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IMPROVING SYSTEM MODEL STRUCTURED RELIABILITY USING INFORMATION MEASURES AND CONCOMITANT APPROACHES WITH APPLICATIONS TO ENERGY DATA IN SAUDI ARABIA

HANAN H. SAKR[1,*], MOHAMED SAID MOHAMED[2]

[1]Department of Management Information Systems, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia

[2]Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Hawtat Bani Tamim 16511, Saudi Arabia

[*]Corresponding author: h.sakr@psau.edu.sa

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Abstract. Traditional entropy measures and their fractional extensions are closely related to the informational generating function, which provides a versatile mathematical framework for quantifying uncertainty. In this paper, we introduce the residual cumulative informational generating function associated with concomitants of generalized order statistics arising from the Huang–Kotz family of distributions. Several analytical properties of the proposed measure are established, including series expansions and illustrative examples. In addition, upper and lower bounds are derived, and stochastic comparisons are investigated under classical stochastic orderings. Two nonparametric estimation procedures are developed to estimate the proposed measure, and their finite-sample behavior is examined. The practical applicability and comparative performance of the estimators are illustrated through both simulated and real datasets.

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1. Introduction

Entropy represents one of the central pillars connecting information theory and statistical inference, offering a mathematical framework for quantifying uncertainty, randomness, and the underlying structure of probability distributions. Following the seminal contribution of Claude E. Shannon [22], a wide range of information measures have been proposed to describe various dimensions of uncertainty inherent in random phenomena and probabilistic systems. The study of entropy gains additional significance when considered within transformed distributional settings arising from dependent bivariate models and their applications in reliability analysis. One important framework in this direction is provided by the Farlie–Gumbel–Morgenstern family of bivariate distributions, often referred to simply

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as the Morgenstern family. This class was originally introduced by Morgenstern [20] while examining joint distributions possessing Cauchy marginals. Since then, it has served as an effective mechanism for constructing multivariate probability models in situations where the marginal distributions are specified beforehand, offering a flexible yet mathematically manageable representation of dependence.

Subsequently, Johnson and Kotz [13] extended the Morgenstern framework to the multivariate setting and examined in detail its theoretical distributional characteristics as well as its statistical implications. To provide greater flexibility in modeling dependence, Huang and Kotz [9] later developed a modified version of the family by introducing an extra parameter capable of enlarging the attainable range of association between the variables.

Motivated by the need for richer dependence structures, numerous extensions of the Morgenstern class have appeared in the literature. Among these developments, Huang and Kotz [10] proposed a polynomial-based generalization characterized by the inclusion of a single shape parameter. For this model, the joint cumulative distribution function and the associated probability density function take the forms

$$K_{X,Y}(x,y) = K_X(x)K_Y(y)\left[1+\eta\big(1-K_X^{\theta}(x)\big)\big(1-K_Y^{\theta}(y)\big)\right], \tag{1}$$

and

$$k_{X,Y}(x,y) = k_X(x)k_Y(y)\left[1+\eta\Big((1+\theta)K_X^{\theta}(x)-1\Big)\Big((1+\theta)K_Y^{\theta}(y)-1\Big)\right], \quad \theta \geq 1, \tag{2}$$

respectively. The admissible parameter space for the vector (η, θ) is

$$Q = \{(\eta,\theta) : -\theta^{-2} \leq \eta \leq \theta^{-1},\ \theta \geq 1\},$$

while the corresponding correlation coefficient is restricted to the interval

$$-3(\theta+2)^{-2}\min(1,\theta^2) \leq \text{Corr.} \leq 3\theta(\theta+2)^{-2}.$$

Concomitants of order statistics were first examined by David et al. [6]. Let (X_i, Y_i) $i = 1, 2, ..., n$ be n pairs of independent, randomly selected variables from a bivariate population with CDF $K_{X,Y}(x,y)$. The concomitant of lth order statistics, indicated by $Y_{[l:n]}$, is Y associated with $X_{(l:n)}$ if $X_{(l:n)}$ is the lth order statistics. $Y_{[l:n]}$'s PDF and CDF are provided by:

$$k_{Y,[l:n]}(y) = \int_{-\infty}^{\infty} k_{Y|X}(y \mid x)\, k_{X,(l:n)}(x)\, dx, \tag{3}$$

$$K_{Y,[l:n]}(y) = \int_{-\infty}^{\infty} K_{Y|X}(y \mid x)\, k_{X,(l:n)}(x)\, dx, \tag{4}$$

with noting that $k_{X,(l:n)}(x)$ is the PDF of $X_{(l:n)}$. The study of concomitants of order statistics has continued to attract considerable attention owing to their usefulness in reliability theory, ranked set sampling, information theory, and inferential procedures involving dependent variables. Recent developments have focused on information-theoretic measures, estimation methods, and generalized bivariate models. For example, Almaspoor et al. [1] derived extropy measures for concomitants of generalized order statistics in the Morgenstern family, while Husseiny et al. [11] investigated entropy and Fisher information measures associated with concomitants under the Sarmanov family. Barakat and Syam [4] studied some Comparison for Huang–Kotz distribution. More recently, Al-Zaydi [2] obtained distributional properties and best linear unbiased estimators based on concomitants arising from a bivariate generalized Weibull model, and Al-Zayd [3] developed theoretical and practical results for concomitants from a bivariate generalized linear exponential distribution. These contributions highlight the growing importance of concomitant-based methodologies in modern statistical inference.

