Asia Pac. J. Math. 2026 **13**:128

ENHANCING ORDERED SYSTEM MODELS THROUGH INFORMATION-THEORETIC AND CONCOMITANT METHODS WITH MECHANICAL ENGINEERING DATA APPLICATION

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Received Jul. 5, 2026

Abstract. This article develops a generating-informational function associated with the concomitant of generalized order statistics within the Huang–Kotz family of distributions. Explicit expansions of the proposed function are derived, and several well-known lifetime distributions are examined as illustrative special cases. A collection of theoretical results is established, including new upper and lower bounds obtained through Minkowski's inequality. In addition, preservation properties under various stochastic orderings are investigated, and a characterization of the parent distribution based on the cumulative generating-informational function is presented. To support statistical implementation, two nonparametric estimators of the proposed generating-informational function are constructed. Their performance is evaluated through an extensive simulation study and further illustrated using real data sets, allowing a comparative assessment of the two estimation procedures.

2020 Mathematics Subject Classification. 60G05; 94A17.

Key words and phrases. generating-informational function; generalized order statistics; concomitants; k-out-of-n systems; stochastic ordering; non-parametric estimation.

1. Introduction

Following the pioneering work of Claude E. Shannon [30], the field of information theory has witnessed the development of numerous measures designed to quantify uncertainty in random variables and probability distributions. The investigation of entropy-based quantities becomes even more meaningful when the underlying random variables exhibit dependence, especially in bivariate models arising in reliability and survival studies. A widely used dependence model is the Farlie–Gumbel–Morgenstern family of bivariate distributions, commonly known as the Morgenstern family. Morgenstern [27] originally introduced this family while studying bivariate distributions with prescribed Cauchy marginals. Owing to its analytical simplicity and tractable dependence structure, the Farlie–Gumbel–Morgenstern

DOI: 10.28924/APJM/13-128

model has become an attractive framework for constructing multivariate distributions when the marginal distributions are known in advance.

The original formulation was subsequently generalized by Johnson and Kotz [17], who extended the model to higher-dimensional settings and investigated its mathematical properties together with its statistical applications. Later, Huang and Kotz [14] proposed a modified version of the family by incorporating an additional parameter that substantially increases the range of attainable dependence between the component variables.

To overcome the limited dependence allowed by the classical Farlie–Gumbel–Morgenstern model, several generalized versions have subsequently been proposed. Among these, the polynomial extension introduced by Huang and Kotz [15] has attracted considerable attention. Their construction introduces a shape parameter that provides additional flexibility while preserving mathematical tractability. For this generalized model, the joint cumulative distribution function and the corresponding probability density function are respectively given by

$$H_{X,Y}(x,y) = H_X(x)H_Y(y)\left[1+\xi\big(1-H_X^{\vartheta}(x)\big)\big(1-H_Y^{\vartheta}(y)\big)\right], \tag{1}$$

and

$$h_{X,Y}(x,y) = h_X(x)h_Y(y)\left[1+\xi\left((1+\vartheta)H_X^{\vartheta}(x)-1\right)\left((1+\vartheta)H_Y^{\vartheta}(y)-1\right)\right], \quad \vartheta \geq 1, \tag{2}$$

where the admissible parameter set is $Q^* = \{(\xi,\vartheta) : -\vartheta^{-2} \leq \xi \leq \vartheta^{-1},\ \vartheta \geq 1\}$, and the resulting correlation coefficient satisfies $-3(\vartheta+2)^{-2}\min(1,\vartheta^2) \leq \text{Corr.} \leq 3\vartheta(\vartheta+2)^{-2}$.

Concomitants of order statistics were originally introduced by David *et al.* [7] as an important tool for investigating dependent random variables. Consider a random sample of size n consisting of independent observations (X_p, Y_p), $p = 1, 2, \ldots, n$, drawn from a bivariate population with joint cumulative distribution function $H_{X,Y}(x,y)$. If $X_{(k:n)}$ denotes the kth order statistic of the X-sample, then the corresponding observation of the second variable is called the concomitant of the kth order statistic and is denoted by $Y_{[k:n]}$. The probability density function and cumulative distribution function of $Y_{[k:n]}$ are respectively given by

$$h_{Y,[k:n]}(y) = \int_{-\infty}^{\infty} h_{Y|X}(y \mid x)\, h_{X,(k:n)}(x)\, dx, \tag{3}$$

and

$$H_{Y,[k:n]}(y) = \int_{-\infty}^{\infty} H_{Y|X}(y \mid x)\, h_{X,(k:n)}(x)\, dx, \tag{4}$$

where $h_{X,(k:n)}(x)$ denotes the probability density function of the order statistic $X_{(k:n)}$. Because concomitants preserve the dependence between associated variables, they have become an important component of modern statistical methodology. Their applications span numerous disciplines, including reliability and survival analysis, environmental modeling, economics, quality control, and ranked set sampling.

In recent years, research on concomitant distributions has expanded considerably, particularly in the areas of information-theoretic measures, estimation techniques, and generalized bivariate models. For instance, Mohamed [25] presented the measure of inaccuracy and its residual between distributions of concomitants of generalized order statistics and parent random variable. Almaspoor *et al.* [1] investigated extropy measures for concomitants of generalized order statistics under the Morgenstern family of distributions. Gamal *et al.* [11] examined Residual and Past Entropies of Concomitants from Lai And Xie Extensions of Case-II of Generalized Order Statistics and its Dual. More recently, Al-Zaydi [2] established several distributional properties together with best linear unbiased estimators for concomitants derived from a bivariate generalized Weibull model. In a subsequent contribution, Al-Zayd [3] developed both theoretical results and practical applications concerning concomitants from a bivariate generalized linear exponential distribution. Collectively, these studies demonstrate the increasing role of concomitant-based approaches in contemporary statistical inference and reliability analysis.

A unified theory for ordered observations was introduced by Kamps [18] through the framework of generalized order statistics. This formulation provides a common probabilistic model that includes several well-known ordered random variables as special cases, such as ordinary order statistics, sequential order statistics, record values, Pfeifer's records, and progressively Type-II censored order statistics. Let $n \in \mathbb{N}$, $m \geq 1$, and $d_1, \ldots, d_{n-1} \in \mathbb{R}$. For $1 \leq s \leq n-1$, define $D_s = \sum_{j=s}^{n-1} d_j$, and let $\beta_s^* = m + n - s + D_s$. Throughout this paper, we assume that $\beta_s^* \geq 1,\ s = 1, 2, \ldots, n-1$. Furthermore, write $\tilde{d} = (d_1, d_2, \ldots, d_{n-1}) \in \mathbb{R}^{n-1}$. A number of commonly used ordered sampling schemes can be obtained as particular cases of the class of d-generalized order statistics. An important special case is obtained by taking all the parameters to be equal, namely, $d_1 = d_2 = \cdots = d_{n-1} = d$. Under this assumption, the probability density function of the lth d-generalized order statistic, denoted by $Y_{(k,n,d,m)}$, is expressed as (see Kamps [18])

$$h_{Y,(k,n,d,m)}(y) = \frac{c_{l-1}^*}{(l-1)!}\big(1 - H_Y(y)\big)^{\beta_l^* - 1} h_Y(y) f_d^{\,l-1}\big(H_Y(y)\big), \tag{5}$$

where $1 \leq l \leq m$, $\beta_i^* = m + (n-l)(d+1)$, and $c_{l-1}^* = \prod_{j=1}^{l} \beta_j^*$. The auxiliary function $f_d(\cdot)$ is defined as $f_d(y) = h_{Y,d}(y) - h_{Y,d}(0)$, $0 < y < 1$, where

$$h_{Y,d}(y) = \begin{cases} \dfrac{-(1-y)^{d+1}}{d+1}, & d \neq -1, \\[2ex] -\ln(1-y), & d = -1. \end{cases}$$

Throughout the remainder of this paper, it is assumed that the parameter ϑ is a positive integer satisfying $\vartheta \geq 1$. Based on the Huang–Kotz dependence model, Mohamed [24] derived closed-form representations for the probability density function and cumulative distribution function of the concomitant corresponding to the lth d-generalized order statistic. These functions are given, respectively, by

$$h_{Y,[k,n,d,m]}(y) = h_Y(y)\left[1 + \xi\, \Xi_{[k,n,d,m;\vartheta]}\Big(1 - (1+\vartheta)H_Y^{\vartheta}(y)\Big)\right], \tag{6}$$

and

$$H_{Y,[k,n,d,m]}(y) = H_Y(y)\left[1 + \xi\, \Xi_{[k,n,d,m;\vartheta]}\Big(1 - H_Y^{\vartheta}(y)\Big)\right]. \tag{7}$$

With knowing $\beta_i^* = m + (n-l)(d+1)$, $1 \leq l \leq n$, the quantity $\Xi_{[k,n,d,m;\vartheta]}$ introduced in the preceding expressions is computed as

$$\Xi_{[k,n,d,m;\vartheta]} = 1 - (1+\vartheta)\sum_{j=0}^{\vartheta}\binom{\vartheta}{j}(-1)^j \prod_{i=1}^{l}\frac{\beta_i^*}{\beta_i^* + j}. \tag{8}$$

Generating functions play a central role in many areas of probability and statistics due to their ability to summarize distributional properties in a compact form. One of the best-known examples is the moment generating function, from which the moments of a random variable can be obtained through successive differentiation at the origin. In a similar spirit, information theory makes use of the *generating-informational function*, a quantity that provides a unified framework for constructing and studying various measures of uncertainty and information. This concept was originally proposed by Golomb [12]. For an absolutely continuous random variable with support $(0,\tau)$, where $\tau \in [0,+\infty)$ and probability density function h_Y, the generating-informational function is defined by

$$GIF_\eta(Y) = \int_0^{\tau} h_Y^{\eta}(y)\, dy, \tag{9}$$

where $\eta > 0$. It satisfies the normalization condition $GI_1(Y) = 1$. Furthermore,

$$\frac{\partial}{\partial\eta} GIF_\eta(Y)\big|_{\eta=1} = \int_0^{\tau} h_Y(y)\ln h_Y(y)\, dy = -Dn(Y), \tag{10}$$

where $Dn(Y)$ represents the differential entropy of Y; see Cover and Thomas [6]. Owing to its close relationship with entropy and information-theoretic quantities, this function has received considerable attention in the literature. As an illustration, Kharazmi and Balakrishnan [19] utilized the generating-informational function to formulate two classes of divergence measures that include a number of well-established divergence metrics as special cases. Additional contributions involving applications to order statistics and record-value distributions can be found in the works of Kharazmi and Balakrishnan [20] and Zamani *et al.* [33].

