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LIE TRIPLE NILALGEBRAS OF NILINDEX FOUR AND DIMENSION AT MOST FIVE

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ABSTRACT. In this paper, we first present some interesting equalities in Lie triple nilalgebras with nilindex four, that enables us to deal with the dimension. Secondly we determine the possible multiplication tables according to the dimension. In dimension three, we find one possible table, in dimension four, two tables and finally we find four tables in dimension five.

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1. INTRODUCTION

In our paper, we study commutative algebras satisfying identity $3(x^2y)x - x^3y - 2x(x(xy)) = 0$. They are called Lie triple or almost Jordan algebras and have been introduced in 1965 by Osborn [7]. He is the precursor of the study of this class of algebras throughout [7,8]. Subsequently, other authors such as Petersson [9,10], Sidorov [11,12], Hentzel and Peresi [6] were interested in Lie triple algebras. More recent results have also been obtained in the studies of Bayara and al. [1,2], and Dembega and al. [3].

Here, we are interested in the possible multiplication tables for Lie triple nilalgebras of nilindex four, and dimension at most five, which are not power associative.

2. PRELIMINARIES

A non-associative algebra A is said to be a nilalgebra of nilindex n, if $x^n = 0$ for all $x \in A$ and if there exists $a \in A$ such that $a^{n-1} \neq 0$.

In what follows, K is an infinite commutative field of characteristics not 2, 3, 5 and A is a Lie triple nilalgebra of nilindex 4, which is not power associative.

Lemma 2.1. *Let A be a Lie triple nilalgebra of nilindex 4. Then, for all $x, y, z \in A$ we have*

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(i) $x^2x^3 = 0$,

(ii) $z(yx^2) + 2x(y(xz)) = 0$,

(iii) $z(x(xy)) + y(x(xz)) + x(x(yz)) = 0$,

(iv) $x^2(x(xy)) = 0$,

(v) $x^2(x^2y) = -2x(yx^3) = -2x^3(yx) = 4R_x^4(y)$,

(vi) $R_x^5 = 0$,

(vii) $x^3(yx^2) = 0$,

$(viii)$ $z(x^2x^2) = (zx^2)(yx^2) = z(x^2(xy)) = 0$.

Proof. A being a Lie triple nilalgebra of nilindex 4, then it satisfies the following identities:

$$x^4 = 0 \tag{1}$$

$$3(x^2y)x - x^3y - 2x(x(xy)) = 0 \tag{2}$$

A partial linearization of (1) gives

$$yx^3 + x(yx^2) + 2x(x(xy)) = 0 \tag{3}$$

The sum of (3) and (2) gives $4(x^2y)x = 0$, which means

$$(x^2y)x = 0 \tag{4}$$

Identity (2) then gives

$$x^3y + 2x(x(xy)) = 0 \tag{5}$$

so result (i) is obtained by putting $y = x^2$ in (5), and (ii) by linearizing (4).

Similarly, linearizing (5) and using (ii) we find (iii). If we take $z = x^2$ in (iii) and using (4), we have the result (iv).

Multiplying (5) by x gives $4R_x^4(y) = -2x(yx^3)$. In (5), replacing y by yx gives $4R_x^4(y) = -2x^3(yx)$. Let's take $z = x^2$ in (ii), it gives $x^2(x^2y) = -2x(yx^3)$. We thus obtain (v).

For (vii), we just need to set $y = yx^2$ in (5) and use (4).

Putting $y = x^2$ in (ii) gives $z(x^2x^2) = 0$, and $z = zx^2$ in (ii) gives $(zx^2)(yx^2) = 0$. Linearizing $z(x^2x^2) = 0$, gives us $z(x^2(xy)) = 0$ and finally we have $(viii)$.

For (vi) replace y by yx in (v) and use $(viii)$. □

Since $R_x^5(y) = 0$, if $xy = \alpha y$ for a non-zero y in A, then $\alpha^5 y = 0$ and $\alpha = 0$. We then have the following lemma.