A comprehensive framework for ordered random variables was developed by Kamps [14] through the concept of generalized order statistics. This approach unifies a variety of ordered-data models within a single theoretical structure, encompassing ordinary order statistics, sequential order statistics, record values, Pfeifer's records, and progressively Type-II censored order statistics as special instances. Let $n \in \mathbb{N}$, $m \geq 1$, and $d_1, \ldots, d_{n-1} \in \mathbb{R}$. For $1 \leq s \leq n-1$, define $D_s = \sum_{j=s}^{n-1} d_j$, and introduce the quantities $\beta_s = m + n - s + D_s$. Throughout, it is assumed that $\beta_s \geq 1$, $s = 1, 2, \ldots, n-1$. Moreover, let $\tilde{d} = (d_1, d_2, \ldots, d_{n-1}) \in \mathbb{R}^{n-1}$. Several important ordered-sample schemes can be recovered from the subclass of d-generalized order statistics. A particularly important situation arises when all parameters coincide, namely, $d_1 = d_2 = \cdots = d_{n-1} = d$. In this case, the probability density function of the lth d-generalized order statistic, denoted by $Y_{(l,n,d,m)}$, takes the form (see Kamps [14])

$$k_{Y,(l,n,d,m)}(y) = \frac{c_{l-1}}{(l-1)!}\big(1 - K_Y(y)\big)^{\beta_l - 1} k_Y(y) g_d^{l-1}\big(K_Y(y)\big), \tag{5}$$

where $1 \leq l \leq m$, $\beta_i = m + (n-l)(d+1)$, and $c_{l-1} = \prod_{j=1}^{l} \beta_j$. The function $g_d(\cdot)$ is defined by $g_d(y) = k_{Y,d}(y) - k_{Y,d}(0)$, $0 < y < 1$, with

$$k_{Y,d}(y) = \begin{cases} \dfrac{-(1-y)^{d+1}}{d+1}, & d \neq -1, \\[2ex] -\ln(1-y), & d = -1. \end{cases}$$

In the sequel, the parameter θ is assumed to satisfy $\theta \in \mathbb{N}$ with $\theta \geq 1$. Extending the dependence structure proposed by Huang and Kotz, Mohamed [19] obtained explicit expressions for both the density and distribution functions of the concomitant associated with the lth d-generalized order statistic. These quantities are respectively given by

$$k_{Y,[l,n,d,m]}(y) = k_Y(y)\left[1 + \eta\, \Xi_{[l,n,d,m;\theta]}\Big(1 - (1+\theta)K_Y^{\theta}(y)\Big)\right], \tag{6}$$

and

$$K_{Y,[l,n,d,m]}(y) = K_Y(y)\left[1 + \eta\, \Xi_{[l,n,d,m;\theta]}\Big(1 - K_Y^{\theta}(y)\Big)\right]. \tag{7}$$

Moreover, the corresponding survival function is

$$\overline{K}_{Y,[l,n,d,m]}(y) = \overline{K}_Y(y) - \eta\, \Xi_{[l,n,d,m;\theta]} K_Y(y)\Big(1 - K_Y^{\theta}(y)\Big), \tag{8}$$

where $\overline{K}_Y(y) = 1 - K_Y(y)$. The coefficient $\Xi_{[l,n,d,m;\theta]}$ appearing in the above representations is defined by

$$\Xi_{[l,n,d,m;\theta]} = 1 - (1+\theta)\sum_{j=0}^{\theta}\binom{\theta}{j}(-1)^j \prod_{i=1}^{l}\frac{\beta_i}{\beta_i + j}, \tag{9}$$

where $\beta_i = m + (n-l)(d+1)$, $1 \le l \le n$.

Generating functions constitute one of the most powerful tools in probability and statistical theory. A classical example is the moment generating function, whose successive derivatives evaluated at the origin yield the moments of the underlying random variable. An analogous construction appears in information theory through the *informational generating function*, which serves as a basis for deriving a variety of uncertainty and information measures. Introduced by Golomb [8], this function is given, for a non–negative random variable with PDF k_Y, by

$$GI_\delta(Y) = \int_0^\infty k_Y^{\delta}(y)\, dy, \tag{10}$$

where $\delta > 0$, with the normalization property $GI_1(Y) = 1$. Moreover,

$$\frac{\partial}{\partial\delta} GI_\delta(Y)\big|_{\delta=1} = \int_0^\infty k_Y(y) \ln k_Y(y)\, dy = -ESn(Y), \tag{11}$$

where $ESn(Y)$ denotes Shannon entropy; see [22]. Because of its fundamental connection with entropy-based methodologies, the informational generating function has been investigated extensively in recent years. For instance, Kharazmi and Balakrishnan [15] employed this framework to develop two divergence measures encompassing several classical divergence indices as special cases. Related developments for order statistics and record-value models were subsequently reported by Kharazmi and Balakrishnan [16] and Zamani *et al.* [25].