Capaldo *et al.* [5] introduced the residual-cumulative generating-informational function as a general framework that encompasses both classical and fractional entropy measures through the cumulative distribution function and the survival function. Let

$$l = \inf\{y \in \mathbb{R} : H_Y(y) > 0\}, \qquad \tau = \sup\{y \in \mathbb{R} : \bar{H}_Y(y) > 0\}, \tag{11}$$

represent the lower and upper endpoints of the support of the random variable Y, respectively, where either endpoint may be finite or infinite. The residual-cumulative generating-informational function associated with Y is given by

$$\begin{aligned} &RCGIF_{\eta,\theta}(Y) : D_Y \subseteq \mathbb{R}^2 \longrightarrow [0,\infty), \\ &\qquad (\eta,\theta) \longmapsto RCGIF_{\eta,\theta}(Y) = \int_l^\tau [H_Y(y)]^\eta [\bar{H}_Y(y)]^\theta \, dy, \end{aligned} \tag{12}$$

where $D_Y = \left\{(\eta,\theta) \in \mathbb{R}^2 : RCGIF_{\eta,\theta}(Y) < \infty\right\}$, $\bar{H}_Y(y) = 1 - H_Y(y)$ is the survival function. Since l and τ denote the endpoints of the support of Y, it follows that $\Pr(l \le Y \le \tau) = 1$. Furthermore, the residual-cumulative generating-informational function satisfies $RCGIF_{\eta,\theta}(Y) = 0$ whenever Y is degenerate, whereas $RCGIF_{\eta,\theta}(Y) > 0$ for every nondegenerate random variable Y.

This article investigates the generating-informational function associated with the concomitant of d-generalized order statistics under the Huang–Kotz extension, with the parameters satisfying $\eta \ge 1$ and $\vartheta \ge 1$. The expansion of the proposed function is derived, and its theoretical and inferential properties are examined. Specifically, Section 2 presents the expansion of the generating-informational function together with illustrative examples based on several well-known lifetime distributions. Section 3 establishes new bounds, investigates preservation properties under stochastic orderings, and derives characterization results for the parent distribution. Finally, Section 4 develops two nonparametric estimation procedures for the proposed function and assesses their performance through simulation experiments and applications to real data sets.

2. Cumulative generating-informational function of the concomitant of d-generalized order statistics

This section is devoted to developing the cumulative generating-informational function associated with the concomitant of d-generalized order statistics under the Huang–Kotz family of distributions.

We begin by recalling the cumulative generating-informational function introduced by Capaldo *et al.* [5]. Using (12), it is defined as

$$RCGIF_{\eta,0}(Y) = CGIF_\eta(Y) = \int_l^\tau [H_Y(y)]^\eta \, dy = \int_0^1 \frac{[H_Y(u)]^\eta}{h_Y(H_Y^{-1}(u))} \, du, \tag{13}$$

where $\eta > 0$, and $H_Y^{-1}(u) = \inf\{y : H_Y(y) \geq u\}$, for all $(\eta, \theta) \in D_Y$. From this definition, the following property is obtained directly:

$$\frac{\partial}{\partial \eta} CGIF_\eta(Y)\big|_{\eta=1} = \int_l^\tau H_Y(y) \ln H_Y(y)\, dy = -CSn(Y),$$

where $CSn(Y)$ denotes the cumulative entropy; see Di Crescenzo and Longobardi [8].

Now, let $Y_{[k,n,d,m]}$ denote the concomitant of kth d-generalized order statistics whose cumulative distribution function is given by (7) under the Huang–Kotz family. By applying (13), the cumulative generating-informational function corresponding to $Y_{[k,n,d,m]}$ is expressed as

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_l^\tau \left(H_{Y,[k,n,d,m]}(y)\right)^\eta\, dy \\ &= \int_l^\tau \left(H_Y(y) + \xi\, \Xi_{[k,n,d,m;\vartheta]} H_Y(y) \left(1 - H_Y^\vartheta(y)\right)\right)^\eta dy, \end{aligned} \tag{14}$$

where $\eta \geq 1$ and $\vartheta \geq 1$. To facilitate the subsequent derivations, we employ the probability integral transformation by defining $H_{U,[k,n,d,m]} = H_{Y,[k,n,d,m]}$. Accordingly, the random variables Y_i are transformed into $U_i = H_{Y_i}$, $i = 1, 2, \ldots, n$, so that each U_i follows the standard uniform distribution on $(0, 1)$. Assuming that $\eta \geq 1$ and $\vartheta \geq 1$, the cumulative distribution function of $U_{[k,n,d,m]}$ is therefore given by

$$H_{U,[k,n,d,m]}(u) = u + \xi\, \Xi_{[k,n,d,m;\vartheta]} u \left(1 - u^\vartheta\right), \forall\, 0 < u < 1. \tag{15}$$

Using (7), (13), and (15), together with the change of variable $u = H_Y(y)$, the cumulative generating-informational function can be rewritten as

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_l^\tau \left(H_{Y,[k,n,d,m]}(y)\right)^\eta\, dy \\ &= \int_0^1 \frac{\left(u + \xi\, \Xi_{[k,n,d,m;\vartheta]} u \left(1 - u^\vartheta\right)\right)^\eta}{h_Y(H_Y^{-1}(u))}\, du \\ &= \int_0^1 \frac{\left(H_{U,[k,n,d,m]}(u)\right)^\eta}{h_Y(H_Y^{-1}(u))}\, du, \end{aligned} \tag{16}$$

with noting that $H_Y^{-1}(u) = \inf\{y : H_Y(y) \geq u\}$.

The following theorem provides the expansions of the cumulative generating-informational function for the concomitant of d-generalized order statistics under the Huang–Kotz family.

Theorem 2.1. *Let $Y_{[k,n,d,m]}$ denote the concomitant of the kth d-generalized order statistic whose cumulative distribution function is given in (7). If*

$$\left|\xi\, \Xi_{[k,n,d,m;\vartheta]} \left(1 - H_Y^\vartheta(y)\right)\right| < 1,$$

then the cumulative generating-informational function admits the expansion

$$CGIF_\eta(Y_{[k,n,d,m]}) = \sum_{j=0}^{\infty} \binom{\eta}{j} \xi^j \Xi^j_{[k,n,d,m;\vartheta]} \int_l^\tau H_Y^\eta(y) \left(1 - H_Y^\vartheta(y)\right)^j dy, \tag{17}$$

where $\eta \geq 1$ *and* $\vartheta \geq 1$.

Proof. From (14),

$$CGIF_\eta(Y_{[k,n,d,m]}) = \int_l^\tau \left[H_Y(y) + \xi\, \Xi_{[k,n,d,m;\vartheta]} H_Y(y) \left(1 - H_Y^\vartheta(y)\right)\right]^\eta dy.$$

Let

$$A = H_Y(y), \qquad B = \xi\, \Xi_{[k,n,d,m;\vartheta]} H_Y(y) \left(1 - H_Y^\vartheta(y)\right).$$

Applying the generalized binomial theorem, which converges for $\left|\frac{B}{A}\right| < 1$, yields

$$CGIF_\eta(Y_{[k,n,d,m]}) = \int_l^\tau \sum_{j=0}^{\infty} \binom{\eta}{j} H_Y^{\eta-j}(y) \left[\xi\, \Xi_{[k,n,d,m;\vartheta]} H_Y(y) \left(1 - H_Y^\vartheta(y)\right)\right]^j dy.$$

By simplifications, we obtain

$$CGIF_\eta(Y_{[k,n,d,m]}) = \int_l^\tau \sum_{j=0}^{\infty} \binom{\eta}{j} \xi^j \Xi^j_{[k,n,d,m;\vartheta]} H_Y^\eta(y) \left(1 - H_Y^\vartheta(y)\right)^j dy.$$

Interchanging the order of summation and integration gives

$$CGIF_\eta(Y_{[k,n,d,m]}) = \sum_{j=0}^{\infty} \binom{\eta}{j} \xi^j \Xi^j_{[k,n,d,m;\vartheta]} \int_l^\tau H_Y^\eta(y) \left(1 - H_Y^\vartheta(y)\right)^j dy,$$

which completes the proof. □

Remark 2.1. *The class of* d*-generalized order statistics includes several familiar models as special cases. Two important examples are presented below.*

(1) *If* $d = 0$ *and* $m = 1$*, then the* d*-generalized order statistics coincide with the ordinary order statistics. Accordingly, for the concomitant corresponding to the kth order statistic* $Y_{[k:n]}$*, equation* (8) *becomes*

$$\begin{aligned} \Xi_{[k,n;\vartheta]} &= 1 - (1+\vartheta) \sum_{j=0}^{\vartheta} \binom{\vartheta}{j} (-1)^j \left(\prod_{i=1}^{k} \frac{n-i+1}{n-i+1+j} \right) \\ &= 1 - \frac{(1+\vartheta) Beta(k+\vartheta, n-k+1)}{Beta(k, n-k+1)}. \end{aligned} \tag{18}$$

Here, $Beta(a,b)$ *denotes the Beta function, which is defined by* $Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$*, where* $a, b > 0$.

(2) *Choosing* $d = -1$ *and* $m = 1$ *yields the record-value model. In this case, the quantity associated with the concomitant of the kth record value* $Y_{[k]}$ *follows directly from* (8) *and is given by*

$$\Xi_{[k;\vartheta]} = 1 - (1+\vartheta)\sum_{j=0}^{\vartheta}\binom{\vartheta}{j}(-1)^j\frac{1}{(j+1)^k}. \tag{19}$$

Equation (16) is applied in the example that follows.

Example 2.1. *The k–out–of–n configuration is a traditional system model that appears in numerous real-world applications in reliability engineering. Let n independent components with lifetimes $Y_1, Y_2, \ldots, Y_n$ make up the system. When at least k components are working, the system is still functional. As a result, the $(n-k+1)$th order statistic, represented by $Y_{n-k+1:n}$, provides the system lifetime.*

(1) *Assume that the component lifetimes are independent and identically distributed according to the power distribution, whose cumulative distribution function is given by*

$$H_Y(y) = y^{\lambda}, \qquad 0 < y < 1,\ \lambda > 0.$$

The corresponding quantile representation implies that

$$h_Y\big(H_Y^{-1}(u)\big) = \lambda u^{\frac{\lambda-1}{\lambda}}, \qquad 0 < u < 1.$$

By inserting this identity into (16), it follows that, for $\eta \geq 1$ and $\vartheta \geq 1$, the cumulative generating-informational function of the system lifetime is obtained as

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_0^1 \frac{\left(u + \xi\,\Xi_{[k,n,d,m;\vartheta]}u\left(1-u^{\vartheta}\right)\right)^{\eta}}{\lambda u^{\frac{\lambda-1}{\lambda}}}\,du \\ &= \int_0^1 \frac{\left(H_{U,[k,n,d,m]}(u)\right)^{\eta}}{\lambda u^{\frac{\lambda-1}{\lambda}}}\,du. \end{aligned} \tag{20}$$

Figure 1, corresponding to the k–out–of–n system defined in Eq. (20) with $n = 70$ and $k = 30$, demonstrates that the cumulative generating-informational function is a decreasing function of λ. Therefore, an increase in λ is accompanied by a reduction in the value of the cumulative generating-informational function, reflecting a lower level of uncertainty in the lifetime of the system. Likewise, for any fixed value of λ, increasing η causes the cumulative generating-informational function to decrease, indicating that greater values of η are associated with reduced uncertainty. This pattern is consistently observed for the parameter settings $(\vartheta,\xi) = (2, 0.5)$ and $(\vartheta,\xi) = (4, 0.25)$.