Lemma 2.2. *If x and y are two non-zero elements of A such that $xy = \alpha y$ then $\alpha = 0$.*

Theorem 2.3. *Let A be a Lie triple nilalgebra of nilindex 4.*

(i) *If there exist elements x and y of A such that $R_x^4(y) \neq 0$ (i.e $x^2(x^2y) \neq 0$) then y, $R_x(y)$, $R_x^2(y)$, $R_x^3(y)$, $R_x^4(y)$, x, x^2, x^3, yx^2 are linearly independant.*

(ii) *If there exist elements x and y in A such that $R_x^4(y^2) \neq 0$ (i.e $x^2(x^2y^2) \neq 0$) then y, $R_x(y)$, $R_x^2(y)$, $R_x^3(y)$, $R_x^4(y)$, x, x^2, x^3, yx^2, z are linearly independant with $z \in \{x^2x^2, x^2(xy)\}$.*

Proof. (i) Assume that $R_x^4(y) \neq 0$ and consider

$$\sum_{i=0}^{4} \alpha_i R_x^i(y) + \sum_{j=1}^{3} \beta_j x^j + \gamma yx^2 = 0 \tag{6}$$

Applying R_x^4 to it gives $\alpha_0 = 0$.

Applying R_x^3 to it gives $\alpha_1 = 0$.

Applying R_x^2 to it and using result (v) of Lemma 2.1 we have $\alpha_2 R_x^4(y) + \beta_1 x^3 = -\frac{1}{2}\alpha_2 x^3(yx) + \beta_1 x^3 = 0$, then Lemma 2.2 gives $\alpha_2 = \beta_1 = 0$.

Applying R_x to it and using result (v) of Lemma 2.1 we have $\alpha_3 R_x^4(y) + \beta_2 x^3 = 0$, then Lemma 2.2, gives $\alpha_3 = \beta_2 = 0$.

Now we have $\alpha_4 R_x^4(y) + \beta_3 x^3 + \gamma yx^2 = 0$. Multiplying this equality by x^2 we obtain $R_x^4(y) = 0$ which gives $\gamma = 0$. Consequently $\alpha_4 R_x^4(y) + \beta_3 x^3 = 0$ i.e. $-\frac{1}{2}\alpha_4 x^3(yx) + \beta_3 x^3 = 0$. Lemma 2.2 gives $\beta_3 = \alpha_4 = 0$, and we have the result.

(ii) Let's assume now that $R_x^4(y^2) \neq 0$ and show that y, $R_x(y)$, $R_x^2(y)$, $R_x^3(y)$, $R_x^4(y)$, x, x^2, x^3, yx^2, z with $z \in \{x^2(xy), x^2x^2\}$ are linearly independent.

Let's consider

$$\sum_{i=0}^{4} \alpha_i R_x^i(y) + \sum_{j=1}^{3} \beta_j x^j + \gamma_1 yx^2 + \gamma_2 x^2(xy) + \gamma_3 x^2x^2 = 0. \tag{7}$$

In the same way as previously in (6), we have $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$. It remains

$$\alpha_4 R_x^4(y) + \beta_3 x^3 + \gamma_1 yx^2 + \gamma_2 x^2(yx) + \gamma_3 x^2x^2 = 0. \tag{8}$$

Multiplying (8) by x^2 and using result (v) of Lemma 2.1, it gives $\gamma_1 x^2(x^2y) = 4\gamma_1 R_x^4(y) = 0$, then $\gamma_1 = 0$. Then, successively multiplying, first by y,second by x, we have $\alpha_4 x(yR_x^4(y)) + \beta_3 x(yx^3) = \beta_3 x(yx^3)) = 0$. This leads to $\beta_3 = 0$. Now we have $\alpha_4 R_x^4(y) + \gamma_2 x^2(xy) + \gamma_3 x^2x^2 = 0$. Multiplying this by y we have $\alpha_4 y R_x^4(y) = 0$, i.e. $\alpha_4 R_x^4(y^2) = 0$, and this leads to $\alpha_4 = 0$. We have used $R_x^4(y^2) = yR_x^4(y)$. To demonstrate it, let's consider the total linearization of 5

$$z(y(xw)) + w(y(xz)) + x(y(wz)) = 0.$$

Putting $x = z = y$, $y = x^3$ and $w = x$, we obtain $x(x^3y^2) + 2y(x^3(xy)) = 0$, which means $-2R_x^4(y^2) + 2yR_x^4(y) = 0$. Since what is remaining is $\gamma_2 x^2(yx) + \gamma_3 x^2x^2 = 0$, we can see that y, $R_x(y)$, $R_x^2(y)$, $R_x^3(y)$, $R_x^4(y)$, x, x^2, x^3, yx^2, z are linearly independant. □

Example 2.4. Let's consider the commutative algebra A in dimension 11 which multiplication table in the basis $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}\}$ is given by $b_1^2 = b_4$, $b_1b_2 = b_3$, $b_1b_3 = b_5$, $b_1b_4 = b_7$, $b_1b_5 = -\frac{1}{2}b_9$, $b_1b_9 = b_{11}$, $b_2b_4 = b_6$, $b_2b_7 = b_9$, $b_3b_4 = b_8$, $b_3b_7 = b_{11}$, $b_4^2 = b_{10}$, $b_4b_6 = -2b_{11}$, all not written products being zero. Then A is a Lie triple nilalgebra of nilindex 4.