Kharazmi and Balakrishnan [17] introduced the cumulative residual informational generating function of the survival function $\overline{K}_Y(y) = 1 - K_Y(y)$ as follows

$$\begin{aligned} RGI_\delta(Y) &= \int_0^\infty \overline{K}_Y^\delta(y)\, dy \\ &= \int_0^1 \frac{\overline{K}_Y^\delta(u)}{k_Y(K_Y^{-1}(u))}\, du, \end{aligned} \tag{12}$$

where $\delta > 0$, and $K_Y^{-1}(u) = \inf\{y : K_Y(y) \geq u\}$. Which easily notice the following characteristics: $RGI_1(Y) = \mathbb{E}[Y] = \int_0^\infty \overline{K}_Y(y)\, dy$, and

$$\frac{\partial}{\partial\delta} RGI_\delta(Y)\big|_{\delta=1} = \int_0^\infty \overline{K}_Y(y) \ln \overline{K}_Y(y)\, dy = -RESn(Y),$$

where $RESn(Y)$ denotes the cumulative residual entropy; see Rao *et al.* [23].

The primary objective of this work is to investigate the residual cumulative informational generating function associated with the concomitant of d-generalized order statistics from the Huang–Kotz family under the conditions $\delta \geq 1$ and $\theta \geq 1$. The remainder of the paper is organized as follows. Section 2 develops several representations and expansions of the proposed measure and provides illustrative examples for a number of lifetime distributions. Furthermore, various bounds and stochastic comparisons based on classical stochastic orderings are established. Section 3 is devoted to computational investigations that support the theoretical findings. In particular, two nonparametric estimation procedures for the residual cumulative informational generating function are constructed and studied. Their practical applicability and finite-sample performance are illustrated through analyses of both simulated and real datasets.

2. Residual cumulative informational generating function of the concomitant of d-generalized order statistics

In this section, we will present the informational generating function of the concomitant of d-generalized order statistics following Huang–Kotz distribution. If $Y_{[l,n,d,m]}$ is the concomitant of lth d-generalized order statistics with survival function defined in (8) of Huang–Kotz family. Then, the residual cumulative informational generating function of $Y_{[l,n,d,m]}$ can be considered as

$$\begin{aligned} RGI_\xi(Y_{[l,n,d,m]}) &= \int_0^\infty \left(\overline{K}_{Y,[l,n,d,m]}(y)\right)^\delta dy \\ &= \int_0^\infty \left(\overline{K}_Y(y) - \eta\, \Xi_{[l,n,d,m;\theta]} K_Y(y)\Big(1 - K_Y^\theta(y)\Big)\right)^\delta dy, \end{aligned} \tag{13}$$

where $\delta \geq 1$ and $\theta \geq 1$. To derive the residual cumulative informational generating function of the system, we apply the probability integral transformation by setting $K_{U,[l,n,d,m]} = K_{Y,[l,n,d,m]}$. Consequently, the original random variables Y_i are mapped into $U_i = K_{Y_i}$, $i = 1, 2, \ldots, n$, whose distributions are uniform on the interval $(0, 1)$. Under the assumptions $\delta \geq 1$ and $\theta \geq 1$, the survival function associated with $U_{[l,n,d,m]}$ takes the form

$$\overline{K}_{U,[l,n,d,m]}(u) = (1-u) - \eta\,\Xi_{[l,n,d,m;\theta]}u\Big(1-u^{\theta}\Big), \forall\, 0 < u < 1. \tag{14}$$

Combining Equations (10), (8), and (14), and introducing the transformation $u = K_Z(z)$, the proposed information measure can be represented as

$$\begin{aligned} RGI_{\xi}(Y_{[l,n,d,m]}) &= \int_0^{\infty} \left(\overline{K}_{Y,[l,n,d,m]}(y)\right)^{\delta} dy \\ &= \int_0^1 \frac{\Big((1-u) - \eta\,\Xi_{[l,n,d,m;\theta]}u\Big(1-u^{\theta}\Big)\Big)^{\delta}}{k_Y(K_Y^{-1}(u))} du \\ &= \int_0^1 \frac{\left(\overline{K}_{U,[l,n,d,m]}(u)\right)^{\delta}}{k_Y(K_Y^{-1}(u))} du, \end{aligned} \tag{15}$$

with noting that $K_Y^{-1}(u) = \inf\{y : K_Y(y) \geq u\}$.

Theorem 2.1. *Let $Y_{[l,n,d,m]}$ be the concomitant of the lth d-generalized order statistic from the Huang–Kotz family with survival function defined in (13), where $\delta > 0$ and $\theta \geq 1$. Assume that*

$$\left| \frac{\eta\,\Xi_{[l,n,d,m;\theta]}K_Y(y)\left(1-K_Y^{\theta}(y)\right)}{\overline{K}_Y(y)} \right| < 1, \qquad y > 0.$$

Then the corresponding residual cumulative informational generating function can be expressed as

$$RGI_{\xi}(Y_{[l,n,d,m]}) = \sum_{j=0}^{\infty}(-1)^j\binom{\delta}{j}\eta^j\Xi^j_{[l,n,d,m;\theta]}\int_0^{\infty}\overline{K}_Y^{\delta-j}(y)K_Y^j(y)\left(1-K_Y^{\theta}(y)\right)^j dy, \tag{16}$$

provided that the interchange of summation and integration is valid.

