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ALGEBRAIC ANALYSIS AND STRUCTURAL PROPERTIES OF $(\in, \in \vee q_k)$-ANTI-INTUITIONISTIC FUZZY SOFT BOOLEAN RINGS

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Abstract. In this paper, we introduce $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean rings and $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideals over Boolean rings. These structures are formulated by using generalized anti-relations for the membership and non-membership components of intuitionistic fuzzy soft sets. Equivalent characterizations in terms of threshold inequalities are established, and finite examples are provided. We prove that the OR operation and the union preserve $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean rings. A counterexample shows that the AND operation and the intersection do not preserve this structure in general, while sufficient conditions based on comparability are obtained for these operations. We also characterize ordinary ideals by means of their anti-characteristic intuitionistic fuzzy soft sets and prove that every $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideal is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean ring. The converse implication is shown not to hold in general. Finally, closure properties of these ideals under soft set operations are investigated.

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Key words and phrases. Boolean ring; intuitionistic fuzzy soft set; generalized quasi-coincidence; $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean ring; $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideal.

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1. Introduction

Soft set theory was introduced by Molodtsov [9] as a parameterized framework for dealing with uncertain and imprecise information. Maji et al. [8] further developed the basic theory and operations of soft sets. By combining fuzzy sets with soft sets, Maji et al. [6] introduced fuzzy soft sets, whereas Ahmad and Kharal [2] subsequently investigated several properties and operations of fuzzy soft sets. In another direction, Atanassov [3] introduced intuitionistic fuzzy sets by assigning both membership and non-membership degrees to each element. Maji et al. [7] later combined this concept with soft set theory and introduced intuitionistic fuzzy soft sets.

The interaction between fuzzy structures and algebraic systems has also received considerable attention. Dixit et al. [4] studied fuzzy rings, while Acar et al. [1] introduced soft rings. These developments provided a basis for extending fuzzy soft methods to Boolean rings and related structures. In particular, fuzzy soft Boolean rings were studied in [12], and fuzzy soft Boolean near-rings together with their idealistic forms were investigated in [15]. Generalized fuzzy soft Boolean near-rings were considered in [11]. Further extensions include intuitionistic fuzzy soft Boolean rings [17], generalized intuitionistic fuzzy soft Boolean near-rings [18], fuzzy soft Γ-Boolean rings [16], and bipolar fuzzy soft Boolean rings [19]. Soft Boolean near-rings and soft intersection Boolean near-rings were also studied in [13,14].

Anti-fuzzy structures provide conditions that are dual in direction to the usual fuzzy conditions. In the intuitionistic setting, both membership and non-membership functions are considered. Intuitionistic anti-fuzzy ideals in rings were investigated in [20], while anti-intuitionistic fuzzy soft structures in BCK/BCI-algebras were studied in [5]. More recently, anti-fuzzy soft Boolean rings were introduced and investigated in [10]. However, the cited studies do not directly examine anti-intuitionistic fuzzy soft Boolean rings under the generalized $(\in, \in \vee q_k)$ conditions or their corresponding soft ideal structures.

Motivated by these developments, we introduce the notion of an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean ring, abbreviated as an $(\in, \in \vee q_k)$-AIFSBR. We also introduce the corresponding notion of an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideal, abbreviated as an $(\in, \in \vee q_k)$-AIFSI. These structures are formulated by using generalized anti-relations separately for the membership and non-membership components of intuitionistic fuzzy soft sets. The generalized quasi-coincidence parameter $k \in [0, 1)$ gives rise to the threshold

$$\tau_k = \frac{1-k}{2}.$$

Equivalent descriptions in terms of threshold inequalities are established for both structures.

The main contributions of this paper are summarized as follows. First, we define $(\in, \in \vee q_k)$-AIFSBRs through componentwise fuzzy-point relations and establish an equivalent characterization by means of threshold inequalities. A finite example over the Boolean ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is provided. We prove that the OR operation and the union preserve $(\in, \in \vee q_k)$-AIFSBRs. A counterexample shows that the AND operation and the intersection do not preserve this structure in general. Sufficient conditions based on the comparability of the corresponding intuitionistic fuzzy sets are then obtained for these two operations.

Second, we introduce $(\in, \in \vee q_k)$-AIFSIs through generalized fuzzy-point relations and obtain their equivalent threshold characterization. A family of ordinary ideals is characterized by means of anti-characteristic intuitionistic fuzzy soft sets. We prove that every $(\in, \in \vee q_k)$-AIFSI is an $(\in, \in \vee q_k)$-AIFSBR and provide an example showing that the converse implication does not hold in general. The OR operation and the union are shown to preserve $(\in, \in \vee q_k)$-AIFSIs, whereas comparability conditions are established for their AND operation and intersection.

The remainder of this paper is organized as follows. Section 2 recalls the basic concepts concerning Boolean rings, fuzzy soft sets, intuitionistic fuzzy soft sets, and soft set operations. It also introduces the componentwise generalized anti-relations and presents the threshold relations used in the subsequent sections. Section 3 introduces $(\in, \in \vee q_k)$-AIFSBRs, establishes their threshold characterization, and investigates their closure properties. Section 4 develops $(\in, \in \vee q_k)$-AIFSIs, characterizes them through threshold inequalities and ordinary ideals, and examines their relationship with $(\in, \in \vee q_k)$-AIFSBRs. The final section presents the conclusion.

Abbreviations

Boolean ring	BR
Fuzzy soft set	FSS
Intuitionistic fuzzy set	IFS
Intuitionistic fuzzy soft set	IFSS
Fuzzy soft Boolean ring	FSBR
Intuitionistic fuzzy soft Boolean ring	IFSBR
Anti-fuzzy soft Boolean ring	AFSBR
Anti-intuitionistic fuzzy soft Boolean ring	AIFSBR
Anti-intuitionistic fuzzy soft ideal	AIFSI

2. Preliminaries

In this section, we recall the definitions and notation required in the subsequent sections. Let $\mathcal{U}$ be a non-empty universe, and let $\mathcal{E}$ be a non-empty set of parameters. For any $a, b \in [0, 1]$, we write

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}.$$

Definition 2.1. *A* ring *is a non-empty set $\mathfrak{N}$ equipped with two binary operations, $+$ and $\cdot$, satisfying the following conditions:*

(i) $(\mathfrak{N}, +)$ *is an abelian group;*

(ii) $(\mathfrak{N}, \cdot)$ *is a semigroup;*

(iii)

$$x(y+z) = xy + xz$$

and

$$(x+y)z = xz + yz$$

for all $x, y, z \in \mathfrak{N}$.

Unless otherwise stated, a ring is not required to have a multiplicative identity.

Definition 2.2. *A ring* $\mathfrak{N}$ *is called a* Boolean ring (BR) *if*

$$x^2 = x$$

for every $x \in \mathfrak{N}$.

Every Boolean ring is commutative and satisfies

$$x + x = 0$$

for every $x \in \mathfrak{N}$. Consequently,

$$-x = x \quad \text{and} \quad x - y = x + y$$

for all $x, y \in \mathfrak{N}$.

Definition 2.3. *Let* $\mathcal{K} \subseteq \mathcal{E}$. *A pair* $(F, \mathcal{K})$ *is called a* soft set *over* $\mathcal{U}$ *if*

$$F : \mathcal{K} \longrightarrow \mathcal{P}(\mathcal{U})$$

is a mapping, where $\mathcal{P}(\mathcal{U})$ *denotes the power set of* $\mathcal{U}$ [8,9].

Definition 2.4. *A* fuzzy set *on* $\mathcal{U}$ *is a mapping*

$$\mu : \mathcal{U} \longrightarrow [0,1].$$

The family of all fuzzy sets on $\mathcal{U}$ *is denoted by*

$$\mathcal{F}(\mathcal{U}) = \{\mu \mid \mu : \mathcal{U} \longrightarrow [0,1]\}.$$

Definition 2.5. *Let* $\mathcal{K} \subseteq \mathcal{E}$. *A pair* $(\mathcal{O}, \mathcal{K})$ *is called a* fuzzy soft set (FSS) *over* $\mathcal{U}$ *if*

$$\mathcal{O} : \mathcal{K} \longrightarrow \mathcal{F}(\mathcal{U})$$

is a mapping. For every $e \in \mathcal{K}$, *we write*

$$\mathcal{O}_e = \mathcal{O}(e),$$

where $\mathcal{O}_e(x)$ *is the membership degree of* $x \in \mathcal{U}$ *with respect to the parameter* e [2,6].