Indeed, if we consider two elements $x = \sum_{i=1}^{11} x_ib_i$ and $y = \sum_{i=1}^{11} y_ib_i$ of A, we have :

$x(x(xy)) = \frac{1}{2}(x_1^2x_2y_1 - x_1^3y_2)b_9 + (-x_1y_1x_2x_4 + \frac{1}{2}x_3x_1^2y_1 - \frac{1}{2}x_1^3y_3 + x_1^2x_2y_4)b_{11}$, $x^2 = 2x_1x_2b_3 + x_1^2b_4 + 2x_1x_3b_5 + 2x_2x_4b_6 + 2x_1x_4b_7 + 2x_3x_4b_8 + (-x_1x_5 + 2x_2x_7)b_9 + x_4^2b_{10} + (-4x_4x_6 + 2x_3x_7 + 2x_1x_9)b_{11}$, $(x^2y)x = 0$ and $x^3y = (-y_1x_1^2x_2 + y_2x_1^3)b_9 + (y_3x_1^3 - 2y_4x_1^2x_2 - y_1x_1^2x_3 + 2y_1x_1x_2x_4)b_{11}$.

Since $3(x^2y)x - x^3y - 2x(x(xy)) = 0$, we can say that A is a Lie triple algebra. Furthermore, since $x(x(xy)) = \frac{1}{2}(x_1^2x_2y_1 - x_1^3y_2)b_9 + (-x_1y_1x_2x_4 + \frac{1}{2}x_3x_1^2y_1 - \frac{1}{2}x_1^3y_3 + x_1^2x_2y_4)b_{11}$, putting $y = x$ we obtain $x^4 = 0$, then A is a nilalgebra of nilindex 4.

Because $x^2x^2 \neq 0$, we can say that A is not power-associative. Moreover we have x^2x^2, $x^2(xy) \in J = \langle b_8, b_{10}\rangle$.

From Theorem 2.3, follow these corollaries.

Corollary 2.5. *Let A be a Lie triple nilalgebra of nilindex 4. Assume that for all $a, b \in A$ we have $R_a^4(b) = 0$. If there are two elements $x, y \in A$ such that $R_x^3(y) \neq 0$, then $\{y, x, x^2, x^3, R_x(y), R_x^2(y), R_x^3(y)\}$ is a linearly independant family. In this case, we have $dimA \geq 7$.*

Corollary 2.6. *If there are elements x and y in A such that $R_x^4(y^2) \neq 0$ with $x^2(xy)$ and x^2x^2 being linearly independant, then y, $R_x(y)$, $R_x^2(y)$, $R_x^3(y)$, $R_x^4(y)$, x, x^2, x^3, yx^2, $x^2(xy)$ and x^2x^2 are linearly independants.*

In this case the dimension of A is at least 11.

Remark 2.7. With the conditions of Corollary 2.5, if $dimA \leq 6$ then for all $a, b \in A$ we have $R_a^3(b) = 0$.

Since the nilindex is 4, there exists $x \in A$ satisfying $x^3 \neq 0$. Moreover, by considering $\alpha, \beta, \gamma \in K$ such that $\alpha x + \beta x^2 + \gamma x^3 = 0$, we just need to apply R_x twice successively to this equality to obtain $\alpha = 0$. Next we apply R_x once to have $\beta = 0$. This gives $\gamma = 0$. Then x, x^2 and x^3 are linearly independants, that means the dimension of this algebra is at least three.

In [4], A. Elduque and A. Labra studied commutative nilalgebras of nilindex four and dimension at most four. Therefore, their results also apply to Lie triple nilalgebras in the conditions of our study, when the dimension is 3 or 4.

In the rest of our work, when S denotes a set, we will denote $\langle S\rangle$, the sub-vector space generated by S and $alg(S)$ the sub-algebra generated by S.