Proof. Substituting the survival function of $Y_{[l,n,d,m]}$ in (8) into the definition of $RGI_{\xi}(Y_{[l,n,d,m]})$ in (13), we obtain

$$RGI_{\xi}(Y_{[l,n,d,m]}) = \int_0^{\infty}\left[\overline{K}_Y(y) - \eta\,\Xi_{[l,n,d,m;\theta]}K_Y(y)\left(1-K_Y^{\theta}(y)\right)\right]^{\delta} dy.$$

Let

$$A = \overline{K}_Y(y), \qquad B = \eta\,\Xi_{[l,n,d,m;\theta]}K_Y(y)\left(1-K_Y^{\theta}(y)\right).$$

Applying the generalized binomial theorem

$$(A-B)^{\delta} = \sum_{j=0}^{\infty}(-1)^j\binom{\delta}{j}A^{\delta-j}B^j,$$

which converges for $\left|\frac{B}{A}\right| < 1$, gives

$$RGI_{\xi}(Y_{[l,n,d,m]}) = \int_0^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} \overline{K}_Y^{\,\delta-j}(y) \left[\eta\, \Xi_{[l,n,d,m;\theta]} K_Y(y) \left(1 - K_Y^{\theta}(y)\right)\right]^j dy.$$

After simplification,

$$RGI_{\xi}(Y_{[l,n,d,m]}) = \int_0^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} \eta^j \Xi^j_{[l,n,d,m;\theta]} \overline{K}_Y^{\,\delta-j}(y) K_Y^j(y) \left(1 - K_Y^{\theta}(y)\right)^j dy.$$

Interchanging the order of summation and integration yields

$$RGI_{\xi}(Y_{[l,n,d,m]}) = \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} \eta^j \Xi^j_{[l,n,d,m;\theta]} \int_0^{\infty} \overline{K}_Y^{\,\delta-j}(y) K_Y^j(y) \left(1 - K_Y^{\theta}(y)\right)^j dy,$$

which completes the proof. □

Remark 2.1. *Several well-known models can be obtained from the d-generalized order statistics framework as particular cases:*

(1) *Setting $d = 0$ and $m = 1$ reduces the d-generalized order statistics to the classical order statistics. Consequently, for the concomitant associated with the lth order statistic $Y_{[l:n]}$, expression (9) simplifies to*

$$\begin{aligned} \Xi_{[l,n;\theta]} &= 1 - (1+\theta) \sum_{j=0}^{\theta} \binom{\theta}{j} (-1)^j \left(\prod_{i=1}^{l} \frac{n-i+1}{n-i+1+j} \right) \\ &= 1 - \frac{(1+\theta) Beta(l+\theta, n-l+1)}{Beta(l, n-l+1)}. \end{aligned} \tag{17}$$

It is worth mentioning that $Beta(a,b)$ denotes the Beta function, given by $Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, where $a, b > 0$.

(2) *When $d = -1$ and $m = 1$, the corresponding model coincides with the record-value scheme. In this situation, the measure associated with the concomitant of the lth record value $Y_{[l]}$ is obtained from (9) as*

$$\Xi_{[l;\theta]} = 1 - (1+\theta) \sum_{j=0}^{\theta} \binom{\theta}{j} (-1)^j \frac{1}{(j+1)^l}. \tag{18}$$

The following example demonstrates an application of Equation (15).

Example 2.1. *In reliability engineering, the l–out–of–n configuration is a classical system model that arises in many practical applications. Consider a system consisting of n independent components with lifetimes $Y_1, Y_2, \ldots, Y_n$. The system remains operative whenever at least l components are functioning. Consequently, the system lifetime is given by the $(n-l+1)$th order statistic, denoted by $Y_{n-l+1:n}$.*

(1) *Suppose that the component lifetimes are independent and identically distributed following the common Lomax distribution, also known as the Pareto Type II distribution with scale parameter τ and α scale*

parameter. Its PDF is

$$k_Y(y) = \frac{\alpha}{\tau}\left(1+\frac{y}{\tau}\right)^{-(\alpha+1)}, \qquad y>0,\ \tau,\alpha>0.$$

Using the corresponding quantile function, one obtains

$$k_Y\big(K_Y^{-1}(u)\big) = \frac{\alpha}{\tau}(1-u)^{\frac{1+\alpha}{\alpha}}, \qquad 0<u<1.$$

Substituting this expression into (15), for $\delta \geq 1$ and $\theta \geq 1$, yields the residual cumulative informational generating function associated with the system lifetime:

$$\begin{aligned} RGI_\delta(Y_{[l,n,d,m]}) &= \frac{\tau}{\alpha}\int_0^1 \frac{\Big((1-u) - \eta\,\Xi_{[l,n,d,m;\theta]}u\Big(1-u^\theta\Big)\Big)^\delta}{(1-u)^{\frac{1+\alpha}{\alpha}}}\,du \\ &= \frac{\tau}{\alpha}\int_0^1 \frac{\big(\overline{K}_{U,[l,n,d,m]}(u)\big)^\delta}{(1-u)^{\frac{1+\alpha}{\alpha}}}\,du. \end{aligned} \tag{19}$$

The integral involved does not generally admit a convenient analytical evaluation because of its complicated structure in Eq. (19). As a result, numerical methods are utilized to compute and analyze the values of $RGI_\delta(Y_{[n-l+1:n]})$ for different choices of the Lomax distribution parameters τ and α in Figure 1, which illustrates the relationship between $RGI_\delta(Y_{[n-l+1:n]})$ and the Lomax distribution parameters τ and α with $n=50$ and $l=20$. The numerical results indicate that the lifetime quantified via the residual cumulative informational generating function decreases as the shape parameter α becomes larger. On the other hand, increasing the scale parameter τ results in a corresponding increase in the lifetime measure. Therefore, the residual cumulative informational generating function of the l–out–of–n system is highly sensitive to variations in the Lomax shape parameter α, reflecting its substantial influence on the associated uncertainty characteristics of the system.