Definition 2.6. *An* intuitionistic fuzzy set *(IFS) $\mathcal{A}$ on $\mathcal{U}$ is an object of the form*

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \mid x \in \mathcal{U}\},$$

where

$$\mu_{\mathcal{A}}, \nu_{\mathcal{A}} : \mathcal{U} \longrightarrow [0,1]$$

satisfy

$$0 \leqslant \mu_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leqslant 1$$

for every $x \in \mathcal{U}$. The values $\mu_{\mathcal{A}}(x)$ and $\nu_{\mathcal{A}}(x)$ are called the membership degree and the non-membership degree of x, respectively [3].

The family of all intuitionistic fuzzy sets on $\mathcal{U}$ is denoted by $\mathcal{IF}(\mathcal{U})$.

Definition 2.7. *Let $\mathcal{K} \subseteq \mathcal{E}$. A pair $(\mathcal{O}, \mathcal{K})$ is called an* intuitionistic fuzzy soft set *(IFSS) over $\mathcal{U}$ if*

$$\mathcal{O} : \mathcal{K} \longrightarrow \mathcal{IF}(\mathcal{U})$$

is a mapping.

For every $e \in \mathcal{K}$, we write

$$\mathcal{O}(e) = \left\{\left\langle x, \mathcal{O}_e(x), \mathcal{O}'_e(x)\right\rangle \mid x \in \mathcal{U}\right\},$$

where $\mathcal{O}_e(x)$ and $\mathcal{O}'_e(x)$ are the membership degree and the non-membership degree of x, respectively, and

$$0 \leqslant \mathcal{O}_e(x) + \mathcal{O}'_e(x) \leqslant 1$$

for all $x \in \mathcal{U}$ and $e \in \mathcal{K}$ [7].

Definition 2.8. *Let*

$$\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}}) \quad \text{and} \quad \mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$$

be two intuitionistic fuzzy sets on $\mathcal{U}$. We write

$$\mathcal{A} \preceq \mathcal{B}$$

if

$$\mu_{\mathcal{A}}(x) \leqslant \mu_{\mathcal{B}}(x)$$

and

$$\nu_{\mathcal{A}}(x) \geqslant \nu_{\mathcal{B}}(x)$$

for every $x \in \mathcal{U}$.

The intuitionistic fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ are called comparable *if*

$$\mathcal{A} \preceq \mathcal{B} \quad \text{or} \quad \mathcal{B} \preceq \mathcal{A}.$$

Definition 2.9. *Let*

$$\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}}) \quad \textit{and} \quad \mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$$

be two intuitionistic fuzzy sets on $\mathcal{U}$*.*

The intersection of $\mathcal{A}$ *and* $\mathcal{B}$ *is defined by*

$$\mathcal{A} \cap \mathcal{B} = (\mu_{\mathcal{A}} \wedge \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \vee \nu_{\mathcal{B}}).$$

The union of $\mathcal{A}$ *and* $\mathcal{B}$ *is defined by*

$$\mathcal{A} \cup \mathcal{B} = (\mu_{\mathcal{A}} \vee \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \wedge \nu_{\mathcal{B}}),$$

where all operations are taken pointwise on $\mathcal{U}$*.*

Definition 2.10. *Let* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *be two IFSSs over* $\mathcal{U}$*, and suppose that*

$$\mathcal{K} \cap \mathcal{G} \neq \emptyset.$$

The intersection *of* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *is the IFSS* $(\mathcal{S}, \mathcal{R})$*, where*

$$\mathcal{R} = \mathcal{K} \cap \mathcal{G}$$

and

$$\mathcal{S}_e(x) = \mathcal{O}_e(x) \wedge \mathcal{N}_e(x),$$

$$\mathcal{S}'_e(x) = \mathcal{O}'_e(x) \vee \mathcal{N}'_e(x)$$

for all $e \in \mathcal{R}$ *and* $x \in \mathcal{U}$*.*

It is denoted by

$$(\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}).$$

Definition 2.11. *Let* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *be two IFSSs over* $\mathcal{U}$*. The* union *of* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *is the IFSS* $(\mathcal{S}, \mathcal{R})$*, where*

$$\mathcal{R} = \mathcal{K} \cup \mathcal{G},$$

$$\mathcal{S}_e(x) = \begin{cases} \mathcal{O}_e(x), & e \in \mathcal{K} \backslash \mathcal{G}, \\ \mathcal{N}_e(x), & e \in \mathcal{G} \backslash \mathcal{K}, \\ \mathcal{O}_e(x) \vee \mathcal{N}_e(x), & e \in \mathcal{K} \cap \mathcal{G}, \end{cases}$$

and

$$\mathcal{S}'_e(x) = \begin{cases} \mathcal{O}'_e(x), & e \in \mathcal{K} \backslash \mathcal{G}, \\ \mathcal{N}'_e(x), & e \in \mathcal{G} \backslash \mathcal{K}, \\ \mathcal{O}'_e(x) \wedge \mathcal{N}'_e(x), & e \in \mathcal{K} \cap \mathcal{G}, \end{cases}$$

for all $e \in \mathcal{R}$ *and* $x \in \mathcal{U}$*.*

It is denoted by

$$(\mathcal{O},\mathcal{K}) \cup (\mathcal{N},\mathcal{G}) = (\mathcal{S},\mathcal{R}).$$

Definition 2.12. *Let $(\mathcal{O},\mathcal{K})$ and $(\mathcal{N},\mathcal{G})$ be two IFSSs over $\mathcal{U}$. The* AND operation *of $(\mathcal{O},\mathcal{K})$ and $(\mathcal{N},\mathcal{G})$ is the IFSS $(\mathcal{S},\mathcal{K}\times\mathcal{G})$ defined by*

$$\mathcal{S}_{(e,f)}(x) = \mathcal{O}_e(x) \wedge \mathcal{N}_f(x)$$

and

$$\mathcal{S}'_{(e,f)}(x) = \mathcal{O}'_e(x) \vee \mathcal{N}'_f(x)$$

for all $(e,f)\in\mathcal{K}\times\mathcal{G}$ and $x\in\mathcal{U}$.

It is denoted by

$$(\mathcal{O},\mathcal{K}) \wedge (\mathcal{N},\mathcal{G}) = (\mathcal{S},\mathcal{K}\times\mathcal{G}).$$

Definition 2.13. *Let $(\mathcal{O},\mathcal{K})$ and $(\mathcal{N},\mathcal{G})$ be two IFSSs over $\mathcal{U}$. The* OR operation *of $(\mathcal{O},\mathcal{K})$ and $(\mathcal{N},\mathcal{G})$ is the IFSS $(\mathcal{S},\mathcal{K}\times\mathcal{G})$ defined by*

$$\mathcal{S}_{(e,f)}(x) = \mathcal{O}_e(x) \vee \mathcal{N}_f(x)$$

and

$$\mathcal{S}'_{(e,f)}(x) = \mathcal{O}'_e(x) \wedge \mathcal{N}'_f(x)$$

for all $(e,f)\in\mathcal{K}\times\mathcal{G}$ and $x\in\mathcal{U}$.

It is denoted by

$$(\mathcal{O},\mathcal{K}) \vee (\mathcal{N},\mathcal{G}) = (\mathcal{S},\mathcal{K}\times\mathcal{G}).$$

Definition 2.14. *The* null fuzzy set *on $\mathcal{U}$ is the fuzzy set $0_{\mathcal{U}}$ defined by*

$$0_{\mathcal{U}}(x) = 0$$

for every $x\in\mathcal{U}$.

The null intuitionistic fuzzy set *on $\mathcal{U}$ is*

$$\widetilde{0}_{\mathcal{U}} = \{\langle x,0,1\rangle \mid x\in\mathcal{U}\}.$$

Definition 2.15. *Let $(\mathcal{O},\mathcal{K})$ be an FSS over $\mathcal{U}$. The support of $(\mathcal{O},\mathcal{K})$ is defined by*

$$\mathrm{Supp}(\mathcal{O},\mathcal{K}) = \{e\in\mathcal{K} \mid \mathcal{O}(e) \neq 0_{\mathcal{U}}\}.$$

If $(\mathcal{O},\mathcal{K})$ is an IFSS over $\mathcal{U}$, then its support is defined by

$$\mathrm{Supp}(\mathcal{O},\mathcal{K}) = \left\{e\in\mathcal{K} \mid \mathcal{O}(e) \neq \widetilde{0}_{\mathcal{U}}\right\}.$$

An FSS or an IFSS is called non-null *if*

$$\mathrm{Supp}(\mathcal{O},\mathcal{K}) \neq \varnothing.$$

Otherwise, it is called null.