3. Lie Triple nilalgebras of nilindex 4 and dimension 3

Theorem 3.1 ([4]). *Let A be a Lie triple nilalgebra of nilindex 4 and dimension 3 which is not power associative. Then there exists an element $a \in A$ such that $\{a, a^2, a^3\}$ be a basis of A with the following multiplication table* (*in which the not written products are zero*) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$.

Proof. Let x be an element of A such that $x^3 \neq 0$. Then x, x^2, x^3 are linearly independants. Since $dim(A) = 3$ we can write $A = alg(x, x^2, x^3)$. Let's put $x^2x^2 = \alpha x + \beta x^2 + \gamma x^3$. Apply R_x^2 to this equality and use Lemma 2.1, we have $\alpha = 0$ which gives $x^2x^2 = \beta x^2 + \gamma x^3$, and leads to $0 = x(x^2x^2) = \beta x^3$ and finally $\beta = 0$; So we write $x^2x^2 = \gamma x^3$. We can choose x such that $\gamma \neq 0$ otherwise, for all $x \in A$, we should have $x^4 = 0 = x^2x^2$ and A would be power associative. Let's put $a = \gamma^{-1}x$. We have $a^2a^2 = \gamma^{-4}x^2x^2 = \gamma^{-3}x^3 = a^3$. Then, $A = alg(a, a^2, a^3)$ and admits the following multiplication table (all not written products being zero) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$. □

4. Lie Triple nilalgebras of nilindex 4 and dimension 4

Theorem 4.1 ([4]). *Let A be a Lie triple nilalgebra of nilindex 4 and dimension 4 which is not power associative. Then A admits one of the following multiplication table in which the not written products are zero.*

(1) *A basis of A is $\{a, a^2, a^3, a^2a^2\}$ with the following multiplication table $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^2a^2$,*

(2) *A basis of A is $\{a, a^2, a^3, b\}$ with the following multiplication table $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$, $b^2 = \beta_1 a^2 + \beta_2 a^3$.*

Proof. Let's distinguish two cases.

1^{st} case : If there is $a \in A$ such that $a^3 \neq 0$ and $a^2a^2 \notin \langle a, a^2, a^3\rangle$, then $alg(a) = \langle\, a, a^2, a^3, a^2a^2\rangle$, with $dimA = 4$, we conclude that $A = \langle a, a^2, a^3, a^2a^2\rangle$ and A admits the following multiplication table (in which the not written products are zero) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^2a^2$.

2^{nd} case : Otherwise, for all $x \in A$ such that $x^3 \neq 0$, we have $x^2x^2 \in \langle x, x^2, x^3\rangle$ and $alg(x) = \langle x, x^2, x^3\rangle$. For all $y \in A - alg(x)$, $alg(x) \cap alg(y)$ is a proper subalgebra of $alg(x)$ otherwise $alg(x) \subset alg(y) = A$. What is a contradiction. Then $alg(x) \cap alg(y) \subseteq (alg(x))^2$. Since $alg(x)$ has codimension 1 then $alg(x)\cap alg(y)$ also has codimension 1 in $alg(y)$, and we can write $(alg(y))^2 \subseteq alg(x)\cap alg(y) \subseteq (alg(x))^2$. Therefore $y^2 \in (alg(x))^2$. So $A^2 = (alg(x))^2 = \langle x^2, x^3\rangle$ has codimension 2. Since for all $y \in A$ we have $R_y^3(A) = 0$. Particularly $R_x^3(A) = 0$ and $KerR_x \neq \{0\}$. Then there exists $y_0 \in A - alg(x)$ such that $xy_0 = 0$. We have $A^3 = \langle x^3\rangle$, and then $x^2y_0 = \alpha_0 x^3$, $x^3y_0 = 0$, $y_0^2 = \beta x^2 + \lambda x^3$, $x^2x^2 = \gamma x^3$. Taking $a = \gamma^{-1}x$ on a $a^2a^2 = a^3$, we have $a^2y_0 = \alpha_1 a^3$, $y_0^2 = \beta_0 a^2 + \beta_0' a^3$. From $a^2y_0 = \alpha_1 a^3$ we have $a^2(y_0 - \alpha_1 a) = 0$, and setting $b = y_0 - \alpha_1 a$, it gives $a^2b = 0$. In the basis $\{a, a^2, a^3, b\}$ on a $b^2 = \beta_1 a^2 + \beta_2 a^3$, $ab = 0$ and $a^2b = 0$. Finally A admits the following multiplication table (all not written products being zero) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$, $b^2 = \beta_1 a^2 + \beta_2 a^3$. □

5. Lie Triple nilalgebras of nilindex 4 and dimension 5

Lemma 5.1. *Every Lie triple nilalgebra of nilindex 4 and dimension 5 satisfies $2 \leq dimA^2 \leq 3$.*

Proof. Since A is a nilalgebra of nilindex 4, we necessarily have $dimA^2 \geq 2$.