(2) *Assume that the component lifetimes are independent and identically distributed according to the common Kumaraswamy distribution, whose cumulative distribution function is given by*

$$H_Y(y) = 1 - \left(1 - y^{\lambda_1}\right)^{\lambda_2}, \qquad y > 0,\ \lambda_1, \lambda_2 > 0.$$

The corresponding quantile representation implies that

$$h_Y\big(H_Y^{-1}(u)\big) = \lambda_1\lambda_2\left[1 - (1-u)^{1/\lambda_2}\right]^{\frac{\lambda_1-1}{\lambda_1}}(1-u)^{\frac{\lambda_2-1}{\lambda_2}}, \qquad 0 < u < 1.$$

By inserting this identity into (16), it follows that, for $\eta \geq 1$ and $\vartheta \geq 1$, the cumulative generating-informational function of the system lifetime is obtained as

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_0^1 \frac{\left(u + \xi\, \Xi_{[k,n,d,m;\vartheta]} u \left(1 - u^{\vartheta}\right)\right)^\eta}{\lambda_1\lambda_2 \left[1-(1-u)^{1/\lambda_2}\right]^{\frac{\lambda_1-1}{\lambda_1}} (1-u)^{\frac{\lambda_2-1}{\lambda_2}}}\, du \\ &= \int_0^1 \frac{\left(H_{U,[k,n,d,m]}(u)\right)^\eta}{\lambda_1\lambda_2 \left[1-(1-u)^{1/\lambda_2}\right]^{\frac{\lambda_1-1}{\lambda_1}} (1-u)^{\frac{\lambda_2-1}{\lambda_2}}}\, du. \end{aligned} \tag{21}$$

The integral involved does not generally admit a convenient analytical evaluation because of its complicated structure in Eq. (21). As a result, numerical methods are employed to compute and analyze the values of $CGIF_\eta(Y_{[n-l+1:n]})$ for different choices of the Kumaraswamy distribution parameters λ_1 and λ_2, as illustrated in Figure 2. The figure depicts the variation of $CGIF_\eta(Y_{[n-l+1:n]})$ with respect to the parameters λ_1 and λ_2 for $n = 70$ and $k = 30$. The numerical results reveal that the cumulative generating-informational function decreases as the parameter λ_1 increases, whereas it increases monotonically with increasing values of λ_2. Moreover, the influence of λ_2 is more pronounced when λ_1 is relatively small, while the surface gradually levels off for larger values of λ_1. These findings indicate that the uncertainty measure is jointly governed by both Kumaraswamy shape parameters, with λ_2 contributing to an increase in the cumulative generating-informational function and λ_1 exhibiting an opposite effect.

(A) $\vartheta = 2; \xi = 0.5$ (B) $\vartheta = 4; \xi = 0.25$

FIGURE 1. Under the power distribution, behavior of $CGIF_\eta(Y_{[41:70]})$ in Example 2.1.

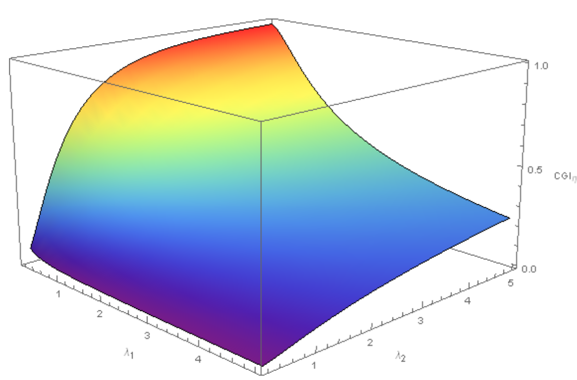

FIGURE 2. 3D plot of $CGIF_3(Y_{[41:70]})$, with respect to $\vartheta = 2$, and $\xi = 0.5$, in the case of Example 2.1.

3. Bounds and stochastic comparison for cumulative generating-informational function

In this section, we will discuss some bounds and stochastic order comparison for cumulative generating-informational function of the concomitant of d-generalized order statistics under the Huang–Kotz family.

3.1. Practical Bounds for cumulative generating-informational function.

Establishing specific bounds for various distributions with complex distribution functions or multiple components is important because there are no closed-form expressions for the cumulative generating-informational function of the concomitant of d-generalized order statistics under the Huang–Kotz family. In light of this issue, we examined how well these bounds describe the cumulative generating-informational function of successive systems. Our preliminary findings demonstrated that the joint cumulative generating-informational function of the system's constituent parts bounds the system's cumulative generating-informational function.

Additional straightforward and useful bounds for the probability density function extremes and the cumulative generating-informational function are given by the following theorem.

Theorem 3.1. *Let $Y_{[k,n,d,m]}$ represent the concomitant corresponding to the kth d-generalized order statistic having common probability density function $h_Y(y)$ and cumulative distribution function $H_Y(y)$. If S is the support of h_Y, and*

$$m = \inf_{y\in S} h_Y(y), \qquad M = \sup_{y\in S} h_Y(y),$$

such that $0 < m < M < 1$, then

$$\frac{CGIF_\eta(U_{[k,n,d,m]})}{M} \le CGIF_\eta(Y_{[k,n,d,m]}) \le \frac{CGIF_\eta(U_{[k,n,d,m]})}{m}, \tag{22}$$

where

$$CGIF_\eta(U_{[k,n,d,m]}) = \int_0^1 \left(H_{U,[k,n,d,m]}(u)\right)^\eta \, du. \tag{23}$$

Proof. Since $0 < m \le h_Y(H_Y^{-1}(u)) \le M < 1, 0 < u < 1$, it follows that

$$\frac{1}{M}\int_0^1 \left(H_{U,[k,n,d,m]}(u)\right)^\eta \, du \le \int_0^1 \frac{H^\eta_{U,[k,n,d,m]}(u)}{h_Y(H_Y^{-1}(u))}\, du \le \frac{1}{m}\int_0^1 \left(H_{U,[k,n,d,m]}(u)\right)^\eta \, du.$$

Using Equation (16), the required inequalities follow immediately. □

The quantity $CGIF_\eta(U_{[k,n,d,m]})$ denotes the cumulative generating-informational function associated with the concomitant of d-generalized order statistics when the parent distribution is uniform over the interval $(0,1)$. The inequalities in Equation (22) depend explicitly on the minimum and maximum values attained by the density function h_Y. Consequently, if $m = 0$, the right-hand inequality is no longer useful because it fails to provide a finite upper bound. Similarly, when $M = \infty$, the left-hand inequality loses its effectiveness, as no finite lower bound can be established under this circumstance.

Theorem 3.2. *The concomitant corresponding to the kth d-generalized order statistic from a sample of size n made up of independent, identically distributed random variables with probability density function $h_Y(y)$ is represented by $Y_{[k,n,d,m]}$. Then,*

$$S_{1;\eta} CGIF_\eta(Y) \le CGIF_\eta(Y_{[k,n,d,m]}) \le S_{2;\eta} CGIF_\eta(Y), \tag{24}$$

where $\eta \ge 1$ and $\vartheta \ge 1$, $S_{1;\eta} = \inf_{u\in(0,1)} \frac{H^\eta_{U,[k,n,d,m]}(u)}{H^\eta_U(u)}$, $S_{2;\eta} = \sup_{u\in(0,1)} \frac{H^\eta_{U,[k,n,d,m]}(u)}{H^\eta_U(u)}$.

Proof. From (16), the upper bound can be found as follows:

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_0^1 \frac{H^\eta_{U,[k,n,d,m]}(u)}{h_Y(H_Y^{-1}(u))}\, du \\ &= \int_0^1 \frac{H^\eta_{U,[k,n,d,m]}(u)}{H^\eta_U(u)} \frac{H^\eta_U(u)}{h_Y(H_Y^{-1}(u))}\, du \\ &\le \sup_{u\in(0,1)} \frac{H^\eta_{U,[k,n,d,m]}(u)}{H^\eta_U(u)} \int_0^1 \frac{H^\eta_U(u)}{h_Y(H_Y^{-1}(u))}\, du = S_{2;\eta} CGIF_\eta(Y). \end{aligned} \tag{25}$$

The result can be obtained by deriving the lower bound in a similar manner. □

Minkowski's inequality is then used to derive appropriate bounds for the measures presented in $CGIF_\eta(Y_{[k,n,d,m]})$ provided by (14).

Theorem 3.3. *The concomitant corresponding to the kth d-generalized order statistic from a sample of size n made up of independent, identically distributed random variables with probability density function $h_Y(y)$ is represented by $Y_{[k,n,d,m]}$.*

$$\left[(\tau-l)^{\frac{1}{\eta}} - \left(RGIF_\eta(Y_{[k,n,d,m]})\right)^{\frac{1}{\eta}}\right]^\eta \le CGIF_\eta(Y_{[k,n,d,m]}) \le \left[(\tau-l)^{\frac{1}{\eta}} + \left(RGIF_\eta(Y_{[k,n,d,m]})\right)^{\frac{1}{\eta}}\right]^\eta,$$

where $\eta \ge 1$ and $\vartheta \ge 1$, and, from (12), we have $RCGIF_{0,\eta}(Y_{[k,n,d,m]}) = RGIF_\eta(Y_{[k,n,d,m]}) = \int_l^\tau \left(\bar{H}_{Y,[k,n,d,m]}(y)\right)^\eta dy$ is the residual cumulative generating-informational function corresponding to $Y_{[k,n,d,m]}$.

Proof. By using Minkowski's inequality, and for $\eta \ge 1$, we have

$$\begin{aligned} \left[CGIF_\eta(Y_{[k,n,d,m]})\right]^{\frac{1}{\eta}} &= \left[\int_l^\tau \left(1-\bar{H}_{Y,[k,n,d,m]}(y)\right)^\eta dy\right]^{\frac{1}{\eta}} \\ &\le (\tau-l)^{\frac{1}{\eta}} + \left[\int_l^\tau \left(\bar{H}_{Y,[k,n,d,m]}(y)\right)^\eta dy\right]^{\frac{1}{\eta}} \\ &= (\tau-l)^{\frac{1}{\eta}} + \left[RGIF_\eta(Y_{[k,n,d,m]})\right]^{\frac{1}{\eta}}. \end{aligned}$$

On the other hand,

$$(\tau - l)^{\frac{1}{\eta}} = \left[\int_l^\tau \left(H_{Y,[k,n,d,m]}(y) + \bar{H}_{Y,[k,n,d,m]}(y) \right)^\eta \, dy \right]^{\frac{1}{\eta}}$$
$$\leq \left[\int_l^\tau \left(H_{Y,[k,n,d,m]}(y) \right)^\eta \, dy \right]^{\frac{1}{\eta}} + \left[\int_l^\tau \left(\bar{H}_{Y,[k,n,d,m]}(y) \right)^\eta \, dy \right]^{\frac{1}{\eta}}$$
$$= \left[CGIF_\eta(Y_{[k,n,d,m]}) \right]^{\frac{1}{\eta}} + \left[RGIF_\eta(Y_{[k,n,d,m]}) \right]^{\frac{1}{\eta}}.$$

The boundaries are obtained by combining the two latter inequalities. □

3.2. **Comparative Results via Stochastic Orders.** This subsection investigates the cumulative generating-informational function associated with the concomitant of d-generalized order statistics when the underlying lifetime distributions satisfy the dispersive order and the location-independent riskier order. Before establishing the main results, we briefly recall these stochastic concepts. A detailed treatment of their definitions and fundamental properties can be found in Shaked and Shanthikumar [29]. Throughout this subsection, let

$$\mathbb{R}_+^* = \{Y : Y \geq 0\}$$

denote the class of absolutely continuous nonnegative random variables whose support is specified in (12). Let Y_1 and Y_2 be two nonnegative random variables with probability density functions h_{Y_1} and h_{Y_2}, cumulative distribution functions H_{Y_1} and H_{Y_2}, survival functions $\bar{H}_{Y_1}$ and $\bar{H}_{Y_2}$, respectively. Their corresponding hazard rate functions are defined by

$$\varsigma_{Y_j}(y) = \frac{h_{Y_j}(y)}{\bar{H}_{Y_j}(y)}, \qquad j = 1, 2.$$

The definitions of several frequently used stochastic orderings and aging classes employed in this work are summarized in Table 1. For the remainder of this subsection, we define

$$\Lambda_\eta(u) = u^\eta, \qquad 0 \leq u \leq 1. \tag{26}$$

TABLE 1. Definitions of several stochastic orderings and aging properties.