Definition 2.16. *Let $x \in \mathcal{U}$ and $t \in [0,1]$. The fuzzy set x_t on $\mathcal{U}$ defined by*

$$x_t(y) = \begin{cases} t, & y = x, \\ 0, & y \neq x \end{cases}$$

is called a fuzzy point *with support x and value t.*

Remark 2.1. *In the present componentwise formulation, the value $t = 0$ is included in the definition of a fuzzy point. Although fuzzy points are usually defined with positive values, the zero-level point is required here to formulate the non-membership component and to obtain the corresponding threshold characterizations.*

Definition 2.17. *Let $(\mathcal{O}, \mathcal{K})$ be an IFSS over $\mathcal{U}$, let $e \in \mathcal{K}$, let $x \in \mathcal{U}$, and let $k \in [0,1)$. Put*

$$\tau_k = \frac{1-k}{2}.$$

The generalized anti-relations are defined separately for the membership and non-membership components as follows.

Membership component. *For $t \in [0,1]$, we write*

$$x_t \in_\mu \mathcal{O}(e)$$

if

$$\mathcal{O}_e(x) \leqslant t,$$

and we write

$$x_t \, q_{k,\mu} \, \mathcal{O}(e)$$

if

$$\mathcal{O}_e(x) + t + k < 1.$$

If

$$x_t \in_\mu \mathcal{O}(e) \quad \text{or} \quad x_t \, q_{k,\mu} \, \mathcal{O}(e),$$

then we write

$$x_t \in_\mu \vee q_{k,\mu} \, \mathcal{O}(e).$$

Non-membership component. *For $s \in [0,1]$, we write*

$$x_s \in_\nu \mathcal{O}(e)$$

if

$$\mathcal{O}'_e(x) \geqslant s,$$

and we write

$$x_s \, q_{k,\nu} \, \mathcal{O}(e)$$

if

$$\mathcal{O}'_e(x) + s + k > 1.$$

If

$$x_s \in_\nu \mathcal{O}(e) \quad \textit{or} \quad x_s\, q_{k,\nu}\, \mathcal{O}(e),$$

then we write

$$x_s \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e).$$

Throughout this paper, the notation $(\in, \in \vee q_k)$ *for an intuitionistic fuzzy soft structure is understood componentwise. That is, the corresponding* $\in_\mu \vee q_{k,\mu}$ *condition is imposed on the membership function, while the corresponding* $\in_\nu \vee q_{k,\nu}$ *condition is imposed on the non-membership function.*

Remark 2.2. *The generalized anti-relations introduced above are interpreted componentwise. Unlike the conventional relation based on a single intuitionistic fuzzy soft point, the membership and non-membership components are treated through independent level values. This componentwise formulation allows the two components to be characterized separately by the corresponding threshold inequalities.*

Lemma 2.1. *Let* $a, c \in [0,1]$*, let* $k \in [0,1)$*, and let*

$$\tau_k = \frac{1-k}{2}.$$

Then the following assertions hold:

(*i*)

$$c \leqslant a \vee \tau_k$$

if and only if, for every $t \in [a,1]$*,*

$$c \leqslant t \quad \textit{or} \quad c + t + k < 1;$$

(*ii*)

$$c \geqslant a \wedge \tau_k$$

if and only if, for every $s \in [0,a]$*,*

$$c \geqslant s \quad \textit{or} \quad c + s + k > 1.$$

Proof. Assume that

$$c \leqslant a \vee \tau_k,$$

and let $t \in [a,1]$. If $c \leqslant t$, there is nothing to prove. Suppose that $c > t$. Since $t \geqslant a$, we have $c > a$. Therefore,

$$c \leqslant \tau_k.$$

Since $t < c$, it follows that

$$c + t + k < 2c + k \leqslant 2\tau_k + k = 1.$$

This proves the forward implication in (i).

Conversely, suppose that the stated condition holds for every $t \in [a,1]$, but

$$c > a \vee \tau_k.$$

Then

$$a < c \quad \text{and} \quad 1-k-c < c.$$

Hence, there exists t such that

$$\max\{a, 1-k-c\} \leqslant t < c.$$

Thus,

$$c > t \quad \text{and} \quad c+t+k \geqslant 1,$$

which is a contradiction. Therefore,

$$c \leqslant a \vee \tau_k.$$

Next, assume that

$$c \geqslant a \wedge \tau_k,$$

and let $s \in [0,a]$. If $c \geqslant s$, there is nothing to prove. Suppose that $c < s$. Then $c < a$. Hence, $a > \tau_k$ and $c \geqslant \tau_k$. Since $s > c$, we obtain

$$c+s+k > 2c+k \geqslant 2\tau_k + k = 1.$$

This proves the forward implication in (ii).

Conversely, suppose that the stated condition holds for every $s \in [0,a]$, but

$$c < a \wedge \tau_k.$$

Then

$$c < a \quad \text{and} \quad c < 1-k-c.$$

Hence, there exists s such that

$$c < s \leqslant \min\{a, 1-k-c\}.$$

It follows that

$$c < s \quad \text{and} \quad c+s+k \leqslant 1,$$

which is a contradiction. Therefore,

$$c \geqslant a \wedge \tau_k.$$

□

Definition 2.18. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null FSS over a Boolean ring $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is called a* fuzzy soft Boolean ring *(FSBR) over $\mathfrak{N}$ if*

$$\mathcal{O}_e(x+y) \geqslant \mathcal{O}_e(x) \wedge \mathcal{O}_e(y)$$

and

$$\mathcal{O}_e(xy) \geqslant \mathcal{O}_e(x) \wedge \mathcal{O}_e(y)$$

for all $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$ [12].

Definition 2.19. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null IFSS over a Boolean ring $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is called an* intuitionistic fuzzy soft Boolean ring *(IFSBR) over $\mathfrak{N}$ if*

$$\mathcal{O}_e(x+y) \geqslant \mathcal{O}_e(x) \wedge \mathcal{O}_e(y),$$

$$\mathcal{O}'_e(x+y) \leqslant \mathcal{O}'_e(x) \vee \mathcal{O}'_e(y),$$

$$\mathcal{O}_e(xy) \geqslant \mathcal{O}_e(x) \wedge \mathcal{O}_e(y),$$

and

$$\mathcal{O}'_e(xy) \leqslant \mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)$$

for all $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$ [17].

Definition 2.20. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null FSS over a Boolean ring $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is called an* anti-fuzzy soft Boolean ring *(AFSBR) over $\mathfrak{N}$ if*

$$\mathcal{O}_e(x+y) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y)$$

and

$$\mathcal{O}_e(xy) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y)$$

for all $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$ [10].

3. $(\in, \in \vee q_k)$-Anti-Intuitionistic Fuzzy Soft Boolean Rings

Ramesh et al. [10] studied anti-fuzzy soft Boolean rings. Motivated by their work and the generalized anti-intuitionistic fuzzy soft approach in [5], we introduce $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean rings and investigate some of their fundamental structural properties.

Throughout this section, $\mathfrak{N}$ denotes a Boolean ring, $k \in [0, 1)$, and

$$\tau_k = \frac{1-k}{2}.$$

All intuitionistic fuzzy soft sets considered in this section are defined over $\mathfrak{N}$.

Definition 3.1. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null IFSS over $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is called an* $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean ring, *briefly an $(\in, \in \vee q_k)$-AIFSBR, if, for every $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$, the following conditions hold:*

(*i*) *for all* $t, u \in [0,1]$,

$$x_t \in_\mu \mathcal{O}(e) \quad \textit{and} \quad y_u \in_\mu \mathcal{O}(e)$$

imply

$$(x+y)_{t\vee u} \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e)$$

and

$$(xy)_{t\vee u} \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e);$$

(*ii*) *for all* $s, r \in [0,1]$,

$$x_s \in_\nu \mathcal{O}(e) \quad \textit{and} \quad y_r \in_\nu \mathcal{O}(e)$$

imply

$$(x+y)_{s\wedge r} \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e)$$

and

$$(xy)_{s\wedge r} \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e).$$

The following theorem gives a convenient characterization of an $(\in, \in \vee q_k)$-AIFSBR in terms of threshold inequalities. This characterization will be used throughout the remainder of this paper.