Now let's show that $dimA^2 \leq 3$. Since $R_x^3 = 0$, according [5] algebra A is nilpotent, that means there is $t \geq 4$ such that $A^t = 0$. We easily see that $A^2 \neq A$. This leads to $dimA^2 \leq 4$. Let's proove now that $dimA^2 < 4$. Assume that $dimA^2 = 4$. Then there should exist $y \in A$ such that $A = Ky + A^2$, that gives $A^2 = Ky^2 + A^3$,and $A = \langle y, y^2 \rangle + A^3$. Therefore we have $A^2 = \langle y^2, y^3, y^2y^2 \rangle + A^4$, which gives $A = \langle y, y^2, y^3, y^2y^2 \rangle + A^4$. In the same way, we have $A = \langle y, y^2, y^3, y^2y^2 \rangle + A^5$, that gives $A^5 = A^4$, because of $A^5 \subseteq A^4$. Since A is nilpotent, we have $A^5 = A^4 = 0$, which gives $A = alg(y)$ and $dimA \leq 4$; this is a contradiction. Finally, we have $dimA^2 \leq 3$. □

Theorem 5.2. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension 5. If there is $a \in A$ such that $alg(a) = \langle a, a^2, a^3, a^2a^2 \rangle$ then there exists $b_0 \in A \setminus alg(a)$ such that $ab_0 = 0$, $a^2b_0 = \alpha_1 a^3 + \alpha_2 a^2a^2$ and $b_0^2 = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^2a^2$.*

Proof. Consider $a \in A$ such that $alg(a) = \langle a, a^2, a^3, a^2a^2 \rangle$. For every $b \in A \setminus alg(a)$, $alg(a) \cap alg(b)$ is a proper subalgebra of $alg(a)$ otherwise we should have $alg(a) \subset alg(b)$ and $A = alg(b)$. That is impossible. Then $alg(a) \cap alg(b) \subset alg(a)^2$.

Since $alg(a)$ has codimension 1 then $alg(a) \cap alg(b)$ also has codimension 1 in $alg(b)$. So we have $alg(b)^2 \subset alg(a) \cap alg(b) \subset alg(a)^2$. Therefore we can write $b^2 \in alg(a)^2$. Then $A^2 = alg(a)^2 = \langle a^2, a^3, a^2a^2 \rangle$ which has codimension 2. Since for every $y \in A$ we have $R_y^3(a) = 0$, particularly $0 = R_a^3(A)$. So $KerR_a \neq 0$ and there exists $b_0 \in A \setminus alg(a)$ such that $ab_0 = 0$. This leads to $A = \langle b_0, a, a^2, a^3, a^2a^2 \rangle$, $A^2 = \langle a^2, a^3, a^2a^2 \rangle$ and $A^3 = \langle a^3, a^2a^2 \rangle$. Finally we have the following products : $ab_0 = 0$, $b_0^2 = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^2a^2$, $a^2b_0 = \alpha_1 a^3 + \alpha_2 a^2a^2$, $a^3b_0 = 0$ and $(a^2a^2)b_0 = 0$. □

Remark 5.3. If A does not contain any element x such that $dim(alg(x)) = 4$, then if $a \in A$ is such that $a^3 \neq 0$, we have $alg(a) = \langle a, a^2, a^3 \rangle$. Then, $a^2a^2 \in alg(a)^3 = \langle a^3 \rangle$ which means $a^2a^2 = \alpha a^3$. We can chose a such that $\alpha \neq 0$, otherwise A should be power associative. Putting $a' = \alpha^{-1}a$ we then have $a'^2a'^2 = a'^3$. In the remainder of this work, when $alg(a) = \langle a, a^2, a^3 \rangle$, we shall consider $a^2a^2 = a^3$.