(2) *Suppose that the component lifetimes are independent and identically distributed following the common exponential distribution with rate parameter τ. Its PDF is*

$$k_Y(y) = 1-e^{-\tau y}, \qquad y>0,\ \tau>0.$$

Using the corresponding quantile function, one obtains

$$k_Y\big(K_Y^{-1}(u)\big) = \tau(1-u), \qquad 0<u<1.$$

Substituting this expression into (15), for $\delta \geq 1$ and $\theta \geq 1$, yields the residual cumulative informational generating function associated with the system lifetime:

$$\begin{aligned} RGI_\delta(Y_{[l,n,d,m]}) &= \int_0^1 \frac{\Big((1-u) - \eta\, \Xi_{[l,n,d,m;\theta]} u \Big(1-u^\theta\Big)\Big)^\delta}{\tau(1-u)}\, du \\ &= \int_0^1 \frac{\big(\overline{K}_{U,[l,n,d,m]}(u)\big)^\delta}{\tau(1-u)}\, du. \end{aligned} \tag{20}$$

Based on l–out–of–n system in Eq. (20) ($n = 50$ and $l = 20$), Figure 2 shows that the residual cumulative informational generating function decreases monotonically as the parameter τ increases. Hence, larger values of τ are associated with lower values of the residual cumulative informational generating function, indicating a reduction in the uncertainty associated with the system lifetime. Furthermore, for a fixed value of τ, the residual cumulative informational generating function decreases as δ increases, implying that higher values of δ correspond to lower uncertainty levels. These observations remain consistent under both parameter settings $(\theta, \eta) = (2, 0.5)$ and $(\theta, \eta) = (4, 0.25)$.

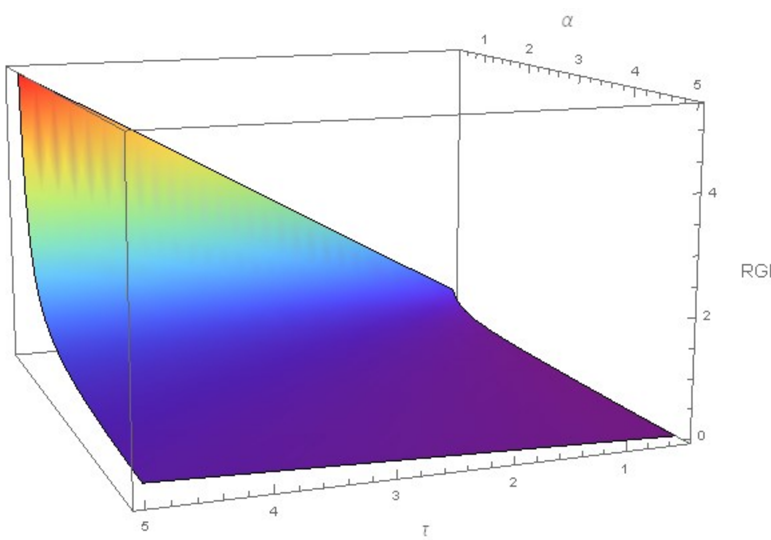
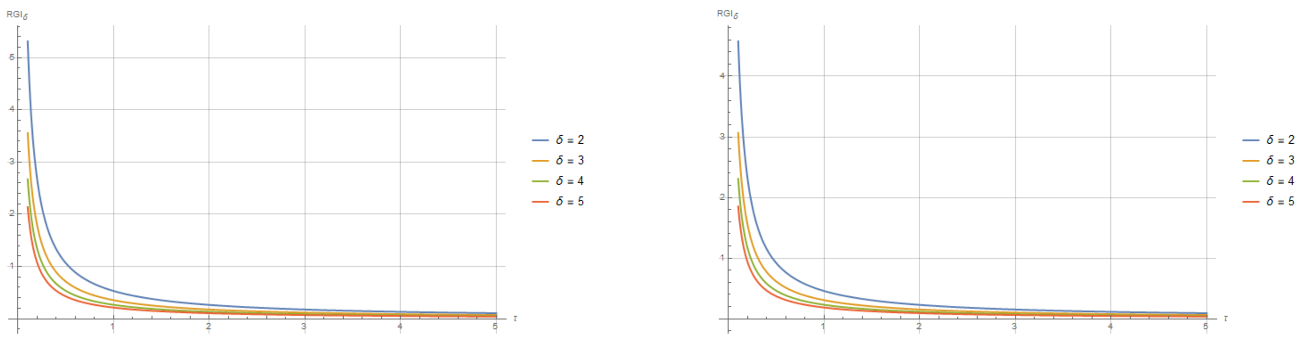


Figure 1. The plot of $RGI_3(Y_{[31:50]})$, with respect to $\theta = 2$, and $\eta = 0.5$, in the case of Example 2.1.

2.1. **Practical Bounds for residual cumulative informational generating function.** For many reliability models, deriving an explicit expression for the residual cumulative informational generating function associated with concomitant ordered systems is often mathematically intractable. This difficulty becomes more pronounced when the component lifetime distributions possess complicated analytical forms or when the number of system components is large. Consequently, the development of suitable upper and lower bounds becomes an important alternative for studying and quantifying the behavior



(A) $\theta = 2; \eta = 0.5$ (B) $\theta = 4; \eta = 0.25$

FIGURE 2. The behavior of $RGI_\delta(Y_{[31:50]})$ for the exponential distribution, in the case of Example 2.1.

of this information measure. Motivated by these considerations, we establish several bounds for the residual cumulative informational generating function that provide useful theoretical understanding of its characteristics in concomitant ordered systems. The corresponding results are stated in the theorem below.