Theorem 3.1. *Let* $(\mathcal{O}, \mathcal{K})$ *be a non-null IFSS over* $\mathfrak{N}$. *Then* $(\mathcal{O}, \mathcal{K})$ *is an* $(\in, \in \vee q_k)$*-AIFSBR if and only if, for every* $e \in \mathcal{K}$ *and* $x, y \in \mathfrak{N}$,

$$\mathcal{O}_e(x+y) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k, \tag{3.1}$$

$$\mathcal{O}'_e(x+y) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k, \tag{3.2}$$

$$\mathcal{O}_e(xy) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k, \tag{3.3}$$

$$\mathcal{O}'_e(xy) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k. \tag{3.4}$$

Proof. Assume first that $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSBR. Fix $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$, and let

$$\diamond \in \{+, \cdot\}.$$

For the membership component, put

$$a = \mathcal{O}_e(x) \vee \mathcal{O}_e(y)$$

and

$$c = \mathcal{O}_e(x \diamond y).$$

Let $v \in [a, 1]$. Then

$$\mathcal{O}_e(x) \leqslant v \quad \text{and} \quad \mathcal{O}_e(y) \leqslant v.$$

Hence,

$$x_v \in_\mu \mathcal{O}(e) \quad \text{and} \quad y_v \in_\mu \mathcal{O}(e).$$

By Definition 3.1,

$$(x \diamond y)_{v \vee v} \in_{\mu} \vee q_{k,\mu} \mathcal{O}(e).$$

Since

$$v \vee v = v,$$

we have

$$c \leqslant v \quad \text{or} \quad c + v + k < 1.$$

This holds for every $v \in [a, 1]$. Therefore, by Lemma 2.1,

$$c \leqslant a \vee \tau_k.$$

Consequently,

$$\mathcal{O}_e(x \diamond y) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k.$$

Taking $\diamond = +$ and $\diamond = \cdot$ gives (3.1) and (3.3), respectively.

For the non-membership component, put

$$b = \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y)$$

and

$$d = \mathcal{O}'_e(x \diamond y).$$

Let $v \in [0, b]$. Then

$$\mathcal{O}'_e(x) \geqslant v \quad \text{and} \quad \mathcal{O}'_e(y) \geqslant v.$$

Thus,

$$x_v \in_{\nu} \mathcal{O}(e) \quad \text{and} \quad y_v \in_{\nu} \mathcal{O}(e).$$

Definition 3.1 yields

$$(x \diamond y)_{v \wedge v} \in_{\nu} \vee q_{k,\nu} \mathcal{O}(e).$$

Since

$$v \wedge v = v,$$

we obtain

$$d \geqslant v \quad \text{or} \quad d + v + k > 1.$$

This holds for every $v \in [0, b]$. Hence, by Lemma 2.1,

$$d \geqslant b \wedge \tau_k.$$

Therefore,

$$\mathcal{O}'_e(x \diamond y) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k.$$

Taking $\diamond = +$ and $\diamond = \cdot$ gives (3.2) and (3.4), respectively.

Conversely, suppose that (3.1)–(3.4) hold. Fix $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$.

Suppose that

$$x_t \in_\mu \mathcal{O}(e) \quad \text{and} \quad y_u \in_\mu \mathcal{O}(e).$$

Then

$$\mathcal{O}_e(x) \leqslant t \quad \text{and} \quad \mathcal{O}_e(y) \leqslant u.$$

Consequently,

$$\mathcal{O}_e(x) \vee \mathcal{O}_e(y) \leqslant t \vee u.$$

Let $\diamond \in \{+, \cdot\}$ and put

$$a = \mathcal{O}_e(x) \vee \mathcal{O}_e(y)$$

and

$$c = \mathcal{O}_e(x \diamond y).$$

By (3.1) and (3.3), according as $\diamond = +$ or $\diamond = \cdot$, we have

$$c \leqslant a \vee \tau_k.$$

Since

$$t \vee u \in [a, 1],$$

Lemma 2.1 implies that

$$c \leqslant t \vee u$$

or

$$c + (t \vee u) + k < 1.$$

Therefore,

$$(x \diamond y)_{t \vee u} \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e).$$

Taking $\diamond = +$ and $\diamond = \cdot$ establishes the two membership conditions in Definition 3.1.

Next, suppose that

$$x_s \in_\nu \mathcal{O}(e) \quad \text{and} \quad y_r \in_\nu \mathcal{O}(e).$$

Then

$$\mathcal{O}'_e(x) \geqslant s \quad \text{and} \quad \mathcal{O}'_e(y) \geqslant r.$$

Hence,

$$\mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \geqslant s \wedge r.$$

Let $\diamond \in \{+, \cdot\}$ and put

$$b = \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y)$$

and

$$d = \mathcal{O}'_e(x \diamond y).$$

By (3.2) and (3.4), according as $\diamond = +$ or $\diamond = \cdot$, we have

$$d \geqslant b \wedge \tau_k.$$

Since

$$s \wedge r \in [0, b],$$

Lemma 2.1 gives

$$d \geqslant s \wedge r$$

or

$$d + (s \wedge r) + k > 1.$$

Thus,

$$(x \diamond y)_{s \wedge r} \in_\nu \vee q_{k,\nu} \mathcal{O}(e).$$

Taking $\diamond = +$ and $\diamond = \cdot$ establishes the two non-membership conditions in Definition 3.1.

Therefore, $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSBR. □

Example 3.1. *Let*

$$\mathfrak{N} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, a, b, c\},$$

where

$$0 = (0, 0), \qquad a = (1, 0), \qquad b = (0, 1), \qquad c = (1, 1).$$

The operations are defined componentwise modulo 2. Equivalently, they are given by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

It follows that

$$x^2 = x$$

for every $x \in \mathfrak{N}$. Hence, $\mathfrak{N}$ is a Boolean ring.

Let

$$\mathcal{K} = \{e_1, e_2, e_3\}.$$

Define an IFSS $(\mathcal{O}, \mathcal{K})$ over $\mathfrak{N}$ by

$$\mathcal{O}(e_1) = \{\langle 0, 0.10, 0.80\rangle, \langle a, 0.40, 0.50\rangle, \langle b, 0.50, 0.40\rangle, \langle c, 0.50, 0.30\rangle\},$$

$$\mathcal{O}(e_2) = \{\langle 0, 0.15, 0.75\rangle, \langle a, 0.35, 0.55\rangle, \langle b, 0.35, 0.50\rangle, \langle c, 0.20, 0.60\rangle\},$$

and

$$\mathcal{O}(e_3) = \{\langle 0, 0.20, 0.70\rangle, \langle a, 0.45, 0.40\rangle, \langle b, 0.30, 0.60\rangle, \langle c, 0.45, 0.45\rangle\}.$$

For every $e \in \mathcal{K}$ and $x \in \mathfrak{N}$,

$$0 \leqslant \mathcal{O}_e(x) + \mathcal{O}'_e(x) \leqslant 1.$$

Thus, $(\mathcal{O}, \mathcal{K})$ is an IFSS over $\mathfrak{N}$.

Let $k = 0.6$. Then

$$\tau_k = \frac{1 - 0.6}{2} = 0.2.$$

A direct verification using the operation tables shows that

$$\mathcal{O}_e(x + y) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee 0.2,$$

$$\mathcal{O}'_e(x + y) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge 0.2,$$

$$\mathcal{O}_e(xy) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee 0.2,$$

and

$$\mathcal{O}'_e(xy) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge 0.2$$

for all $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$. Therefore, by Theorem 3.1, $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_{0.6})$-AIFSBR over $\mathfrak{N}$.

The following example shows that neither the AND operation nor the intersection preserves $(\in, \in \vee q_k)$-AIFSBRs in general.

Example 3.2. *Let $\mathfrak{N}$ be the Boolean ring in Example 3.1, and let*

$$\mathcal{K} = \mathcal{G} = \{e\}.$$

Take $k = 0.6$, so that $\tau_k = 0.2$. Define IFSSs $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ by

$$\mathcal{O}(e) = \{\langle 0, 0.10, 0.50\rangle, \langle a, 0.10, 0.50\rangle, \langle b, 0.40, 0.50\rangle, \langle c, 0.40, 0.50\rangle\}$$

and

$$\mathcal{N}(e) = \{\langle 0, 0.10, 0.50\rangle, \langle a, 0.40, 0.50\rangle, \langle b, 0.10, 0.50\rangle, \langle c, 0.40, 0.50\rangle\}.$$

A direct verification using Theorem 3.1 shows that both $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ are $(\in, \in \vee q_{0.6})$-AIFSBRs.