Property/Order	Definition
Hazard Rate Order	The notation $Y_1 \leq^{ho_y} Y_2$ is used whenever $$\varsigma_{Y_1}(r) \geq \varsigma_{Y_2}(r), \qquad r > 0.$$ In this case, Y_1 is regarded as smaller than Y_2 with respect to the hazard rate ordering.

Continued on next page

Table 1 (*continued*)

Property/Order	Definition
Dispersive Order	The relation $Y_1 \leq^{\mathrm{dpo}_y} Y_2$ holds if $$H_{Y_1}^{-1}(r_1) - H_{Y_1}^{-1}(r_2) \leq H_{Y_2}^{-1}(r_1) - H_{Y_2}^{-1}(r_2),$$ for every $0 < r_1 \leq r_2 < 1$. An equivalent characterization is $$h_{Y_1}\left(H_{Y_1}^{-1}(v)\right) \geq h_{Y_2}\left(H_{Y_2}^{-1}(v)\right), \qquad 0 < v < 1.$$ Thus, the spacing between corresponding quantiles of Y_1 does not exceed that of Y_2.
Location-Independent Riskier Order	The random variable Y_1 is said to precede Y_2 in the location-independent riskier order, denoted by $Y_1 \leq^{lro_y} Y_2$, whenever $$\int_0^{H_{Y_1}^{-1}(r)} H_{Y_1}(y)\,dy \leq \int_0^{H_{Y_2}^{-1}(r)} H_{Y_2}(y)\,dy, \qquad r \in (0,1).$$ This inequality must be satisfied for all probability levels r in the interval $(0,1)$.
InFR/DnFR	The random variable Y_1 is said to exhibit an increasing failure rate (InFR) if $\varsigma_{Y_1}(y)$ increases with y. Likewise, Y_1 possesses a decreasing failure rate (DnFR) whenever $\varsigma_{Y_1}(y)$ decreases as y increases.

The concept of the dispersive order, denoted by $\leq^{\mathrm{dpo}_y}$, was first established by Bickel and Lehmann [4] in studies of nonparametric statistics. Subsequently, Jewitt [16] introduced the location-independent riskier order, represented by $\leq^{lro_y}$, as a useful criterion in expected utility analysis and insurance-related decision making. A well-known characterization of the dispersive order is given by

$$h_{Y_2}\left(H_{Y_2}^{-1}(v)\right) \leq h_{Y_1}\left(H_{Y_1}^{-1}(v)\right), \qquad 0 < v < 1, \tag{27}$$

which holds if and only if $Y_1 \leq^{\mathrm{dpo}_y} Y_2$. In addition, these stochastic orderings satisfy the implication

$$Y_1 \leq^{ho_y} Y_2 \text{ and either } Y_1 \text{ or } Y_2 \text{ is DnFR} \Longrightarrow Y_1 \leq^{\mathrm{dpo}_y} Y_2 \Longrightarrow Y_1 \leq^{lro_y} Y_2. \tag{28}$$

Since $\Lambda_\eta(x) \geq 0$ for all $x \in [0,1]$ whenever $\eta \geq 1$, an application of (13) together with the characterization in (27) shows that $CGIF_\eta(Y_1) \leq CGIF_\eta(Y_2)$, provided that $Y_1 \leq^{\mathrm{dpo}_y} Y_2$. Consequently, by invoking the implication chain in (28), the following result is obtained.

Corollary 3.1. *Suppose that $Y_1 \leq^{ho_y} Y_2$ and that at least one of the random variables Y_1 or Y_2 possesses the DnFR property. Then $CGIF_\eta(Y_1) \leq CGIF_\eta(Y_2), \qquad \forall\, \eta \geq 1$.*

Let T be a random variable having cumulative distribution function H_V. Its cumulative reversed hazard function is defined by

$$\varphi_T(t) = \int_0^t H_V(v)\,dv, \qquad z > 0. \tag{29}$$

A characterization of the location-independent riskier order was established by Landsberger and Meilijson [23], who proved that

$$Y_1 \leq^{lro_y} Y_2 \iff \varphi_{Y_2}^{-1}(t) - \varphi_{Y_1}^{-1}(t) \text{ is increasing for } t > 0, \tag{30}$$

where φ^{-1} denotes the left-continuous inverse associated with φ.

The following result utilizes the preceding ordering relations to establish a comparison between generalized cumulative generating information measures. Assuming that the component lifetime distributions satisfy the DnFR property, it is shown that the $CGIF_\eta$ of a series system with l components is upper bounded by the corresponding $CGIF_\eta$ of the concomitant associated with the kth d-generalized order statistic in the Huang–Kotz family.

Theorem 3.4. *Suppose that $Y_{[k,n,d,m]}$ is the lifetime of the concomitant corresponding to the kth d-generalized order statistic from the Huang–Kotz family, where the underlying distribution has probability density function $h_{Y_1}(x)$ and cumulative distribution function $H_{Y_1}(x)$. If the parent random variable Y_1 is DnFR, then, for every $\eta \geq 1$,*

$$CGIF_\eta(X_{1:n}) \leq CGIF_\eta(Y_{[k,n,d,m]}).$$

Proof. Since Y_1 is assumed to satisfy the DnFR property, the same aging property is inherited by $X_{1:n}$. In addition, Theorem 4.5 of Eryılmaz and Navarro [10] establishes that $X_{1:n} \leq^{ho_y} Y_{[k,n,d,m]}$. Therefore, the stated inequality follows as an immediate consequence of Corollary 3.1. □

The next theorem establishes a condition guaranteeing that the dispersive order is preserved by the concomitants of the kth d-generalized order statistic in the Huang–Kotz family. Its proof is omitted because it is a straightforward application of the preceding results.

Theorem 3.5. *Let $Y_{1,[k,n,d,m]}$ and $Y_{2,[k,n,d,m]}$ denote two concomitant lifetimes associated with the kth d-generalized order statistic from the Huang–Kotz family. Assume that their corresponding probability density functions are $h_{Y_1}(x)$ and $h_{Y_2}(x)$, while their cumulative distribution functions are $H_{Y_1}(x)$ and $H_{Y_2}(x)$, respectively. If $Y_1 \leq^{\mathrm{dpo}_y} Y_2$, then*

$$CGIF_\eta\left(Y_{1,[k,n,d,m]}\right) \leq CGIF_\eta\left(Y_{2,[k,n,d,m]}\right), \qquad \forall\, \eta \geq 0.$$

The next theorem identifies sufficient conditions ensuring that the concomitant of the kth d-generalized order statistic from the Huang–Kotz family preserves the location-independent riskier order.

Theorem 3.6. *Under the assumptions of Theorem 3.5, define the function*

$$\phi_{[k,n,d,m;\vartheta]}(v) = v + \xi\,\Xi_{[k,n,d,m;\vartheta]}v\left(1 - v^{\vartheta}\right), \qquad 0 < v < 1. \tag{31}$$

Assume that $Y_1 \leq^{lro_y} Y_2$, and that the mapping $\frac{\Lambda_\eta(\phi_{[k,n,d,m;\vartheta]}(v))}{v}, 0 \leq v \leq 1$, is monotone decreasing in v for every $\eta \geq 1$. Then

$$CGIF_\eta\big(Y_{1,[k,n,d,m]}\big) \leq CGIF_\eta\big(Y_{2,[k,n,d,m]}\big), \qquad \forall\, \eta \geq 1.$$

Proof. Rewriting (7) yields

$$H_{Y,[k,n,d,m]}(y) = \phi_{[k,n,d,m;\vartheta]}\big(H_Y(y)\big).$$

Given that $Y_1 \leq^{lro_y} Y_2$, (30) and (29) imply that

$$\frac{d}{dz}\left(\varphi_{Y_2}^{-1}(z) - \varphi_{Y_1}^{-1}(z)\right) = H_{Y_2}\big(\varphi_{Y_2}^{-1}(z)\big) - H_{Y_1}\big(\varphi_{Y_1}^{-1}(z)\big) \geq 0.$$

As a result, the mapping $\varphi_{Y_2}^{-1}(z) - \varphi_{Y_1}^{-1}(z)$ is nondecreasing, which produces

$$H_{Y_1}(z) \geq H_{Y_2}\big(\varphi_{Y_2}^{-1}(\varphi_{Y_1}(z))\big), \qquad z > 0. \tag{32}$$

We may write using (7), (16), and (26).

$$\begin{aligned}\int_l^\tau \left(H_{Y_1,[k,n,d,m]}(y)\right)^\eta dy &= \int_l^\tau \Lambda_\eta\left(H_{Y_1,[k,n,d,m]}(y)\right) dy \\ &= \int_l^\tau \frac{\Lambda_\eta\left(\phi_{[k,n,d,m;\vartheta]}(H_{Y_1}(y))\right)}{H_{Y_1}(y)} H_{Y_1}(y)\, dy \\ &\leq \int_l^\tau \frac{\Lambda_\eta\left(\phi_{[k,n,d,m;\vartheta]}\left(H_{Y_2}\big(\varphi_{Y_2}^{-1}(\varphi_{Y_1}(y))\big)\right)\right)}{H_{Y_2}\big(\varphi_{Y_2}^{-1}(\varphi_{Y_1}(y))\big)} H_{Y_1}(y)\, dy.\end{aligned} \tag{33}$$

The aforementioned inequality follows directly from (32) and the presumptive monotonicity of $\Lambda_\eta(\phi_{[k,n,d,m;\vartheta]}(t))/t$. The variable change $u = \varphi_{Y_2}^{-1}(\varphi_{Y_1}(y))$ is then introduced. Differentiation provides

$$dy = \frac{H_{Y_2}(u)}{H_{Y_1}\big(\varphi_{Y_1}^{-1}(\varphi_{Y_2}(u))\big)}\, du.$$