Theorem 2.2. *Let $Y_{[l,n,d,m]}$ represent the concomitant corresponding to the lth d-generalized order statistic obtained from a sample of size n consisting of independent and identically distributed random variables with PDF $k_Y(y)$. Then,*

(1) *We have*

$$RGI_\delta(Y_{[l,n,d,m]}) \leq \mathbb{E}[Y]. \tag{21}$$

(2) *We have*

$$S_{1;\delta} RGI_\delta(Y) \leq RGI_\delta(Y_{[l,n,d,m]}) \leq S_{2;\delta} RGI_\delta(Y), \tag{22}$$

where $\delta \geq 1$ and $\theta \geq 1$, $S_{1;\delta} = \inf_{u\in(0,1)} \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{\overline{K}^\delta_U(u)}$, $S_{2;\delta} = \sup_{u\in(0,1)} \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{\overline{K}^\delta_U(u)}$.

Proof. (1) From (8), since $\overline{K}_{Y,[l,n,d,m]}(y) \geq 0$. Then, $\overline{K}_Y(y) \geq \eta\, \Xi_{[l,n,d,m;\theta]} K_Y(y)\Big(1 - K^\theta_Y(y)\Big)$. Using the fact that $\overline{K}_Y(y) \geq \overline{K}^\delta_Y(y)$, $\forall \delta \geq 1$. From (13), we have

$$\begin{aligned} RGI_\delta(Y_{[l,n,d,m]}) &= \int_0^\infty \overline{K}^\delta_{Y,[l,n,d,m]}(y)\, dy \\ &\leq \int_0^\infty \overline{K}_{Y,[l,n,d,m]}(y)\, dy \leq \int_0^\infty \overline{K}_Y(y)\, dy = \mathbb{E}[Y], \;\; \delta \geq 1. \end{aligned}$$

(2) From (15), the upper bound can be found as follows:

$$\begin{aligned} RGI_\xi(Y_{[l,n,d,m]}) &= \int_0^1 \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{k_Y(K_Y^{-1}(u))}\,du \\ &= \int_0^1 \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{\overline{K}^\delta_U(u)} \frac{\overline{K}^\delta_U(u)}{k_Y(K_Y^{-1}(u))}\,du \\ &\leq \sup_{u\in(0,1)} \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{\overline{K}^\delta_U(u)} \int_0^1 \frac{\overline{K}^\delta_U(u)}{k_Y(K_Y^{-1}(u))}\,du = S_{2;\delta} RGI_\delta(Y). \end{aligned} \tag{23}$$

The result can be obtained by deriving the lower bound in a similar manner.

□

Further helpful bounds based on the PDF's extremes are introduced in the next theorem.

Theorem 2.3. *Let $Y_{[l,n,d,m]}$ represent the concomitant corresponding to the lth d-generalized order statistic having common PDF $k_Y(y)$ and CDF $K_Y(y)$. If S is the support of k_Y, and*

$$m = \inf_{y\in S} k_Y(y), \qquad M = \sup_{y\in S} k_Y(y),$$

such that $0 < m < M < 1$, then

$$\frac{RGI_\delta(U_{[l,n,d,m]})}{M} \leq RGI_\delta(Y_{[l,n,d,m]}) \leq \frac{RGI_\delta(U_{[l,n,d,m]})}{m}, \tag{24}$$

where

$$RGI_\xi(U_{[l,n,d,m]}) = \int_0^1 \left(\overline{K}_{U,[l,n,d,m]}(u)\right)^\delta du. \tag{25}$$

Proof. Since $0 < m \leq k_Y(K_Y^{-1}(u)) \leq M < 1, 0 < u < 1$, we have

$$\frac{1}{M}\int_0^1 \left(\overline{K}_{U,[l,n,d,m]}(u)\right)^\delta du \leq \int_0^1 \frac{\overline{K}^\delta_{U,[l,n,d,m]}(u)}{k_Y(K_Y^{-1}(u))}\,du \leq \frac{1}{m}\int_0^1 \left(\overline{K}_{U,[l,n,d,m]}(u)\right)^\delta du.$$

Recalling Equation (15), we obtain the desired result. □

The notation $RGI_\xi(U_{[l,n,d,m]})$ refers to the residual cumulative informational generating function of the concomitant corresponding to d-generalized order statistics under a uniform parent distribution on $(0,1)$. The bounds presented in Equation (24) are expressed in terms of the lower and upper bounds of the PDF k_Y. It follows that the upper inequality ceases to be informative when $m = 0$, since no finite upper bound exists in this case. Likewise, the lower inequality becomes unavailable whenever $M = \infty$, as a finite lower bound cannot then be derived.

2.2. **Stochastic comparison.** We will now examine the residual cumulative informational generating function of the concomitant corresponding to d-generalized order statistics whose individuated lifetimes satisfy spreading and location-independent hazardous ordering conditions. Let's begin by briefly reviewing these important concepts. The notation $\mathbb{R}_+ = \{Y : Y \geq 0\}$ will be used to indicate the set of fully continuously nonnegative random variables with supporting $(0, \infty)$. For informal definitions and properties of these concepts, readers are urged to see Shaked Shanthikumar [21]. Consider two nonnegative random variables Y_1 and Y_2 with PDFs k_{Y_1} and k_{Y_2}, CDFs K_{Y_1} and K_{Y_2}, survival functions $\bar{K}_{Y_1}$ and $\bar{K}_{Y_2}$, and hazard rate functions

$$\psi_{Y_i}(y) = \frac{k_{Y_i}(y)}{\bar{K}_{Y_i}(y)}, \qquad i = 1, 2.$$

Table 1 presents the definitions of several commonly used aging properties and stochastic order relations. In the following results, we will take

$$\Omega_\delta(u) = (u)^\delta, \ 0 \leq u \leq 1. \tag{26}$$

Table 1. Definitions of several stochastic orderings and aging properties.