Let

$$(\mathcal{S}, \mathcal{K}) = (\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G}).$$

Then

$$\mathcal{S}_e(0) = 0.10, \qquad \mathcal{S}_e(a) = 0.10, \qquad \mathcal{S}_e(b) = 0.10,$$

and

$$\mathcal{S}_e(c) = 0.40.$$

Since

$$a + b = c,$$

we obtain

$$\mathcal{S}_e(a + b) = 0.40 > 0.20 = 0.10 \vee 0.10 \vee 0.20.$$

Thus, $(\mathcal{S}, \mathcal{K})$ does not satisfy (3.1) and, by Theorem 3.1, it is not an $(\in, \in \vee q_{0.6})$-AIFSBR.

Since $\mathcal{K} = \mathcal{G} = \{e\}$, the component associated with (e, e) in

$$(\mathcal{O}, \mathcal{K}) \wedge (\mathcal{N}, \mathcal{G})$$

has the same membership and non-membership functions as $\mathcal{S}(e)$. Hence, the AND operation is also not an $(\in, \in \vee q_{0.6})$-AIFSBR.

Theorem 3.2. *Let $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSBRs. Suppose that $\mathcal{O}(e)$ and $\mathcal{N}(f)$ are comparable for every*

$$(e, f) \in \mathcal{K} \times \mathcal{G}.$$

Then

$$(\mathcal{O}, \mathcal{K}) \wedge (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$-AIFSBR.

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \wedge (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \times \mathcal{G}.$$

For every $(e, f) \in \mathcal{R}$,

$$\mathcal{S}(e, f) = \mathcal{O}(e) \cap \mathcal{N}(f).$$

By the comparability assumption, either

$$\mathcal{O}(e) \preceq \mathcal{N}(f)$$

or

$$\mathcal{N}(f) \preceq \mathcal{O}(e).$$

In the first case,

$$\mathcal{S}(e, f) = \mathcal{O}(e),$$

whereas in the second case,

$$\mathcal{S}(e, f) = \mathcal{N}(f).$$

Thus, each component of $(\mathcal{S}, \mathcal{R})$ is a component of one of the given $(\in, \in \vee q_k)$-AIFSBRs. Consequently, each component satisfies the conditions in Definition 3.1.

It remains to verify that $(\mathcal{S}, \mathcal{R})$ is non-null. Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K})$$

and

$$f_0 \in \mathrm{Supp}(\mathcal{N}, \mathcal{G}).$$

Both $\mathcal{O}(e_0)$ and $\mathcal{N}(f_0)$ are non-null intuitionistic fuzzy sets. Since they are comparable,

$$\mathcal{S}(e_0, f_0) = \mathcal{O}(e_0)$$

or

$$\mathcal{S}(e_0, f_0) = \mathcal{N}(f_0).$$

Hence, $\mathcal{S}(e_0, f_0)$ is non-null. Therefore, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSBR. $\square$

Theorem 3.3. *Let $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSBRs. Then*

$$(\mathcal{O}, \mathcal{K}) \vee (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$-AIFSBR.

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \vee (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \times \mathcal{G}.$$

For every $(e, f) \in \mathcal{R}$ and $x \in \mathfrak{N}$,

$$\mathcal{S}_{(e,f)}(x) = \mathcal{O}_e(x) \vee \mathcal{N}_f(x)$$

and

$$\mathcal{S}'_{(e,f)}(x) = \mathcal{O}'_e(x) \wedge \mathcal{N}'_f(x).$$

Let $\diamond$ denote either $+$ or $\cdot$. For any $x, y \in \mathfrak{N}$, Theorem 3.1 gives

$$\begin{aligned} \mathcal{S}_{(e,f)}(x \diamond y) &= \mathcal{O}_e(x \diamond y) \vee \mathcal{N}_f(x \diamond y) \\ &\leqslant (\mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k) \vee (\mathcal{N}_f(x) \vee \mathcal{N}_f(y) \vee \tau_k) \\ &= \mathcal{S}_{(e,f)}(x) \vee \mathcal{S}_{(e,f)}(y) \vee \tau_k. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{S}'_{(e,f)}(x \diamond y) &= \mathcal{O}'_e(x \diamond y) \wedge \mathcal{N}'_f(x \diamond y) \\ &\geqslant \big(\mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k\big) \wedge \big(\mathcal{N}'_f(x) \wedge \mathcal{N}'_f(y) \wedge \tau_k\big) \\ &= \mathcal{S}'_{(e,f)}(x) \wedge \mathcal{S}'_{(e,f)}(y) \wedge \tau_k. \end{aligned}$$

Therefore, the threshold conditions in Theorem 3.1 hold.

To verify non-nullness, choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K})$$

and any $f \in \mathcal{G}$. Since

$$\mathcal{S}(e_0, f) = \mathcal{O}(e_0) \cup \mathcal{N}(f)$$

and $\mathcal{O}(e_0)$ is non-null, $\mathcal{S}(e_0, f)$ is non-null. Consequently, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSBR. □

Theorem 3.4. *Let* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *be two* $(\in, \in \vee q_k)$*-AIFSBRs. Then*

$$(\mathcal{O}, \mathcal{K}) \cup (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$*-AIFSBR.*

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \cup (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \cup \mathcal{G}.$$

Fix $e \in \mathcal{R}$. We consider the following cases.

Case 1. If

$$e \in \mathcal{K} \backslash \mathcal{G},$$

then

$$\mathcal{S}(e) = \mathcal{O}(e).$$

Hence, $\mathcal{S}(e)$ satisfies all the conditions of an $(\in, \in \vee q_k)$-AIFSBR.

Case 2. If

$$e \in \mathcal{G} \backslash \mathcal{K},$$

then

$$\mathcal{S}(e) = \mathcal{N}(e).$$

Hence, $\mathcal{S}(e)$ satisfies all the required conditions.

Case 3. If

$$e \in \mathcal{K} \cap \mathcal{G},$$

then

$$\mathcal{S}_e(x) = \mathcal{O}_e(x) \vee \mathcal{N}_e(x)$$

and

$$\mathcal{S}'_e(x) = \mathcal{O}'_e(x) \wedge \mathcal{N}'_e(x)$$

for every $x \in \mathfrak{N}$. By the same argument used in the proof of Theorem 3.3, $\mathcal{S}(e)$ satisfies the threshold conditions in Theorem 3.1.

Thus, every component of $(\mathcal{S}, \mathcal{R})$ satisfies the required conditions.

Finally, choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K}).$$

If $e_0 \in \mathcal{K} \backslash \mathcal{G}$, then

$$\mathcal{S}(e_0) = \mathcal{O}(e_0)$$

is non-null. If $e_0 \in \mathcal{K} \cap \mathcal{G}$, then

$$\mathcal{S}(e_0) = \mathcal{O}(e_0) \cup \mathcal{N}(e_0)$$

is also non-null. Hence, $(\mathcal{S}, \mathcal{R})$ is non-null and is therefore an $(\in, \in \vee q_k)$-AIFSBR. □

Theorem 3.5. *Let $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSBRs. Suppose that*

$$\mathrm{Supp}(\mathcal{O}, \mathcal{K}) \cap \mathrm{Supp}(\mathcal{N}, \mathcal{G}) \neq \varnothing$$

and that $\mathcal{O}(e)$ and $\mathcal{N}(e)$ are comparable for every

$$e \in \mathcal{K} \cap \mathcal{G}.$$

Then

$$(\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$-AIFSBR.

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \cap \mathcal{G}.$$

For every $e \in \mathcal{R}$,

$$\mathcal{S}(e) = \mathcal{O}(e) \cap \mathcal{N}(e).$$

By the comparability assumption, either

$$\mathcal{O}(e) \preceq \mathcal{N}(e)$$

or

$$\mathcal{N}(e) \preceq \mathcal{O}(e).$$

Therefore,

$$\mathcal{S}(e) = \mathcal{O}(e)$$

or

$$\mathcal{S}(e) = \mathcal{N}(e).$$

In either case, $\mathcal{S}(e)$ satisfies all the conditions in Definition 3.1.

Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K}) \cap \mathrm{Supp}(\mathcal{N}, \mathcal{G}).$$

Both $\mathcal{O}(e_0)$ and $\mathcal{N}(e_0)$ are non-null. Since they are comparable,

$$\mathcal{S}(e_0) = \mathcal{O}(e_0)$$

or

$$\mathcal{S}(e_0) = \mathcal{N}(e_0).$$

Thus, $\mathcal{S}(e_0)$ is non-null. Consequently, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSBR. □

4. $(\in, \in \vee q_k)$-Anti-Intuitionistic Fuzzy Soft Ideals over Boolean Rings

The classical notion of an intuitionistic anti-fuzzy ideal in a ring was studied in [20]. Motivated by this notion and the generalized anti-intuitionistic fuzzy soft approach in [5], we introduce $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideals over Boolean rings and investigate their structural properties.

Throughout this section, $\mathfrak{N}$ denotes a Boolean ring, $k \in [0, 1)$, and

$$\tau_k = \frac{1-k}{2}.$$

All intuitionistic fuzzy soft sets considered in this section are defined over $\mathfrak{N}$.

Definition 4.1. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null IFSS over $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is called an* $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideal, *briefly an $(\in, \in \vee q_k)$-AIFSI, if, for every $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$, the following conditions hold:*

(i) for all $t, u \in [0, 1]$,

$$x_t \in_\mu \mathcal{O}(e) \quad \textit{and} \quad y_u \in_\mu \mathcal{O}(e)$$

imply

$$(x + y)_{t \vee u} \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e)$$

and

$$(xy)_{t \wedge u} \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e);$$

(ii) for all $s, r \in [0, 1]$,

$$x_s \in_\nu \mathcal{O}(e) \quad \textit{and} \quad y_r \in_\nu \mathcal{O}(e)$$

imply

$$(x + y)_{s \wedge r} \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e)$$

and

$$(xy)_{s \vee r} \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e).$$

The following theorem characterizes $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideals in terms of threshold inequalities.

Theorem 4.1. *Let $(\mathcal{O}, \mathcal{K})$ be a non-null IFSS over $\mathfrak{N}$. Then $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSI if and only if, for every $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$,*

$$\mathcal{O}_e(x+y) \leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k, \tag{4.1}$$

$$\mathcal{O}'_e(x+y) \geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k, \tag{4.2}$$

$$\mathcal{O}_e(xy) \leqslant (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k, \tag{4.3}$$

$$\mathcal{O}'_e(xy) \geqslant \big(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\big) \wedge \tau_k. \tag{4.4}$$

Proof. Assume first that $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSI. Since the fuzzy-point conditions for addition in Definition 4.1 are the same as those in Definition 3.1, the argument in the proof of Theorem 3.1 gives (4.1) and (4.2).

We next consider multiplication. Fix $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$. Put

$$\alpha = \mathcal{O}_e(x), \qquad \beta = \mathcal{O}_e(y), \qquad A = \alpha \wedge \beta$$

and

$$C = \mathcal{O}_e(xy).$$

Let $v \in [A, 1]$. Suppose first that $\alpha \leqslant \beta$. Set

$$t = v \quad \text{and} \quad u = v \vee \beta.$$

Then

$$t \geqslant \alpha, \qquad u \geqslant \beta, \qquad t \wedge u = v.$$

Consequently,

$$x_t \in_\mu \mathcal{O}(e) \quad \text{and} \quad y_u \in_\mu \mathcal{O}(e).$$

Definition 4.1 therefore gives

$$(xy)_v \in_\mu \vee q_{k,\mu}\, \mathcal{O}(e).$$

Thus,

$$C \leqslant v \quad \text{or} \quad C + v + k < 1.$$

If $\beta \leqslant \alpha$, the same conclusion follows by taking

$$t = v \vee \alpha \quad \text{and} \quad u = v.$$

Hence, for every $v \in [A, 1]$,

$$C \leqslant v \quad \text{or} \quad C + v + k < 1.$$

By Lemma 2.1,

$$C \leqslant A \vee \tau_k.$$

Therefore,

$$\mathcal{O}_e(xy) \leqslant (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k,$$

which proves (4.3).

For the non-membership component, put

$$\alpha' = \mathcal{O}'_e(x), \qquad \beta' = \mathcal{O}'_e(y), \qquad B = \alpha' \vee \beta'$$

and

$$D = \mathcal{O}'_e(xy).$$

Let $v \in [0, B]$.

Suppose first that $\alpha' \geqslant \beta'$. Set

$$s = v \quad \text{and} \quad r = v \wedge \beta'.$$

Then

$$s \leqslant \alpha', \qquad r \leqslant \beta', \qquad s \vee r = v.$$

Hence,

$$x_s \in_\nu \mathcal{O}(e) \quad \text{and} \quad y_r \in_\nu \mathcal{O}(e).$$

Definition 4.1 gives

$$(xy)_v \in_\nu \vee q_{k,\nu}\, \mathcal{O}(e).$$

Thus,

$$D \geqslant v \quad \text{or} \quad D + v + k > 1.$$

If $\beta' \geqslant \alpha'$, the same conclusion follows by taking

$$s = v \wedge \alpha' \quad \text{and} \quad r = v.$$

Therefore, for every $v \in [0, B]$,

$$D \geqslant v \quad \text{or} \quad D + v + k > 1.$$

Lemma 2.1 yields

$$D \geqslant B \wedge \tau_k.$$

Consequently,

$$\mathcal{O}'_e(xy) \geqslant \big(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\big) \wedge \tau_k,$$

which proves (4.4).

Conversely, suppose that (4.1)–(4.4) hold. The addition conditions in Definition 4.1 follow from the argument in the converse part of Theorem 3.1.

Suppose that

$$x_t \in_\mu \mathcal{O}(e) \quad \text{and} \quad y_u \in_\mu \mathcal{O}(e).$$

Then

$$\mathcal{O}_e(x) \leqslant t \quad \text{and} \quad \mathcal{O}_e(y) \leqslant u,$$

and hence

$$\mathcal{O}_e(x) \wedge \mathcal{O}_e(y) \leqslant t \wedge u.$$

By (4.3),

$$\mathcal{O}_e(xy) \leqslant (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k.$$

Applying Lemma 2.1 at the level $t \wedge u$, we obtain

$$\mathcal{O}_e(xy) \leqslant t \wedge u$$

or

$$\mathcal{O}_e(xy) + (t \wedge u) + k < 1.$$

Therefore,

$$(xy)_{t \wedge u} \in_\mu \vee q_{k,\mu} \, \mathcal{O}(e).$$

Now suppose that

$$x_s \in_\nu \mathcal{O}(e) \quad \text{and} \quad y_r \in_\nu \mathcal{O}(e).$$

Then

$$\mathcal{O}'_e(x) \geqslant s \quad \text{and} \quad \mathcal{O}'_e(y) \geqslant r.$$

It follows that

$$\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y) \geqslant s \vee r.$$

By (4.4),

$$\mathcal{O}'_e(xy) \geqslant \big(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\big) \wedge \tau_k.$$

Applying Lemma 2.1 at the level $s \vee r$, we obtain

$$\mathcal{O}'_e(xy) \geqslant s \vee r$$

or

$$\mathcal{O}'_e(xy) + (s \vee r) + k > 1.$$

Thus,

$$(xy)_{s \vee r} \in_\nu \vee q_{k,\nu} \, \mathcal{O}(e).$$

Therefore, all the conditions in Definition 4.1 hold. □

Definition 4.2. *Let I be a non-empty subset of $\mathfrak{N}$. The* anti-characteristic intuitionistic fuzzy set *associated with I is the intuitionistic fuzzy set*

$$\chi_I^a = (\mu_I^a, \nu_I^a),$$

where

$$\mu_I^a(x) = \begin{cases} 0, & x \in I, \\ 1, & x \notin I, \end{cases}$$

and

$$\nu_I^a(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I. \end{cases}$$

Theorem 4.2. *Let $\mathcal{K}$ be a non-empty set of parameters, and let*

$$\{I_e \mid e \in \mathcal{K}\}$$

be a family of non-empty subsets of $\mathfrak{N}$. Suppose that at least one I_e is a proper subset of $\mathfrak{N}$. Define an IFSS $(\mathcal{O}, \mathcal{K})$ over $\mathfrak{N}$ by

$$\mathcal{O}(e) = \chi_{I_e}^a$$

for every $e \in \mathcal{K}$. Then $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSI if and only if I_e is an ideal of $\mathfrak{N}$ for every $e \in \mathcal{K}$.

