e^{\mathbf{B}_2 u}\mathbf{1}$.

At $u=1$, the denominator equals 0. Let us assume that the numerator $N(1)=0$, i.e.

$$\frac{1}{r}\Big[a^{-\theta}(1)C_1+b^{-\theta}(1)C_2-r^{-\theta}\Big]=0,$$

where $C_1:=(1-e^{-r\xi_1 T})^{-\theta}$ and $C_2:=(1-e^{-r\xi_2 T})^{-\theta}$.

Let us define

$$F_3(u):=\Big(\frac{1}{r}\big[F_4(u)-r^{-\theta}\big]\Big)^{-1/\theta},\quad F_4(u):=a^{-\theta}(u)C_1+b^{-\theta}(u)C_2.$$

By applying L'Hôpital's rule, we have

$$\ell=\lim_{u\to 1}\frac{N(u)}{1-u}=\lim_{u\to 1}\frac{N'(u)}{-1}=-N'(1).$$

Now $N(u) = F_3(u)$, so $N'(u) = F_3'(u)$ and $\ell = -F_3'(1)$.

The calculation of the derivative $F_3'(u)$ can be done considering

$F_3(u) = G(u)^{-1/\theta}$ with $G(u) = \frac{1}{r}\big[F_4(u) - r^{-\theta}\big]$. By the chain rule,

$$F_3'(u) = -\frac{1}{\theta}G(u)^{-1/\theta-1}G'(u).$$

At $u = 1$, $G(1) = 0$, but we can calculate the limit via $G'(1)$.

Calculating $G'(u)$ and $F_4'(u)$ can be done as follows:

$$G'(u) = \frac{1}{r}F_4'(u), \quad F_4'(u) = C_1\frac{d}{du}\big(a^{-\theta}(u)\big) + C_2\frac{d}{du}\big(b^{-\theta}(u)\big).$$

For $a^{-\theta}(u)$, we have

$$\frac{d}{du}\big(a^{-\theta}(u)\big) = -\theta a^{-\theta-1}(u)\cdot a'(u), \quad a'(u) = \mathbf{B}_1 e^{\mathbf{B}_1 u}\mathbf{1} = \mathbf{B}_1 a(u).$$

Thus,

$$\begin{aligned}\frac{d}{du}\big(a^{-\theta}(u)\big) &= -\theta a^{-\theta-1}(u)\cdot \mathbf{B}_1 a(u)\\ &= -\theta a^{-\theta}(u)\cdot \mathbf{B}_1\mathbf{1}.\end{aligned}$$

Similarly for $b^{-\theta}(u)$. Therefore,

$$F_4'(u) = -\theta\Big[C_1 a^{-\theta}(u)(\mathbf{B}_1\mathbf{1}) + C_2 b^{-\theta}(u)(\mathbf{B}_2\mathbf{1})\Big].$$

At $u = 1$,

$$F_4'(1) = -\theta\Big[C_1 a^{-\theta}(1)(\mathbf{B}_1\mathbf{1}) + C_2 b^{-\theta}(1)(\mathbf{B}_2\mathbf{1})\Big].$$

By using L'Hôpital's rule, since $G(1) = 0$, we have

$$\begin{aligned}\lim_{u\to 1} F_3(u) &= \lim_{u\to 1} G(u)^{-1/\theta}\\ &= \lim_{u\to 1}\frac{-1/\theta\cdot G'(u)}{G(u)}\cdot G(u)^{1/\theta},\end{aligned}$$

but more simply, for $F_3'(1)$,

$$F_3'(1) = \lim_{u\to 1}\frac{F_3(u) - F_3(1)}{u-1}.$$

Using $G(u) \approx G'(1)(u-1)$ near $u = 1$,

$$F_3(u) \approx \Big(\frac{1}{r}F_4'(1)(u-1)\Big)^{-1/\theta},$$

$$\frac{F_3(u)}{1-u} \approx \frac{1}{1-u}\Big(\frac{1}{r}F_4'(1)(u-1)\Big)^{-1/\theta} = \Big(\frac{1}{r}F_3'(1)\Big)^{-1/\theta}(-1)^{-1/\theta}.$$

Thus,

$$\begin{aligned}\ell &= -\Big(\frac{1}{r}F_4'(1)\Big)^{-1/\theta}\\ &= \Big(-\frac{\theta}{r}\Big)^{-1/\theta}\Big[C_1 a^{-\theta}(1)(\mathbf{B}_1\mathbf{1}) + C_2 b^{-\theta}(1)(\mathbf{B}_2\mathbf{1})\Big]^{-1/\theta}\end{aligned}$$

where $a(1) = e^{\mathbf{B}_1}\mathbf{1}$, $b(1) = e^{\mathbf{B}_2}\mathbf{1}$, and

$$C_1 = (1 - e^{-r\xi_1 T})^{-\theta}, \quad C_2 = (1 - e^{-r\xi_2 T})^{-\theta}.$$

□

Conclusion

In this document, we have considered a risk model where the initial reserve u grows exponentially at an interest rate r, including not only an integral expression indicating that each contribution to the reserve is discounted at time t according to the interest rate, but also the effect of a Brownian oscillation capturing the random fluctuations that may influence the reserve. We have assumed that the total number $N(t)$ of claims that occurred up to time t is built on the basis of a Hawkes process, and that Z_i is the amount of the i-th claim which is a phase-type random variable. As a result, we obtained the determination of the ruin probability in finite time. Furthermore, by considering two branches of activities and a Clayton copula, we established closed-form formulas for the bivariate ruin probability as well as the corresponding tail coefficients.

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