Property/Order	Definition
IFRT/DFRT	The random variable Y_1 is said to exhibit an increasing failure rate (IFRT) if $\psi_{Y_1}(y)$ increases with y. Likewise, Y_1 possesses a decreasing failure rate (DFRT) whenever $\psi_{Y_1}(y)$ decreases as y increases.
Hazard Rate Order	The notation $Y_1 \leq_{hro_y} Y_2$ is used whenever $$\psi_{Y_1}(r) \geq \psi_{Y_2}(r), \qquad r > 0.$$ In this case, Y_1 is regarded as smaller than Y_2 with respect to the hazard rate ordering.
Dispersive Order	The relation $Y_1 \leq_{\text{dispo}_y} Y_2$ holds if $$K_{Y_1}^{-1}(r_1) - K_{Y_1}^{-1}(r_2) \leq K_{Y_2}^{-1}(r_1) - K_{Y_2}^{-1}(r_2),$$ for every $0 < r_1 \leq r_2 < 1$. An equivalent characterization is $$k_{Y_1}\left(K_{Y_1}^{-1}(v)\right) \geq k_{Y_2}\left(K_{Y_2}^{-1}(v)\right), \qquad 0 < v < 1.$$ Thus, the spacing between corresponding quantiles of Y_1 does not exceed that of Y_2.
Location-Independent Riskier Order	The random variable Y_1 is said to precede Y_2 in the location-independent riskier order, denoted by $Y_1 \leq_{liro_y} Y_2$, whenever $$\int_0^{K_{Y_1}^{-1}(r)} K_{Y_1}(y)\,dy \leq \int_0^{K_{Y_2}^{-1}(r)} K_{Y_2}(y)\,dy, \qquad r \in (0,1).$$ This inequality must be satisfied for all probability levels r in the interval $(0, 1)$.

The dispersive ordering, denoted by $\leq_{\text{dispo}_y}$, was originally introduced by Bickel and Lehmann [5] within the framework of nonparametric statistical analysis. Later, Jewitt [12] proposed the location-independent riskier ordering $\leq_{liro_y}$ in the context of expected utility theory and insurance decision models. It is well known that the dispersive order admits the characterization

$$k_{Y_2}\left(K_{Y_2}^{-1}(v)\right) \le k_{Y_1}\left(K_{Y_1}^{-1}(v)\right), \qquad 0 < v < 1, \tag{27}$$

which is equivalent to the relation $Y_1 \le_{\text{dispo}_y} Y_2$.

Furthermore, several stochastic orderings are connected through the implication chain

$$Y_1 \le_{hro_y} Y_2 \text{ and either } Y_1 \text{ or } Y_2 \text{ is DFRT} \Longrightarrow Y_1 \le_{\text{dispo}_y} Y_2 \Longrightarrow Y_1 \le_{liro_y} Y_2. \tag{28}$$

Because the function $\Omega_\delta(x)$ remains nonnegative for every $0 \le x \le 1$ and $\delta \ge 1$, it follows directly from (12) and (27) that $RGI_\delta(Y_1) \le RGI_\delta(Y_2)$, whenever the dispersive ordering $Y_1 \le_{\text{dispo}_y} Y_2$ holds. Combining this observation with the implication in (28) immediately yields the following result.

Corollary 2.1. *Suppose that $Y_1 \le_{hro_y} Y_2$ and that at least one of the random variables Y_1 or Y_2 possesses the DFRT property. Then $RGI_\delta(Y_1) \le RGI_\delta(Y_2), \qquad \forall\, \delta \ge 1$.*

Let Z be a random variable having cumulative distribution function K_V. Its cumulative reversed hazard function is defined by

$$\chi_Z(z) = \int_0^z K_V(v)\, dv, \qquad z > 0. \tag{29}$$

A characterization of the location-independent riskier order was established by Landsberger and Meilijson [18], who proved that

$$Y_1 \le_{liro_y} Y_2 \Longleftrightarrow \chi_{Y_2}^{-1}(z) - \chi_{Y_1}^{-1}(z) \text{ is increasing for } z > 0, \tag{30}$$

where χ^{-1} denotes the left-continuous inverse associated with χ.

The next theorem applies these ordering concepts to compare generalized residual generating information measures. Specifically, it establishes that the quantity RGI_δ corresponding to a series system consisting of l components is bounded above by the RGI_δ of the concomitant associated with the lth d-generalized order statistic from the Huang–Kotz family, provided that the component lifetimes in both systems satisfy the DFRT condition.