Proof. Assume first that I_e is an ideal of $\mathfrak{N}$ for every $e \in \mathcal{K}$. Fix $e \in \mathcal{K}$ and let $x, y \in \mathfrak{N}$.

If $x, y \in I_e$, then

$$x + y \in I_e.$$

Therefore,

$$\mathcal{O}_e(x + y) = 0 \leqslant 0 \vee 0 \vee \tau_k$$

and

$$\mathcal{O}'_e(x + y) = 1 \geqslant 1 \wedge 1 \wedge \tau_k.$$

If at least one of x and y does not belong to I_e, the two addition inequalities follow immediately because all membership and non-membership values belong to $\{0, 1\}$.

Since I_e is an ideal, if either $x \in I_e$ or $y \in I_e$, then

$$xy \in I_e.$$

It follows that

$$\mathcal{O}_e(xy) = 0 \leqslant (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k$$

and

$$\mathcal{O}'_e(xy) = 1 \geqslant \left(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\right) \wedge \tau_k.$$

If neither x nor y belongs to I_e, the multiplication inequalities also hold directly.

Thus, all the inequalities in Theorem 4.1 hold. Since at least one I_e is proper, $(\mathcal{O}, \mathcal{K})$ is non-null. Hence, it is an $(\in, \in \vee q_k)$-AIFSI.

Conversely, assume that $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSI. Fix $e \in \mathcal{K}$.

Let $x, y \in I_e$. Then

$$\mathcal{O}_e(x) = \mathcal{O}_e(y) = 0.$$

By Theorem 4.1,

$$\mathcal{O}_e(x + y) \leqslant \tau_k.$$

Since

$$\tau_k < 1$$

and

$$\mathcal{O}_e(x + y) \in \{0, 1\},$$

we obtain

$$\mathcal{O}_e(x + y) = 0.$$

Thus,

$$x + y \in I_e.$$

Because I_e is non-empty, choose $x \in I_e$. Since $\mathfrak{N}$ is a Boolean ring,

$$x + x = 0.$$

Therefore,

$$0 \in I_e.$$

Moreover, every element of a Boolean ring is its own additive inverse. Hence, I_e is an additive subgroup of $(\mathfrak{N}, +)$.

Let $x \in I_e$ and $y \in \mathfrak{N}$. Then

$$\mathcal{O}_e(x) = 0.$$

Using (4.3), we obtain

$$\mathcal{O}_e(xy) \leqslant (0 \wedge \mathcal{O}_e(y)) \vee \tau_k = \tau_k.$$

Since the membership value is either 0 or 1 and $\tau_k < 1$, we have

$$\mathcal{O}_e(xy) = 0.$$

Therefore,

$$xy \in I_e.$$

Since a Boolean ring is commutative, I_e is an ideal of $\mathfrak{N}$. $\square$

Example 4.1. *Let $\mathfrak{N}$ be the Boolean ring in Example 3.1. Consider*

$$I_1 = \{0, a\}$$

and

$$I_2 = \{0, b\}.$$

Both I_1 and I_2 are ideals of $\mathfrak{N}$.

Let

$$\mathcal{K} = \{e_1, e_2\}$$

and define an IFSS $(\mathcal{O}, \mathcal{K})$ over $\mathfrak{N}$ by

$$\mathcal{O}(e_1) = \chi^a_{I_1}$$

and

$$\mathcal{O}(e_2) = \chi^a_{I_2}.$$

Explicitly,

$$\mathcal{O}(e_1) = \{\langle 0, 0, 1\rangle, \langle a, 0, 1\rangle, \langle b, 1, 0\rangle, \langle c, 1, 0\rangle\}$$

and

$$\mathcal{O}(e_2) = \{\langle 0, 0, 1\rangle, \langle a, 1, 0\rangle, \langle b, 0, 1\rangle, \langle c, 1, 0\rangle\}.$$

By Theorem 4.2, $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSI for every $k \in [0, 1)$.

Theorem 4.3. *Every $(\in, \in \vee q_k)$-AIFSI over $\mathfrak{N}$ is an $(\in, \in \vee q_k)$-AIFSBR over $\mathfrak{N}$.*

Proof. Let $(\mathcal{O}, \mathcal{K})$ be an $(\in, \in \vee q_k)$-AIFSI. The addition inequalities in Theorems 4.1 and 3.1 are the same.

For every $e \in \mathcal{K}$ and $x, y \in \mathfrak{N}$,

$$\begin{aligned}\mathcal{O}_e(xy) &\leqslant (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k \\ &\leqslant \mathcal{O}_e(x) \vee \mathcal{O}_e(y) \vee \tau_k.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{O}'_e(xy) &\geqslant \left(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\right) \wedge \tau_k \\ &\geqslant \mathcal{O}'_e(x) \wedge \mathcal{O}'_e(y) \wedge \tau_k.\end{aligned}$$

Thus, all the threshold inequalities in Theorem 3.1 hold. Consequently, $(\mathcal{O}, \mathcal{K})$ is an $(\in, \in \vee q_k)$-AIFSBR. □

The converse of Theorem 4.3 does not hold in general, as shown in the following example.

Example 4.2. *Consider the* $(\in, \in \vee q_{0.6})$*-AIFSBR* $(\mathcal{O}, \mathcal{K})$ *constructed in Example 3.1. For the parameter* e_2*, we have*

$$ac = a,$$

$$\mathcal{O}_{e_2}(a) = 0.35,$$

and

$$\mathcal{O}_{e_2}(c) = 0.20.$$

Therefore,

$$\mathcal{O}_{e_2}(ac) = 0.35,$$

whereas

$$(\mathcal{O}_{e_2}(a) \wedge \mathcal{O}_{e_2}(c)) \vee \tau_{0.6} = (0.35 \wedge 0.20) \vee 0.20 = 0.20.$$

Hence,

$$\mathcal{O}_{e_2}(ac) > (\mathcal{O}_{e_2}(a) \wedge \mathcal{O}_{e_2}(c)) \vee \tau_{0.6}.$$

Thus, (4.3) does not hold, and $(\mathcal{O}, \mathcal{K})$ *is not an* $(\in, \in \vee q_{0.6})$*-AIFSI.*

Remark 4.1. *The two IFSSs constructed in Example 3.2 are also* $(\in, \in \vee q_{0.6})$*-AIFSIs. Indeed, their membership functions take the value* 0.10 *on the ideals* $\{0, a\}$ *and* $\{0, b\}$*, respectively, and take the value* 0.40 *outside these ideals. Their non-membership functions have the constant value* 0.50*. Thus, the four inequalities in Theorem 4.1 are satisfied.*

However, their AND operation and their intersection fail the addition membership condition, as shown in Example 3.2. Therefore, neither the AND operation nor the intersection preserves $(\in, \in \vee q_k)$*-AIFSIs in general.*

Theorem 4.4. *Let* $(\mathcal{O}, \mathcal{K})$ *and* $(\mathcal{N}, \mathcal{G})$ *be two* $(\in, \in \vee q_k)$*-AIFSIs. Suppose that* $\mathcal{O}(e)$ *and* $\mathcal{N}(f)$ *are comparable for every*

$$(e, f) \in \mathcal{K} \times \mathcal{G}.$$

Then

$$(\mathcal{O}, \mathcal{K}) \wedge (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$*-AIFSI.*

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \wedge (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \times \mathcal{G}.$$

For every $(e, f) \in \mathcal{R}$,

$$\mathcal{S}(e, f) = \mathcal{O}(e) \cap \mathcal{N}(f).$$

By the comparability assumption, either

$$\mathcal{O}(e) \preceq \mathcal{N}(f)$$

or

$$\mathcal{N}(f) \preceq \mathcal{O}(e).$$

Therefore,

$$\mathcal{S}(e, f) = \mathcal{O}(e)$$

or

$$\mathcal{S}(e, f) = \mathcal{N}(f).$$

Hence, every component of $(\mathcal{S}, \mathcal{R})$ satisfies all the conditions in Definition 4.1.

Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K})$$

and

$$f_0 \in \mathrm{Supp}(\mathcal{N}, \mathcal{G}).$$

Both $\mathcal{O}(e_0)$ and $\mathcal{N}(f_0)$ are non-null. Since they are comparable,

$$\mathcal{S}(e_0, f_0) = \mathcal{O}(e_0)$$

or

$$\mathcal{S}(e_0, f_0) = \mathcal{N}(f_0).$$

Thus, $\mathcal{S}(e_0, f_0)$ is non-null. Consequently, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSI. $\square$

Theorem 4.5. *Let $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSIs. Then*

$$(\mathcal{O}, \mathcal{K}) \vee (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$-AIFSI.