Proposition 5.4. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension 5. Assume that A does not contain any element x such that $dim(alg(x)) = 4$. If $dimA^2 = 3$ then there are $y, x \in A$ such that $A = \langle y, y^2, x, x^2, x^3 \rangle$.*

Proof. Consider $x \in A$ such that $x^3 \neq 0$. We have $alg(x) = \langle x, x^2, x^3 \rangle$. Let $X = alg(x)$. Let's show that there exists $y \in A$ such that $y^2 \notin \langle y, x, x^2, x^3 \rangle$.

Let's proceed by negation. Assume that for all $z \in A$ we have $z^2 \in \langle z, x, x^2, x^3 \rangle$. Since $dimA^2 = 3$, there exixts $z \in A$ such that $A^2 = \langle z^2, x^2, x^3 \rangle$. (Indeed, if for all $z \in A$ we have $z^2 \in \langle x^2, x^3 \rangle$, then for all $a, b \in A$ we should have $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2] \in \langle x^2, x^3 \rangle$. This leads to $A^2 \subset \langle x^2, x^3 \rangle$, contradicting $dimA^2 = 3$). Since $z \notin \langle x, x^2, x^3 \rangle = X$, then $D = \langle z, x, x^2, x^3 \rangle$ is such that $z^2 \in D$ by hypothesis. We have $A^2 = \langle z^2, x^2, x^3\rangle \subset D$ and then D is an ideal of A. Thus D is a nilalgebra of nilindex 4 in dimension 4. Since, by hypothesis, there is no $x \in A$ such that $dim(alg(x)) = 4$, then this algebra D satisfies Table 3 of [4, Theorem 3]. Therefore, $D^2 = \langle x^2, x^3\rangle = X^2$ and so we have $z^2 \in X^2$. Which is a contradiction. So there exixts $y \in A$ such that $A = \langle y, y^2, x, x^2, x^3\rangle$. □

Theorem 5.5. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension 5. Assume that A does not contain any element x such that $dim(alg(x)) = 4$. If $dimA^2 = 3$ and $dimA^3 = 1$ then there are $x, y \in A$ such that $A = \langle y, yx, x, x^2, x^3\rangle$ and $(xy)x^2 = \alpha x^3$, $(xy)x = \beta x^3$, $(xy)^2 = \gamma x^3$, $y(xy) = \lambda x^3$ and $y^2 = \varepsilon_1(yx) + \varepsilon_2 x^2 + \varepsilon_3 x^3$.*

Proof. Let's consider $x \in A$ such that $x^3 \neq 0$. We have $alg(x) = \langle x, x^2, x^3\rangle$. Let $X = alg(x)$,

we show that there exixts $y \in A$ such that $yx \notin X^2 = \langle x^2, x^3\rangle$. Let's proceed by negation.

Because of Proposition 5.4, we know there exists $y \in A$ such that $A = \langle y, y^2, x, x^2, x^3\rangle$. Because of our hypothesis we then have $xy \in X^2$ which means $xy = \alpha_1 x^2 + \alpha_2 x^3$. If $y' = y - (\alpha_1 x + \alpha_2 x^2)$ then we have $xy' = 0$ and $A = \langle y', y'^2, x, x^2, x^3\rangle$. We can therefore assume in the initial basis $\{y, y^2, x, x^2, x^3\}$ that we have $xy = 0$.

Note that there are $\lambda \in K^*$ such that $(y + \lambda x)^3 \neq 0$. Then, we should have by hypothesis, $y(y + \lambda x) \in \langle (x + \lambda y)^2, (x + \lambda y)^3\rangle$ which means $y^2 = \beta_2(y + \lambda x)^2 + \beta_3(y + \lambda x)^3$. So we have $y^2 = \beta_2 y^2 + \beta_2\lambda^2 x^2 + z_0$ with $z_0 \in A^3 = X^3$. This leads to $\beta_2 = 1$ and $\lambda^2 x^2 = -z_0 \in X^3$. Which is a contradiction. We therefore conclude that there exists $y \in A$ such that $xy \notin X^2 = \langle x^2, x^3\rangle$.

Since $y \notin X$ then $xy \notin \langle y, x, x^2, x^3\rangle$. Indeed, assume that $xy = \gamma_0 y + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3$, we then have $x(xy) - (\gamma_0 xy + \gamma_1 x^2) = \gamma_2 x^3 \in X^3$. However, we have $x(xy) \in X^3$ (because $dimA^3 = 1 = dimX^3$ implies that $A^3 = X^3$), which means that $\gamma_0 xy + \gamma_1 x^2 \in X^3 \subset X^2$. This implies that $\gamma_0 = 0$ and it remains $\gamma_1 x^2 \in X^3$, which means $\gamma_1 = 0$. So, we have $xy = \gamma_2 x^2 + \gamma_3 x^3$. Contradiction.