When this phrase is substituted on the right side of (33), it results in

$$\begin{aligned}&\int_{\varphi_{Y_2}^{-1}(\varphi_{Y_1}(0))}^{\infty} \frac{\Lambda_\eta\left(\phi_{[k,n,d,m;\vartheta]}(H_{Y_2}(u))\right)}{H_{Y_2}(u)} H_{Y_1}\big(\varphi_{Y_1}^{-1}(\varphi_{Y_2}(u))\big) \frac{H_{Y_2}(u)}{H_{Y_1}\big(\varphi_{Y_1}^{-1}(\varphi_{Y_2}(u))\big)}\, du \\ &= \int_{\varphi_{Y_2}^{-1}(\varphi_{Y_1}(0))}^{\infty} \Lambda_\eta\left(\phi_{[k,n,d,m;\vartheta]}(H_{Y_2}(u))\right) du\end{aligned}$$

$$= \int_l^\tau \Lambda_\eta\big(H_{Y_2,[k,n,d,m]}(y)\big)\ dy$$
$$= \int_l^\tau \big(H_{Y_2,[k,n,d,m]}(y)\big)^\eta\ dy.$$

The penultimate equality is obtained by using the identity

$$H_{Y_2,[k,n,d,m]}(y) = \phi_{[k,n,d,m;\vartheta]}\big(H_{Y_2}(y)\big).$$

Moreover, the lower bound of the integral reduces to zero since

$$\varphi_{Y_2}^{-1}(\varphi_{Y_1}(0)) = 0.$$

Consequently, the preceding inequalities imply that

$$CGIF_\eta\big(Y_{1,[k,n,d,m]}\big) \leq CGIF_\eta\big(Y_{2,[k,n,d,m]}\big)\,,$$

for all $\eta \geq 1$, thereby completing the proof. □

3.3. **Characterization of the parent distribution via the cumulative generating-informational function.** This subsection develops characterization results based on the cumulative generating-informational function of the lifetime of the concomitant associated with the kth d-generalized order statistic from the Huang–Kotz family. Several characterization results for symmetric continuous distributions have previously been established through extropy-based uncertainty measures, including cumulative residual extropy and cumulative past extropy; see [13] and [26]. These studies showed that symmetric distributions can be identified by the coincidence of the corresponding measures for upper and lower order statistics. Moreover, the same characterization property was verified for concomitants of order statistics arising from the Farlie–Gumbel–Morgenstern family. Motivated by these findings, we show that the cumulative generating-informational function of the lifetime of the concomitant corresponding to the kth d-generalized order statistic in the Huang–Kotz family provides a unique characterization of the underlying parent lifetime distribution.

Theorem 3.7. *Let $Y_{1,[k,n,d,m]}$ and $Y_{2,[k,n,d,m]}$ denote the concomitants of the kth d-generalized order statistics under the Huang–Kotz family, with probability density functions $h_{Y_1}(y)$ and $h_{Y_2}(y)$ and cumulative distribution functions $H_{Y_1}(y)$ and $H_{Y_2}(y)$, respectively. Then H_{Y_1} and H_{Y_2} belong to the same family of distributions if and only if $Y_1 \leq^{\mathrm{dpo}_y} Y_2$ and*

$$CGIF_\eta\big(Y_{1,[k,n,d,m]}\big) = CGIF_\eta\big(Y_{2,[k,n,d,m]}\big)\,,$$

for all $\eta \geq 1$.

Proof. The necessity is immediate. It remains to establish the sufficiency. Using the representation of the cumulative generating-information function given in (13), we obtain

$$CGIF_\eta\left(Y_{1,[k,n,d,m]}\right) = \int_0^1 \frac{\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)}{h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}\,du, \tag{34}$$

The same argument applies to $CGIF_\eta(Y_{2,[k,n,d,m]})$. Using (34) together with the assumption $CGIF_\eta\left(Y_{1,[k,n,d,m]}\right) = CGIF_\eta\left(Y_{2,[k,n,d,m]}\right)$, we obtain

$$\int_0^1 \Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)\left[\frac{1}{h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)} - \frac{1}{h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}\right]du = 0. \tag{35}$$

When $\eta \geq 1$, we have $0 < H_{U,[k,n,d,m]}(u) < 1$, $0 < u < 1$, Hence, $\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right) > 0, \qquad 0 < u < 1.$ The assumption $X \leq^{\mathrm{dpo}_y} Y$ implies, by relation (27), that

$$h_{Y_1}\left(H_{Y_1}^{-1}(u)\right) \geq h_{Y_2}\left(H_{Y_2}^{-1}(u)\right), \qquad 0 < u < 1.$$

Therefore, the expression inside the brackets in (35) is nonpositive. Since $\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right) > 0$ and the integral is zero, it follows that $h_{Y_1}\left(H_{Y_1}^{-1}(u)\right) = h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)$, $u \in (0,1)$. Consequently, $H_{Y_1}^{-1}(u) = H_{Y_2}^{-1}(u) + d$, where d is a constant. Since

$$\lim_{u\to 0} H_{Y_1}^{-1}(u) = \lim_{u\to 0} H_{Y_2}^{-1}(u) = 0,$$

we conclude that $H_{Y_1}^{-1}(u) = H_{Y_2}^{-1}(u)$. Hence, H_{Y_1} and H_{Y_2} belong to the same family of distributions. □

The properties of a series system are described in the following corollary.

Corollary 3.2. *Under the conditions of Theorem 3.7, H_{Y_1} and H_{Y_2} belong to the same family of distributions if and only if $Y_1 \leq^{\mathrm{dpo}_y} Y_2$ and*

$$CGIF_\eta\left(Y_{1,[n,n,d,m]}\right) = CGIF_\eta\left(Y_{1,[n,n,d,m]}\right),$$

for all $n \geq 1$.

The following theorem provides a further characterization.

Theorem 3.8. *Under the conditions of Theorem 3.7, H_{Y_1} and H_{Y_2} belong to the same family of distributions, up to a change of scale, if and only if $Y_1 \leq^{\mathrm{dpo}_y} Y_2$ and*

$$\frac{CGIF_\eta\left(Y_{1,[k,n,d,m]}\right)}{CGIF_\eta(Y_2)} = \frac{CGIF_\eta\left(Y_{2,[k,n,d,m]}\right)}{CGIF_\eta(Y_1)}, \tag{36}$$

for all $\eta \geq 1$.

Proof. The necessity is obvious, so only the sufficiency needs to be established.

From (36), we obtain

$$\frac{CGIF_\eta\big(Y_{1,[k,n,d,m]}\big)}{CGIF_\eta(Y_2)} = \int_0^1 \frac{\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)}{h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)\, CGIF_\eta(Y)}\, du, \tag{37}$$

where $\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)$ is defined (26). Similarly, $\frac{CGIF_\eta\left(Y_{2,[k,n,d,m]}\right)}{CGIF_\eta(Y_1)}$ admits the analogous representation. Hence, from (36) and (37),

$$\int_0^1 \frac{\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)}{CGIF_\eta(Y_2)h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}\, du = \int_0^1 \frac{\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)}{CGIF_\eta(Y_1)h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)}\, du. \tag{38}$$

Let $\varpi = \frac{CGIF_\eta(Y_2)}{CGIF_\eta(Y_1)}$. By the assumption $Y_1 \leq^{\mathrm{dpo}_y} Y_2$, it follows that $\varpi \geq 1$ for all $\eta \geq 1$. Moreover, (38) can be rewritten as

$$\int_0^1 \left[\frac{1}{h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)} - \frac{1}{\varpi\, h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}\right] \Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right)\, du = 0. \tag{39}$$

Since $Y_1 \leq^{\mathrm{dpo}_y} Y_2$, we have $\frac{h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}{h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)} \geq 1,\ \forall\, 0 < u < 1$. Because $\varpi \geq 1$, it follows that $\frac{\varpi\, h_{Y_1}\left(H_{Y_1}^{-1}(u)\right)}{h_{Y_2}\left(H_{Y_2}^{-1}(u)\right)} \geq 1,\ \forall 0 < u < 1$. Hence, $\varpi\, h_{Y_1}\left(H_{Y_1}^{-1}(u)\right) \geq h_{Y_2}\left(H_{Y_2}^{-1}(u)\right),\ \forall\, 0 < u < 1$, so that the expression inside the brackets of (39) is nonpositive. Since $\Lambda_\eta\left(H_{U,[k,n,d,m]}(u)\right) > 0$ for all $u \in (0,1)$ and the integral in (39) is equal to zero, we conclude that

$$h_{Y_1}\left(H_{Y_1}^{-1}(u)\right) = \varpi\, h_{Y_2}\left(H_{Y_2}^{-1}(u)\right), \qquad u \in (0,1).$$

Therefore, $H_{Y_1}^{-1}(u) = \varpi\, H_{Y_2}^{-1}(u) + d$, where d is a constant. Since $\lim_{u\to 0} H_{Y_1}^{-1}(u) = \lim_{u\to 0} H_{Y_2}^{-1}(u) = 0$, we obtain $H_{Y_1}^{-1}(u) = \varpi\, H_{Y_2}^{-1}(u)$. Consequently, H_{Y_1} and H_{Y_2} belong to the same family of distributions up to a change of scale. □

As an immediate consequence of Theorem 3.8, we obtain the following corollary.

Corollary 3.3. *Suppose that the assumptions of Theorem 3.8 hold. Then H_{Y_1} and H_{Y_2} belong to the same family of distributions, up to a change of scale, if and only if $Y_1 \leq^{\mathrm{dpo}_y} Y_2$ and*

$$\frac{CGIF_\eta\big(Y_{1,[n,n,d,m]}\big)}{CGIF_\eta(Y_2)} = \frac{CGIF_\eta\big(Y_{1,[n,n,d,m]}\big)}{CGIF_\eta(Y_1)},$$

for all $n \geq 1$.

4. Nonparametric estimation applications

This section introduces two nonparametric, distribution-free estimators for CGIF_η corresponding to the concomitant $Y_{[k,n,d,m]}$. Let $Y_1, Y_2, \ldots, Y_N$ be independent and identically distributed nonnegative absolutely continuous random variables, and let their associated order statistics be denoted by $Y_{1:N} \leq$

$Y_{2:N} \leq \cdots \leq Y_{N:N}$. By combining (26) with (31), the cumulative generating-informational function of the concomitant $Y_{[k,n,d,m]}$ can be represented in the following equivalent form:

$$\begin{aligned} CGIF_\eta(Y_{[k,n,d,m]}) &= \int_0^1 \frac{\Lambda_\eta\big(H_{Y,[k,n,d,m]}(u)\big)}{h_Y(H_Y^{-1}(u))}\,du \\ &= \int_0^1 \Lambda_\eta\big(\phi_{[k,n,d,m;\vartheta]}(u)\big)\left[\frac{d}{du}K_Y^{-1}(u)\right]du \\ &= \int_0^1 \left[u + \xi\,\Xi_{[k,n,d,m;\vartheta]}u\left(1-u^\vartheta\right)\right]^\eta \left[\frac{d}{du}K_Y^{-1}(u)\right]du, \end{aligned} \tag{40}$$

under the conditions $\eta \geq 1$ and $\vartheta \geq 1$. Formulation (40) operates as the basis for building the estimators of $\mathrm{CGIF}_\eta(Y_{[k,n,d,m]})$.