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \vee (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \times \mathcal{G}.$$

For every $(e, f) \in \mathcal{R}$ and $x \in \mathfrak{N}$,

$$\mathcal{S}_{(e,f)}(x) = \mathcal{O}_e(x) \vee \mathcal{N}_f(x)$$

and

$$\mathcal{S}'_{(e,f)}(x) = \mathcal{O}'_e(x) \wedge \mathcal{N}'_f(x).$$

Let $x, y \in \mathfrak{N}$. For addition, Theorem 4.1 gives

$$\begin{aligned}\mathcal{S}_{(e,f)}(x+y) &= \mathcal{O}_e(x+y) \vee \mathcal{N}_f(x+y)\\ &\leqslant \mathcal{S}_{(e,f)}(x) \vee \mathcal{S}_{(e,f)}(y) \vee \tau_k\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}'_{(e,f)}(x+y) &= \mathcal{O}'_e(x+y) \wedge \mathcal{N}'_f(x+y)\\ &\geqslant \mathcal{S}'_{(e,f)}(x) \wedge \mathcal{S}'_{(e,f)}(y) \wedge \tau_k.\end{aligned}$$

For multiplication, we have

$$\begin{aligned}\mathcal{S}_{(e,f)}(xy) &= \mathcal{O}_e(xy) \vee \mathcal{N}_f(xy)\\ &\leqslant [(\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee \tau_k] \vee [(\mathcal{N}_f(x) \wedge \mathcal{N}_f(y)) \vee \tau_k]\\ &= (\mathcal{O}_e(x) \wedge \mathcal{O}_e(y)) \vee (\mathcal{N}_f(x) \wedge \mathcal{N}_f(y)) \vee \tau_k\\ &\leqslant [(\mathcal{O}_e(x) \vee \mathcal{N}_f(x)) \wedge (\mathcal{O}_e(y) \vee \mathcal{N}_f(y))] \vee \tau_k\\ &= \left(\mathcal{S}_{(e,f)}(x) \wedge \mathcal{S}_{(e,f)}(y)\right) \vee \tau_k.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{S}'_{(e,f)}(xy) &= \mathcal{O}'_e(xy) \wedge \mathcal{N}'_f(xy)\\ &\geqslant \left[\left(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\right) \wedge \tau_k\right] \wedge \left[\left(\mathcal{N}'_f(x) \vee \mathcal{N}'_f(y)\right) \wedge \tau_k\right]\\ &= \left(\mathcal{O}'_e(x) \vee \mathcal{O}'_e(y)\right) \wedge \left(\mathcal{N}'_f(x) \vee \mathcal{N}'_f(y)\right) \wedge \tau_k\\ &\geqslant \left[\left(\mathcal{O}'_e(x) \wedge \mathcal{N}'_f(x)\right) \vee \left(\mathcal{O}'_e(y) \wedge \mathcal{N}'_f(y)\right)\right] \wedge \tau_k\\ &= \left(\mathcal{S}'_{(e,f)}(x) \vee \mathcal{S}'_{(e,f)}(y)\right) \wedge \tau_k.\end{aligned}$$

Thus, all the inequalities in Theorem 4.1 hold.

Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K})$$

and any $f \in \mathcal{G}$. Then

$$\mathcal{S}(e_0, f) = \mathcal{O}(e_0) \cup \mathcal{N}(f)$$

is non-null. Therefore, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSI. □

Theorem 4.6. *Let $(\mathcal{O}, \mathcal{K})$ and $(\mathcal{N}, \mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSIs. Then*

$$(\mathcal{O}, \mathcal{K}) \cup (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$-AIFSI.

Proof. Let

$$(\mathcal{O},\mathcal{K}) \cup (\mathcal{N},\mathcal{G}) = (\mathcal{S},\mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \cup \mathcal{G}.$$

Fix $e \in \mathcal{R}$.

If

$$e \in \mathcal{K} \backslash \mathcal{G},$$

then

$$\mathcal{S}(e) = \mathcal{O}(e).$$

If

$$e \in \mathcal{G} \backslash \mathcal{K},$$

then

$$\mathcal{S}(e) = \mathcal{N}(e).$$

If

$$e \in \mathcal{K} \cap \mathcal{G},$$

then

$$\mathcal{S}(e) = \mathcal{O}(e) \cup \mathcal{N}(e).$$

In the third case, the same argument as in the proof of Theorem 4.5 shows that $\mathcal{S}(e)$ satisfies the inequalities in Theorem 4.1. Thus, every component of $(\mathcal{S},\mathcal{R})$ satisfies the required conditions.

Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O},\mathcal{K}).$$

If $e_0 \in \mathcal{K} \backslash \mathcal{G}$, then

$$\mathcal{S}(e_0) = \mathcal{O}(e_0)$$

is non-null. If $e_0 \in \mathcal{K} \cap \mathcal{G}$, then

$$\mathcal{S}(e_0) = \mathcal{O}(e_0) \cup \mathcal{N}(e_0)$$

is also non-null. Hence, $(\mathcal{S},\mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSI. □

Theorem 4.7. *Let $(\mathcal{O},\mathcal{K})$ and $(\mathcal{N},\mathcal{G})$ be two $(\in, \in \vee q_k)$-AIFSIs. Suppose that*

$$\mathrm{Supp}(\mathcal{O},\mathcal{K}) \cap \mathrm{Supp}(\mathcal{N},\mathcal{G}) \neq \varnothing$$

and that $\mathcal{O}(e)$ and $\mathcal{N}(e)$ are comparable for every

$$e \in \mathcal{K} \cap \mathcal{G}.$$

Then

$$(\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G})$$

is an $(\in, \in \vee q_k)$*-AIFSI.*

Proof. Let

$$(\mathcal{O}, \mathcal{K}) \cap (\mathcal{N}, \mathcal{G}) = (\mathcal{S}, \mathcal{R}),$$

where

$$\mathcal{R} = \mathcal{K} \cap \mathcal{G}.$$

For every $e \in \mathcal{R}$,

$$\mathcal{S}(e) = \mathcal{O}(e) \cap \mathcal{N}(e).$$

By the comparability assumption, either

$$\mathcal{O}(e) \preceq \mathcal{N}(e)$$

or

$$\mathcal{N}(e) \preceq \mathcal{O}(e).$$

Therefore,

$$\mathcal{S}(e) = \mathcal{O}(e)$$

or

$$\mathcal{S}(e) = \mathcal{N}(e).$$

In either case, $\mathcal{S}(e)$ satisfies all the conditions in Definition 4.1.

Choose

$$e_0 \in \mathrm{Supp}(\mathcal{O}, \mathcal{K}) \cap \mathrm{Supp}(\mathcal{N}, \mathcal{G}).$$

Both $\mathcal{O}(e_0)$ and $\mathcal{N}(e_0)$ are non-null. Since they are comparable,

$$\mathcal{S}(e_0) = \mathcal{O}(e_0)$$

or

$$\mathcal{S}(e_0) = \mathcal{N}(e_0).$$

Thus, $\mathcal{S}(e_0)$ is non-null. Consequently, $(\mathcal{S}, \mathcal{R})$ is an $(\in, \in \vee q_k)$-AIFSI. □

5. Conclusion

In this paper, we introduced and investigated $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean rings and $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideals over Boolean rings. The two structures were formulated by using generalized anti-relations separately for the membership and non-membership components of intuitionistic fuzzy soft sets. Equivalent characterizations in terms of the threshold

$$\tau_k = \frac{1-k}{2}$$

were established, providing practical criteria for verifying the newly defined structures.

For $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean rings, we provided a finite example and proved that the OR operation and the union preserve the structure. A counterexample demonstrated that the AND operation and the intersection do not preserve it in general. Nevertheless, sufficient conditions based on comparability were obtained under which these two operations preserve the structure.

For $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideals, we characterized families of ordinary ideals through their anti-characteristic intuitionistic fuzzy soft sets. We proved that every $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft ideal is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft Boolean ring. An explicit example showed that the converse implication does not hold in general. We also established closure under the OR operation and the union and obtained sufficient comparability conditions for closure under the AND operation and the intersection.

These results provide a consistent algebraic framework for studying generalized anti-intuitionistic fuzzy soft structures over Boolean rings. Further research may consider homomorphisms, level subsets, quotient structures, products, and corresponding constructions in other classes of rings and related algebraic systems.

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