Finally, $\{y, yx, x, x^2, x^3\}$ is a basis of A.

Let $yx^2 = \sigma x^3$. Then we have $(y - \sigma x)x^2 = 0$. We can then assume that in the basis $\{y, yx, x, x^2, x^3\}$ we have $yx^2 = 0$. The other products are written $(yx)x = \alpha x^3$, $(yx)x^2 = \beta x^3$, $(yx)^2 = \gamma x^3$, $y(yx) = \lambda x^3$ and $y^2 = \varepsilon_1 x^2 + \varepsilon_2 x^3 + \varepsilon_3(yx)$. □

Lemma 5.6. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension ≤ 6. Then, for all $x, y \in A$, we have $x(xy^2) = y(yx^2)$*

Proof. Because of Remark 2.7, in dimension ≤ 6, A satisfies $R_x^3(y) = 0$ for all $x, y \in A$.

Linearizing $x(x(xy)) = 0$, gives

$$z(x(xy)) + x(z(xy)) + x(x(zy)) = 0. \tag{9}$$

Take $z = y$ in (9), it gives

$$y(x(xy)) + x(y(xy)) + x(x(y^2)) = 0. \tag{10}$$

Let's interchange x and y in (10), we have

$$x(y(xy)) + y(x(xy)) + y(y(x^2)) = 0. \tag{11}$$

Difference between (10) and (11) gives $x(x(y^2) = y(yx^2)$. □

Theorem 5.7. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension 5. Assume that A does not contain any element x such that $dim(alg(x)) = 4$. If $dimA^2 = 3$ and $dimA^3 = 2$, then there exist $x_0, y_0 \in A$ such that $A = \langle y_0, y_0x_0^2, x_0, x_0^2, x_0^3 \rangle$ and $y_0^2 = \alpha_1 y_0 x_0^2 + \alpha_2 x_0^2 + \alpha_3 x_0^3$.*

Proof. Let $x \in A$ such that $x^3 \neq 0$. We have $alg(x) = \langle x, x^2, x^3 \rangle$. Let's set $X = alg(x)$.

We first show that there exists $y \in A$ such that $yx^2 \notin X^3$. Let's proceed by negation, that means for all $z \in A$ we have $zx^2 \in X^3$.

Since $X^3 = \langle x^3 \rangle$ is an ideal because $x^3y = 0$ for all $y \in A$, the hypothesis implies that $X^2 = \langle x^2, x^3 \rangle$ is also an ideal and the algebra A/X^2 is in dimension 3. We know because of Proposition5.4 that there exists $z \in A$ such that $A = \langle z, z^2, x, x^2, x^3 \rangle$. That implies that $A/X^2 = \langle \overline{z}, \overline{z}^2, \overline{x} \rangle$ is a nilalgebra of nilindex 3. Thus, [4, Theorem 1] implies that $\overline{A}^3 = (A/X^2)^3 = 0$ which means $A^3 \subset X^2$. However $dimA^3 = 2 = dimX^2$, it gives $A^3 = X^2$.

Since X^3 is an ideal, let's consider $\widetilde{A} = A/X^3$ which is in dimension 4. $\widetilde{A}$ is a nilalgebra of nilindex 4. Indeed, if the nilindex was 3, because of [4, Theorem 1] we should have $\widetilde{A}^3 = 0$ which means $A^3 = X^2 \subset X^3$. Contradiction.

The nilindex is then 4. So there exists $\overline{y} \in \widetilde{A}$ such that $\overline{y}^3 \neq 0$, that means $y^3 \notin X^3$. Since $y^3 \in A^3 = X^2$, then $y^3 = \alpha_2 x^2 + \alpha_3 x^3$ with $\alpha_2 \neq 0$. However $xy^3 = 0 = \alpha_2 x^3$, this implies $\alpha_2 = 0$. Contradiction.