The first proposed estimator is obtained by approximating the derivative of the quantile function at the observed order statistics. Adopting the finite-difference approach of Vasicek [32], the derivative $\frac{d}{du}K_Y^{-1}(u)$ is estimated by

$$\frac{d}{du}K_Y^{-1}(u) \approx \frac{N\big(Y_{i+\nu:N} - Y_{i-\nu:N}\big)}{2\nu},$$

where the end-point corrections are defined by setting $Y_{i:N} = Y_{1:N}$ for $i < 1$ and $Y_{i:N} = Y_{N:N}$ for $i > N$. In this approximation, N represents the sample size, whereas ν is an integer satisfying $\nu \leq N/2$, referred to as the window parameter. Incorporating the boundary correction proposed by Ebrahimi *et al.* [9] yields a refined estimator of $CGIF_\eta(Y_{[k,n,d,m]})$, which is given by

$$\begin{aligned} \widehat{CGIF}_{\eta;1}(Y_{[k,n,d,m]}) &= \frac{1}{N}\sum_{i=1}^{N} \Lambda_\eta\left(\phi_{[k,n,d,m;\vartheta]}\left(\frac{i}{N+1}\right)\right)\left(\frac{N\big(Y_{i+\nu:N} - Y_{i-\nu:N}\big)}{\Delta_j \nu}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{i}{N+1}\right) + \xi\,\Xi_{[k,n,d,m;\vartheta]}\left(\frac{i}{N+1}\right)\left(1-\left(\frac{i}{N+1}\right)^\vartheta\right)\right]^\eta \\ &\quad \times \left(\frac{N\big(Y_{i+\nu:N} - Y_{i-\nu:N}\big)}{\Delta_j \nu}\right), \end{aligned} \tag{41}$$

with noting that the weights Δ_j are defined by

$$\Delta_j = \begin{cases} 1 + \dfrac{j-1}{\nu}, & 1 \leq j \leq \nu, \\ 2, & \nu+1 \leq j \leq N-\nu, \\ 1 + \dfrac{N-j}{\nu}, & N-\nu+1 \leq j \leq N. \end{cases} \tag{42}$$

A second distribution-free estimator is constructed by utilizing the empirical distribution function corresponding to $H_Y(y)$, defined as

$$H_{Y_N}(y) = \sum_{i=1}^{N-1} \frac{N-i}{N}\, I_{[Y_{i:N},\, Y_{i+1:N})}(y), \qquad y \geq 0,$$

where the indicator function is given by

$$I_A(y) = \begin{cases} 1, & y \in A, \\ 0, & y \notin A. \end{cases}$$

Substituting this empirical distribution function into (16) yields the empirical counterpart of the cumulative generating-informational function for $Y_{[k,n,d,m]}$, namely,

$$\begin{aligned} \widehat{CGIF}_{\eta;2}(Y_{[k,n,d,m]}) &= \int_0^\infty \left(H_{Y,[k,n,d,m]}(y) \right)^\eta dy \\ &= \sum_{i=1}^{N-1} \int_{Y_{i:N}}^{Y_{i+1:N}} \Lambda_\eta \left(\phi_{[k,n,d,m;\vartheta]} \left(\frac{i}{N+1} \right) \right) dy \\ &= \sum_{i=1}^{N-1} \left[\left(\frac{i}{N+1} \right) + \xi \, \Xi_{[k,n,d,m;\vartheta]} \left(\frac{i}{N+1} \right) \left(1 - \left(\frac{i}{N+1} \right)^\vartheta \right) \right]^\eta S_{i+1}, \end{aligned} \tag{43}$$

with noting that

$$S_{i+1} = Y_{i+1:N} - Y_{i:N}, \qquad i = 1, 2, \ldots, N-1,$$

denotes the difference between consecutive order statistics, commonly referred to as the sample spacings.

A Monte Carlo simulation based on the power distribution,

$$H_Y(y) = y^\lambda, \qquad 0 < y < 1, \ \lambda > 0,$$

is carried out to investigate the finite-sample performance of the proposed estimators $\widehat{CGIF}_{\eta;1}(Y_{[k,n,d,m]})$ and $\widehat{CGIF}_{\eta;2}(Y_{[k,n,d,m]})$ introduced in (41) and (43), respectively. The numerical study evaluates their statistical properties in terms of bias, root mean square error (RME), and standard deviation (SD). The investigation is performed for sample sizes $N = 50$, 70, and 100, several combinations of the parameters ν and η, and the concomitant associated with the 15th order statistic, namely $Y_{[15,40,0,1]}$. The simulation outcomes for $\widehat{CGIF}_{\eta;1}(Y_{[k,n,d,m]})$ and $\widehat{CGIF}_{\eta;2}(Y_{[k,n,d,m]})$ are summarized in Tables 2 and 3, respectively. The numerical findings indicate the following:

(1) For every considered combination of ν and η, increasing the sample size N leads to smaller values of bias, RME, and SD for both estimators.

(2) When the sample size is held fixed, larger values of ν and η result in higher bias, RME, and SD for both estimation methods.

The simulation results, therefore, demonstrate that the estimation accuracy is influenced by the parameters ν and η as well as the sample size N. In particular, larger samples improve the performance of the proposed estimators, whereas increasing either ν or η tends to reduce their precision.

TABLE 2. The estimator $\widehat{CGIF}_{\eta;1}(Y_{[15,40,0,1]})$ with power distribution (at $\lambda = 3$) component lifetimes for various values of ν and η with $\vartheta = 2$ and $\xi = 0.5$ yielded bias, RME, and SD results.

ν	η	$N = 50$			$N = 70$			$N = 100$		
		Bias	RME	SD	Bias	RME	SD	Bias	RME	SD
2	2	0.0058742	0.0265642	0.0259195	0.0022442	0.0179754	0.0178437	0.0012713	0.0150110	0.0149645
	3	0.0049039	0.0227711	0.0222479	0.0027438	0.0161132	0.0158858	0.0013971	0.0130485	0.0129800
	4	0.0041603	0.0208568	0.0204479	0.0016990	0.0141653	0.0140701	0.0016432	0.0110943	0.0109774
3	2	0.0052846	0.0262016	0.0256760	0.0032504	0.0175477	0.0172527	0.0014120	0.0139872	0.0139227
	3	0.0057911	0.0225562	0.0218111	0.0021015	0.0155519	0.0154170	0.0017828	0.0130694	0.0129537
	4	0.0067836	0.0212313	0.0201285	0.0032610	0.0143514	0.0139830	0.0024040	0.0118359	0.0115950
4	2	0.0084560	0.0256852	0.0242655	0.0046108	0.0187182	0.0181506	0.0028094	0.0148519	0.0145910
	3	0.0073442	0.0227105	0.0215009	0.0047827	0.0167773	0.0160892	0.0019089	0.0126934	0.0125553
	4	0.0074793	0.0211652	0.0198096	0.0038330	0.0144782	0.0139686	0.0023640	0.0120075	0.0117784
5	2	0.0109399	0.0272835	0.0250067	0.0046485	0.0182005	0.0176057	0.0038083	0.0149571	0.0144714
	3	0.0092818	0.0243788	0.0225539	0.0038049	0.0154442	0.0149756	0.0031453	0.0131741	0.0127995
	4	0.0102075	0.0222162	0.0197422	0.0050022	0.0153103	0.0144773	0.0033799	0.0124953	0.0120356
6	2	0.0140027	0.0284858	0.0248190	0.0059647	0.0188571	0.0178979	0.0039191	0.0153257	0.0148235
	3	0.0116092	0.0244410	0.0215187	0.0048686	0.0160569	0.0153087	0.0037436	0.0134539	0.0129290
	4	0.0107704	0.0223862	0.0196347	0.0052553	0.0148062	0.0138490	0.0036635	0.0120286	0.0114629
7	2	0.0149545	0.0288118	0.0246391	0.0064680	0.0193279	0.0182227	0.0045677	0.0152840	0.0145928
	3	0.0134080	0.0257726	0.0220213	0.0068023	0.0168003	0.0153692	0.0043107	0.0136992	0.0130098
	4	0.0117708	0.0230300	0.0198046	0.0063592	0.0154278	0.0140632	0.0043808	0.0121326	0.0113198
8	2	0.0163860	0.0294253	0.0244529	0.0080034	0.0195067	0.0177982	0.0047656	0.0151262	0.0143631
	3	0.0142774	0.0256392	0.0213068	0.0080286	0.0179302	0.0160403	0.0047473	0.0133224	0.0124541
	4	0.0147684	0.0240413	0.0189799	0.0079971	0.0164910	0.0144294	0.0055987	0.0130596	0.0118045
9	2	0.0180225	0.0297784	0.0237172	0.0085431	0.0193923	0.0174178	0.0054770	0.0157127	0.0147346
	3	0.0176176	0.0267393	0.0201250	0.0090731	0.0177070	0.0152134	0.0046738	0.0131941	0.0123447
	4	0.0172296	0.0255645	0.0188955	0.0085202	0.0166165	0.0142730	0.0055180	0.0133252	0.0121350
10	2	0.0216807	0.0324773	0.0241932	0.0091660	0.0199117	0.0176853	0.0063246	0.0152343	0.0138664
	3	0.0176301	0.0275155	0.0211359	0.0094643	0.0184816	0.0158823	0.0060547	0.0139270	0.0125483
	4	0.0188238	0.0259588	0.0178843	0.0100790	0.0171066	0.0138291	0.0063941	0.0129637	0.0112828
11	2	0.0226303	0.0324075	0.0232090	0.0106033	0.0212210	0.0183913	0.0064870	0.0155456	0.0141344
	3	0.0208240	0.0293585	0.0207053	0.0099344	0.0183144	0.0153936	0.0069901	0.0146889	0.0129255
	4	0.0195071	0.0266327	0.0181411	0.0102396	0.0173600	0.0140255	0.0072165	0.0137052	0.0116572
12	2	0.0250065	0.0345674	0.0238779	0.0113262	0.0210037	0.0176970	0.0077058	0.0160162	0.0140476
	3	0.0223476	0.0301983	0.0203209	0.0113809	0.0192548	0.0155391	0.0070739	0.0144645	0.0126231
	4	0.0194126	0.0267392	0.0183977	0.0110765	0.0175734	0.0136499	0.0073159	0.0135501	0.0114111
13	2	0.0273122	0.0353104	0.0223913	0.0127020	0.0214896	0.0173425	0.0079243	0.0164692	0.0144447
	3	0.0241855	0.0309396	0.0193052	0.0118104	0.0195352	0.0155685	0.0075734	0.0147916	0.0127121
	4	0.0229172	0.0292406	0.0181699	0.0118519	0.0185763	0.0143114	0.0079591	0.0140995	0.0116441
14	2	0.0287950	0.0369296	0.0231338	0.0135615	0.0214031	0.0165666	0.0085237	0.0166164	0.0142708
	3	0.0260673	0.0327150	0.0197778	0.0136427	0.0206328	0.0154864	0.0084380	0.0150735	0.0124966
	4	0.0248747	0.0303523	0.0174015	0.0135401	0.0191575	0.0135594	0.0089424	0.0145160	0.0114402
15	2	0.0330804	0.0397805	0.0221058	0.0146521	0.0227234	0.0173773	0.0093137	0.0167218	0.0138949
	3	0.0277979	0.0339384	0.0194800	0.0135877	0.0204072	0.0152336	0.0085889	0.0153312	0.0127058
	4	0.0255250	0.0311034	0.0177823	0.0135971	0.0192578	0.0136444	0.0092779	0.0146116	0.0112937
16	2	0.0352674	0.0418544	0.0225502	0.0157492	0.0232529	0.0171159	0.0097827	0.0169031	0.0137914
	3	0.0321747	0.0372685	0.0188171	0.0143626	0.0205510	0.0147062	0.0095093	0.0157532	0.0125656
	4	0.0278456	0.0324998	0.0167672	0.0139189	0.0193719	0.0134803	0.0097023	0.0150989	0.0115747
17	2	0.0360178	0.0425102	0.0225909	0.0167475	0.0234177	0.0163762	0.0101350	0.0173656	0.0141082
	3	0.0332594	0.0381770	0.0187522	0.0158527	0.0216709	0.0147830	0.0108008	0.0168505	0.0129403
	4	0.0308429	0.0348832	0.0163040	0.0155978	0.0206728	0.0135742	0.0098309	0.0146797	0.0109071
18	2	0.0386893	0.0442398	0.0214652	0.0180809	0.0249041	0.0171345	0.0114383	0.0183787	0.0143927
	3	0.0346909	0.0395085	0.0189162	0.0159853	0.0219753	0.0150869	0.0108183	0.0166955	0.0127227
	4	0.0324283	0.0365669	0.0169063	0.0165202	0.0214191	0.0136399	0.0110275	0.0158857	0.0114403
19	2	0.0425627	0.0476076	0.0213390	0.0193576	0.0255776	0.0167266	0.0113656	0.0180758	0.0140625
	3	0.0381574	0.0418543	0.0172071	0.0178546	0.0230722	0.0146204	0.0112058	0.0169271	0.0126932
	4	0.0339643	0.0375440	0.0160073	0.0169216	0.0214587	0.0132026	0.0113821	0.0161258	0.0114289
20	2	0.0457393	0.0502150	0.0207338	0.0198415	0.0257240	0.0163801	0.0127056	0.0189194	0.0140253
	3	0.0380250	0.0422864	0.0185089	0.0184619	0.0232923	0.0142089	0.0118855	0.0171313	0.0123438
	4	0.0352470	0.0386666	0.0159062	0.0176030	0.0219423	0.0131061	0.0118051	0.0159218	0.0106892