So, there is $y \in A$ such that $yx^2 \notin X^3$. Since $\widetilde{A} = A/X^3$ is a nilalgebra of nilindex 4 and dimension 4, because of [4, Theorem 3], we have $\widetilde{A}^4 = 0$ that means $A^4 \subset X^3$. However, we have $y \in A$ such that $yx^2 \notin X^3$, which implies that $y \notin X$. In addition, $yx^2 \notin \langle y, x, x^2, x^3 \rangle$. Indeed, if we assume that $yx^2 = \beta_0 y + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$, then multiplying by x^2 we have $0 = x^2(yx^2) = \beta_0 yx^2 + (\beta_1 + \beta_2)x^3$ by Lemma 2.1. It implies that $\beta_0 = 0$ and $\beta_1 + \beta_2 = 0$. So, we have $yx^2 = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$. Multiplying by x, equality (4) gives $0 = x(yx^2) = \beta_1 x^2 + \beta_2 x^3$ and $\beta_1 = \beta_2 = 0$. Then we have $yx^2 = \beta_3 \in X^3$. Contradiction.

Thus, $yx^2 \notin \langle y, x, x^2, x^3 \rangle$ and $A = \langle y, yx^2, x, x^2, x^3 \rangle$. Since $yx \in A^2$, then $yx = \delta_1 yx^2 + \delta_2 x^2 + \delta_3 x^3$. If $x_0 = x - \delta_1 x^2$ then $yx_0 = \gamma_2 x_0^2 + \gamma_3 x_0^3$. Let $y_0 = y - (\gamma_2 x_0 + \gamma_3 x_0^2)$. We have $x_0 y_0 = 0$ and $\{y_0, y_0x_0^2, x_0, x_0^2, x_0^3\}$ is a basis of A.

Let's take $y_0^2 = \alpha_1 x_0^2 + \alpha_2 x_0^3 + \alpha_3 y_0 x_0^2$ and $(y_0x_0^2)^2 = \lambda x_0^3$.

Lemma 5.6 gives $y_0(y_0x_0^2) = x_0(x_0y_0^2) = x_0 \left(\alpha_3 x_0(y_0x_0^2) + \alpha_1 x_0^3\right) = 0$.

Taking $z = y_0$ in Lemma 2.1 $(viii)$, we have $\lambda = 0$. □

Theorem 5.8. *Let's consider A a Lie triple nilalgebra of nilindex 4 and dimension 5. Assume that A does not contain any element x such that $dim(alg(x)) = 4$. If $dimA^2 = 2$, then there exist $x_0, y_0, a \in A$ such that $A = \langle x_0, y_0, a, a^2, a^3\rangle$ and $x_0^2 = \alpha_1 a^2 + \alpha_2 a^3$, $y_0^2 = \lambda_1 a^2 + \lambda_2 a^3$, $x_0 y_0 = \beta_1 a^2 + \beta_2 a^3$.*

Proof. Let $a \in A$ such that $a^3 \neq 0$. Since $dimA^2 = 2$, then we have $A^2 = \langle a^2, a^3\rangle$, $A^3 = \langle a^3\rangle$. There exist $x, y \in A \setminus alg(a)$ such that $A = \langle x, y, a, a^2, a^3\rangle$ and $x^2 = \alpha_1 a^2 + \alpha_2 a^3$, $ax = \gamma_1 a^2 + \gamma_2 a^3$, $ya = \varepsilon_1 a^2 + \varepsilon_2 a^3$, $xy = \beta_1 a^2 + \beta_2 a^3$, $xa^2 = \delta a^3$, $ya^2 = \mu a^3$ and $y^2 = \lambda_1 a^2 + \lambda_2 a^3$.

In equality $ax = \gamma_1 a^2 + \gamma_2 a^3$, if we take $x' = x - \gamma_1 a - \gamma_2 a^2$, we have $ax' = 0$.

In equality $ay = \varepsilon_1 a^2 + \varepsilon_2 a^3$, taking $y' = y - \varepsilon_1 a - \varepsilon_2 a^2$, gives $ay' = 0$.

Thus, in the basis $\{x, y, a, a^2, a^3\}$, we can assume that $ax = 0 = ay$.

In the same way, in equalities $xa^2 = \delta a^3$ and $ya^2 = \mu a^3$, by respectively setting $x_0 = x - \delta a$ and $y_0 = y - \mu a$, we have $x_0 a^2 = 0 = y_0 a^2$. Thus, in the basis $\{x_0, y_0, a, a^2, a^3\}$ we have $x_0^2 = \alpha_1 a^2 + \alpha_2 a^3$, $x_0 y_0 = \beta_1 a^2 + \beta_2 a^3$, $y_0^2 = \lambda_1 a^2 + \lambda_2 a^3$. □

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