Table 3. The estimator $\widehat{CGIF}_{\eta;2}(Y_{[15,40,0,1]})$ with power distribution (at $\lambda = 3$) component lifetimes for various values of ν and η with $\vartheta = 2$ and $\xi = 0.5$ yielded bias, RME, and SD results.

ν	η	$N = 50$			$N = 70$			$N = 100$		
		Bias	RME	SD	Bias	RME	SD	Bias	RME	SD
	2	-0.0026105	0.0252274	0.0251045	-0.0016757	0.0177646	0.0176942	-0.0016359	0.0148378	0.0147547
2	3	-0.0069325	0.0216360	0.0205056	-0.0028009	0.0156475	0.0154025	-0.0024657	0.0128473	0.0126148
	4	-0.0070682	0.0202198	0.0189536	-0.0041838	0.0139620	0.0133270	-0.0023742	0.0110163	0.0107628
	2	-0.0001724	0.0245265	0.0245382	0.0003700	0.0170590	0.0170635	-0.0005096	0.0137845	0.0137819
3	3	-0.0042079	0.0216628	0.0212609	-0.0030685	0.0154904	0.0151911	-0.0018264	0.0127749	0.0126500
	4	-0.0052456	0.0192986	0.0185813	-0.0029435	0.0139241	0.0136162	-0.0016865	0.0115358	0.0114176
	2	0.0042671	0.0243525	0.0239877	0.0025291	0.0181256	0.0179573	0.0015865	0.0145929	0.0145136
4	3	-0.0027504	0.0209614	0.0207906	-0.0008206	0.0157267	0.0157131	-0.0016523	0.0123634	0.0122586
	4	-0.0044388	0.0198588	0.0193660	-0.0024524	0.0136767	0.0134618	-0.0020232	0.0120991	0.0119347
	2	0.0104144	0.0272881	0.0252352	0.0039157	0.0177839	0.0173562	0.0033696	0.0150741	0.0147000
5	3	0.0008174	0.0220067	0.0220026	-0.0007720	0.0149568	0.0149444	-0.0003299	0.0124975	0.0124994
	4	-0.0019689	0.0194838	0.0193938	-0.0016307	0.0139266	0.0138377	-0.0011072	0.0116531	0.0116062
	2	0.0158251	0.0306319	0.0262406	0.0067921	0.0191054	0.0178662	0.0043474	0.0156781	0.0150708
6	3	0.0037359	0.0218900	0.0215796	0.0009216	0.0149654	0.0149445	0.0007365	0.0128187	0.0128040
	4	-0.0008933	0.0195045	0.0194938	-0.0009717	0.0135062	0.0134780	-0.0007783	0.0113691	0.0113481
	2	0.0190131	0.0317064	0.0253859	0.0088550	0.0197404	0.0176517	0.0058966	0.0155465	0.0143920
7	3	0.0081443	0.0241389	0.0227348	0.0031772	0.0159488	0.0156369	0.0014851	0.0130838	0.0130057
	4	0.0009252	0.0194237	0.0194114	-0.0004855	0.0133332	0.0133310	-0.0004387	0.0112221	0.0112191
	2	0.0236627	0.0352006	0.0260738	0.0108638	0.0210317	0.0180176	0.0062505	0.0153361	0.0140115
8	3	0.0096497	0.0243624	0.0223810	0.0051402	0.0167395	0.0159387	0.0021343	0.0125297	0.0123528
	4	0.0038059	0.0199550	0.0195985	0.0015872	0.0144797	0.0143997	0.0012525	0.0119633	0.0119036
	2	0.0288142	0.0387524	0.0259262	0.0134391	0.0224825	0.0180327	0.0083545	0.0171857	0.0150258
9	3	0.0151113	0.0271452	0.0225614	0.0058592	0.0164913	0.0154231	0.0022521	0.0124784	0.0122796
	4	0.0079497	0.0213048	0.0197759	0.0019670	0.0141595	0.0140293	0.0008303	0.0118346	0.0118113
	2	0.0346379	0.0436688	0.0266061	0.0152566	0.0237279	0.0181818	0.0107292	0.0176866	0.0140676
10	3	0.0170572	0.0285248	0.0228744	0.0067834	0.0172784	0.0158992	0.0043035	0.0133761	0.0126713
	4	0.0094492	0.0224664	0.0203928	0.0033541	0.0151234	0.0147542	0.0018669	0.0113429	0.0111938
	2	0.0385855	0.0470040	0.0268563	0.0184105	0.0265569	0.0191491	0.0113237	0.0188562	0.0150850
11	3	0.0204102	0.0310026	0.0233480	0.0082568	0.0180268	0.0160328	0.0049504	0.0140322	0.0131366
	4	0.0111140	0.0225402	0.0196195	0.0039545	0.0150308	0.0145085	0.0022893	0.0118216	0.0116037
	2	0.0440982	0.0524196	0.0283543	0.0211491	0.0284024	0.0189675	0.0137593	0.0202361	0.0148459
12	3	0.0244718	0.0336277	0.0230757	0.0098025	0.0188345	0.0160906	0.0056447	0.0140347	0.0128560
	4	0.0130012	0.0238242	0.0199740	0.0048344	0.0150663	0.0142767	0.0025624	0.0118682	0.0115941
	2	0.0480643	0.0556198	0.0280032	0.0232957	0.0298259	0.0186344	0.0151195	0.0213661	0.0151043
13	3	0.0291393	0.0372860	0.0232742	0.0112004	0.0191574	0.0155499	0.0066031	0.0139321	0.0122741
	4	0.0180391	0.0277106	0.0210455	0.0061969	0.0155914	0.0143141	0.0029310	0.0118618	0.0114997
	2	0.0536477	0.0612971	0.0296672	0.0254070	0.0311694	0.0180650	0.0162864	0.0216928	0.0143364
14	3	0.0321967	0.0399280	0.0236258	0.0138918	0.0214508	0.0163531	0.0071184	0.0145044	0.0126438
	4	0.0206664	0.0292860	0.0207605	0.0071810	0.0160111	0.0143176	0.0041191	0.0126716	0.0119894
	2	0.0594015	0.0667144	0.0303841	0.0283799	0.0340295	0.0187867	0.0177535	0.0229796	0.0145975
15	3	0.0371187	0.0450859	0.0256047	0.0142243	0.0214775	0.0161000	0.0086007	0.0157393	0.0131882
	4	0.0242319	0.0323162	0.0213918	0.0077191	0.0159239	0.0139349	0.0043492	0.0124924	0.0117168
	2	0.0635049	0.0704997	0.0306313	0.0313622	0.0361371	0.0179617	0.0191507	0.0240949	0.0146297
16	3	0.0425775	0.0496121	0.0254787	0.0159122	0.0229235	0.0165095	0.0095719	0.0157363	0.0124965
	4	0.0280137	0.0346621	0.0204233	0.0085333	0.0163369	0.0139381	0.0050690	0.0125816	0.0115211
	2	0.0671413	0.0739216	0.0309420	0.0338664	0.0385094	0.0183405	0.0219795	0.0264968	0.0148054
17	3	0.0475222	0.0542054	0.0260874	0.0184869	0.0244732	0.0160447	0.0111766	0.0175586	0.0135488
	4	0.0326513	0.0390993	0.0215200	0.0116623	0.0187257	0.0146581	0.0056462	0.0127076	0.0113900
	2	0.0720035	0.0788513	0.0321566	0.0371442	0.0419576	0.0195227	0.0239123	0.0283527	0.0152417
18	3	0.0517905	0.0575767	0.0251685	0.0193233	0.0255169	0.0166734	0.0116006	0.0176603	0.0133225
	4	0.0365858	0.0428707	0.0223578	0.0117314	0.0186467	0.0145011	0.0063281	0.0133703	0.0117838
	2	0.0755385	0.0824733	0.0331190	0.0392144	0.0438067	0.0195355	0.0250483	0.0292649	0.0151409
19	3	0.0580917	0.0642274	0.0274092	0.0221635	0.0274322	0.0161731	0.0124550	0.0184904	0.0136731
	4	0.0401129	0.0461461	0.0228240	0.0123690	0.0188226	0.0141950	0.0065983	0.0135458	0.0118360
	2	0.0787533	0.0856943	0.0338019	0.0428455	0.0471807	0.0197655	0.0261677	0.0303430	0.0153683
20	3	0.0612430	0.0668023	0.0266937	0.0233346	0.0287226	0.0167560	0.0139173	0.0189763	0.0129063
	4	0.0453888	0.0515756	0.0245050	0.0155273	0.0214723	0.0148386	0.0078293	0.0141190	0.0117553

4.1. **Mechanical engineering data application.** In reliability analysis, the number of shocks sustained by a component prior to failure provides valuable information about its resistance to repeated loading and its underlying failure mechanism. Such shock-count data are frequently employed to investigate lifetime behavior and to develop stochastic models describing cumulative damage processes; see Murthy *et al.* [28]. To illustrate the proposed methodology, we consider the dataset consisting of the failure times of shocks for 20 mechanical systems, as reported in Table 4.

TABLE 4. Failure times of shocks before the failure of 20 mechanical systems.

0.2	0.3	0.6	0.6	0.7
0.9	0.9	1.0	1.0	1.1
1.2	1.2	1.2	1.3	1.3
1.3	1.5	1.6	1.6	1.8

To justify the use of the proposed procedures, it is first necessary to verify the assumption that the data arise from a uniform distribution. Consider a random variable U following the $U(a, b)$ distribution with probability density function

$$h_Y(y) = \frac{1}{b-a}, \qquad a < y < b.$$

When the parameters a and b are known, the transformation

$$Z = \frac{U-a}{b-a}$$

maps the observations onto the standard uniform distribution $U(0, 1)$, allowing the application of goodness-of-fit tests for the standard case.

In practice, the parameters are typically unknown and are estimated using the maximum likelihood method. Let $U_1, U_2, \ldots, U_n$ denote a random sample from $U(a, b)$. The maximum likelihood estimators of a and b are given by the minimum and maximum order statistics, namely $U_{(1)}$ and $U_{(n)}$, respectively. The transformed observations are then obtained as

$$Z_i = \frac{U_{(i+1)} - U_{(1)}}{U_{(n)} - U_{(1)}}, \qquad i = 1, 2, \ldots, n-2,$$

which constitute a sample from the standard uniform distribution. Consequently, the original goodness-of-fit problem for the $U(a, b)$ model is reduced to testing uniformity on $U(0, 1)$ based on a transformed sample of size $n-2$; see Figure 3, which reveals the Q-Q plot of the original and transformed mechanical engineering data.

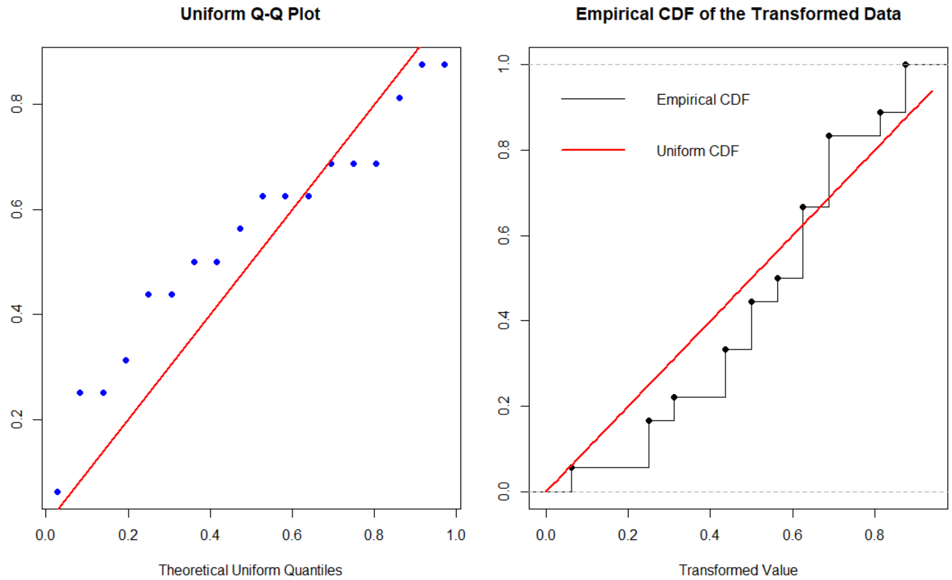

FIGURE 3. The Q-Q plot of the original and transformed mechanical engineering data.

To assess the adequacy of the uniform model, several classical goodness-of-fit tests were performed, including the Kolmogorov–Smirnov, Pearson chi-square, Anderson–Darling, and Cramér–von Mises tests. The corresponding p-values were 0.3746, 0.1247, 0.3258, and 0.2899, respectively. Since all of these values exceed the conventional significance level of 0.05, there is insufficient evidence to reject the null hypothesis of uniformity. Therefore, the transformed observations are consistent with the standard uniform distribution, supporting the suitability of the uniform model for the subsequent analysis; see Algorithm 1.

Algorithm 1 Mechanical engineering data analysis

small

1: Collect the failure-time data.
2: Estimate $\hat{a} = U_{(1)}$ and $\hat{b} = U_{(n)}$.
3: Compute

$$Z_i = \frac{U_{(i+1)} - \hat{a}}{\hat{b} - \hat{a}}, \qquad i = 1, \ldots, n-2.$$

4: Perform the Kolmogorov–Smirnov, Pearson chi-square, Anderson–Darling, and Cramér–von Mises tests.
5: **if** all p-values > 0.05 **then**
6: Compute the theoretical $CGIF_\eta(Y_{[15,30,0,1]})$.
7: Compute $\widehat{CGIF}_{\eta;1}$ and $\widehat{CGIF}_{\eta;2}$.
8: Calculate the absolute and relative biases.
9: **else**
10: Choose another probability model.
11: **end if**
12: Compare the results.

Based on the concomitant of the 15th order statistic, $Y_{[15,30,0,1]}$, Tables 5 and 6 illustrate the absolute biases associated with the theoretical value and the proposed estimators $\widehat{CGIF}_{\eta;1}(Y_{[15,30,0,1]})$ and $\widehat{CGIF}_{\eta;2}(Y_{[15,30,0,1]})$ for $\vartheta = 2$ and $\xi = 0.5$. The numerical results indicate that, for both estimators, the absolute bias decreases consistently as the parameter η increases, demonstrating an improvement in estimation accuracy for larger values of η. Furthermore, a comparison of the two estimators reveals that $\widehat{CGIF}_{\eta;1}(Y_{[15,30,0,1]})$ consistently produces smaller absolute biases than $\widehat{CGIF}_{\eta;2}(Y_{[15,30,0,1]})$ for all considered combinations of ν and η, indicating its superior estimation accuracy.

Table 5. Performance of $\widehat{CGIF}_{\eta;1}(Y_{[15,30,0,1]})$ estimator for different values of ν and η.

ν	η	Estimator	True Value	Absolute Bias	Relative Bias (%)
2	2	0.304807	0.371325	0.066517	17.9135
	4	0.199609	0.234346	0.034737	14.8230
	6	0.154695	0.172516	0.017821	10.3300
	8	0.127676	0.136929	0.009253	6.7576
3	2	0.306775	0.371325	0.064550	17.3837
	4	0.195702	0.234346	0.038645	16.4904
	6	0.148214	0.172516	0.024301	14.0865
	8	0.120443	0.136929	0.016486	12.0398
4	2	0.309587	0.371325	0.061738	16.6264
	4	0.193463	0.234346	0.040883	17.4456
	6	0.142179	0.172516	0.030337	17.5848
	8	0.112488	0.136929	0.024441	17.8493
5	2	0.334797	0.371325	0.036527	9.8370
	4	0.214859	0.234346	0.019487	8.3154
	6	0.158832	0.172516	0.013684	7.9320
	8	0.125778	0.136929	0.011151	8.1438

Table 6. Performance of $\widehat{CGIF}_{\eta;2}(Y_{[15,30,0,1]})$ estimator for different values of ν and η.

ν	η	Estimator	True Value	Absolute Bias	Relative Bias (%)
2	2	0.186297	0.371325	0.185028	49.8291
	4	0.084832	0.234346	0.149514	63.8004
	6	0.048442	0.172516	0.124074	71.9203
	8	0.031001	0.136929	0.105928	77.3600
3	2	0.201444	0.371325	0.169880	45.7498
	4	0.090961	0.234346	0.143386	61.1854
	6	0.048152	0.172516	0.124364	72.0885
	8	0.028003	0.136929	0.108926	79.5494
4	2	0.273908	0.371325	0.097417	26.2350
	4	0.156261	0.234346	0.078085	33.3203
	6	0.100902	0.172516	0.071614	41.5116
	8	0.070911	0.136929	0.066018	48.2134
5	2	0.233071	0.371325	0.138253	37.2324
	4	0.131750	0.234346	0.102596	43.7797
	6	0.078688	0.172516	0.093827	54.3877
	8	0.049300	0.136929	0.087629	63.9959

Conclusion

This article introduced the generating-informational function associated with the concomitant of d-generalized order statistics under the Huang–Kotz extension with parameters satisfying $\eta \geq 1$ and $\vartheta \geq 1$. Explicit expansions of the proposed measure were derived, and several well-known lifetime distributions were presented to illustrate its applicability. In addition, new theoretical properties were established, including bounds obtained through Minkowski's inequality, preservation properties under stochastic orderings, and characterization results for the parent distribution based on the cumulative generating-informational function.

To facilitate statistical inference, two nonparametric estimators of the proposed generating-informational function were developed. The first estimator incorporates a correction factor designed to improve estimation accuracy, whereas the second estimator provides an alternative estimation procedure. Their finite-sample performance was investigated through extensive Monte Carlo simulations. The simulation results showed that both estimators are consistent in the sense that their bias, root mean squared error, and standard deviation decrease as the sample size increases. Conversely, larger values of the parameters ν and η generally lead to increases in these performance measures, indicating a reduction in estimation precision.

The proposed methods were further examined using real data sets. The empirical results demonstrated that the absolute bias of both estimators decreases as η increases. Moreover, the estimator incorporating the correction factor consistently outperformed the second estimator by producing smaller absolute biases across all considered combinations of the model parameters, confirming its superior estimation accuracy.

Overall, the theoretical developments and numerical investigations demonstrate that the proposed generating-informational function provides a useful framework for analyzing concomitants of d-generalized order statistics under the Huang–Kotz family. The derived properties and the proposed estimation procedures extend the existing literature on information measures and provide effective tools for applications in reliability analysis, lifetime modeling, and related areas of mathematical statistics. Future research may consider extending these results to dependent data structures, censored observations, or broader classes of generalized order statistics and information measures.

Author Contributions. Conceptualization, Formal Analysis, Investigation, Methodology, Software, Visualization, Resources, Validation, Writing–Original Draft, and Writing–Review & Editing: M.S.M.

Data Availability. The datasets generated and/or analyzed during the current study are included in this published article.

Funding. This research was funded by Prince Sattam bin Abdulaziz University under project number (PSAU/2026/01/44879).

Acknowledgments. The author extends sincere appreciation to Prince Sattam bin Abdulaziz University for funding this research through project number (PSAU/2026/01/44879).

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

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