Theorem 2.4. *Let $Y_{[l,n,d,m]}$ denote the lifetime of the concomitant associated with the lth d-generalized order statistic from the Huang–Kotz family, with common pdf $k_{Y_1}(x)$ and cdf $K_{Y_1}(x)$. If the baseline random variable Y_1 possesses the DFRT property, then*

$$RGI_\delta(X_{1:l}) \le RGI_\delta(Y_{[l,n,d,m]}), \qquad \forall\, \delta \ge 1.$$

Proof. The DFRT assumption on Y_1 implies that $X_{1:l}$ also enjoys the DFRT property. Furthermore, by invoking Theorem 4.5 of Eryılmaz and Navarro [7], it follows that $X_{1:l} \le_{hro_y} Y_{[l,n,d,m]}$. The desired inequality is then obtained directly from Corollary 2.1. □

The theorem below provides a sufficient condition under which the dispersive ordering is retained for concomitants of the lth d-generalized order statistic arising from the Huang–Kotz family. Since the argument follows immediately from previously established results, the proof is not presented.

Theorem 2.5. *Let $Y_{1,[l,n,d,m]}$ and $Y_{2,[l,n,d,m]}$ represent the lifetimes of two concomitants of the lth d-generalized order statistic from the Huang–Kotz family, having pdfs $k_{Y_1}(x)$ and $k_{Y_2}(x)$ and cdfs $K_{Y_1}(x)$ and $K_{Y_2}(x)$, respectively. If $Y_1 \leq_{\text{dispo}_y} Y_2$, then $RGI_\delta\big(Y_{1,[l,n,d,m]}\big) \leq RGI_\delta\big(Y_{2,[l,n,d,m]}\big)\,, \qquad \forall\,\delta \geq 0.$*

The following example demonstrates how Theorem 2.5 can be employed in practice.

Example 2.2. *Consider two systems whose lifetimes are given by the concomitants $Y_{[l,n,d,m]}$ and $Y^*_{[l,n,d,m]}$ of the lth d-generalized order statistic from the Huang–Kotz family. Assume that the components of the first system, $Y_1, Y_2, \ldots, Y_n$, are independent and identically distributed according to the Makeham distribution with survival function*

$$\bar{K}_Y(y) = \exp\big\{-y - a\big(y + e^{-y} - 1\big)\big\}\,, \qquad y > 0,\ a > 0.$$

For the second system, suppose that the component lifetimes $Y_1^*, Y_2^*, \ldots, Y_n^*$ are i.i.d. exponential random variables with survival function

$$\bar{K}_{Y^*}(y) = e^{-y}, \qquad y > 0.$$

The associated hazard rate functions are given by

$$\lambda_Y(y) = 1 + a\big(1 - e^{-y}\big), \qquad \lambda_{Y^*}(y) = 1.$$

Since

$$\lambda_Y(y) > \lambda_{Y^*}(y), \qquad y > 0,\ a > 0,$$

we conclude that $Y \leq_{hro_y} Y^*$. In addition, the exponential distribution is known to satisfy the DFRT condition. Hence, using relation (28), we deduce that $Y \leq_{\text{dispo}_y} Y^*$. An application of Theorem 2.5 therefore yields

$$RGI_\delta\big(Y_{[l,n,d,m]}\big) \leq RGI_\delta\big(Y_{[l,n,d,m]^*}\big)\,, \qquad \forall\,\delta \geq 1.$$

Accordingly, under the FGCRE criterion, the lifetime uncertainty of the system represented by $Y_{[l,n,d,m]}$ does not exceed that of the system represented by $Y^*_{[l,n,d,m]}$.

The following result establishes sufficient conditions under which the location-independent riskier order is retained by the concomitant of the lth d-generalized order statistic associated with the Huang–Kotz family.

Theorem 2.6. *In the setting of Theorem 2.5, let*

$$\phi_{[l,n,d,m;\theta]}(t) = (1-t) - \eta\,\Xi_{[l,n,d,m;\theta]}\,t\left(1 - t^{\theta}\right),\ 0 < t < 1. \tag{31}$$

If $Y_1 \leq_{liro_y} Y_2$ and

$$\frac{\Omega_\delta\big(\phi_{[l,n,d,m;\theta]}(t)\big)}{t},\ 0 \leq t \leq 1,$$

is a decreasing function of t for all $\delta \geq 1$, then $RGI_\delta\big(Y_{1,[l,n,d,m]}\big) \leq RGI_\delta\big(Y_{2,[l,n,d,m]}\big)$, $\forall\,\delta \geq 1$.

Proof. By rewriting (8), one obtains

$$\overline{K}_{Y,[l,n,d,m]}(y) = \phi_{[l,n,d,m;\theta]}\big(K_Y(y)\big).$$

Since $Y_1 \leq_{liro_y} Y_2$, it follows from (30) together with (29) that

$$\frac{d}{dz}\Big(\chi_{Y_2}^{-1}(z) - \chi_{Y_1}^{-1}(z)\Big) = K_{Y_2}\big(\chi_{Y_2}^{-1}(z)\big) - K_{Y_1}\big(\chi_{Y_1}^{-1}(z)\big) \geq 0.$$

Consequently, the mapping $\chi_{Y_2}^{-1}(z) - \chi_{Y_1}^{-1}(z)$ is nondecreasing, which yields

$$K_{Y_1}(z) \geq K_{Y_2}\big(\chi_{Y_2}^{-1}(\chi_{Y_1}(z))\big), \qquad z > 0. \tag{32}$$

Making use of (8), (15), and (26), we can write

$$\begin{aligned}
\int_0^\infty \big(\overline{K}_{Y_1,[l,n,d,m]}(y)\big)^\delta\, dy &= \int_0^\infty \Omega_\delta\big(\overline{K}_{Y_1,[l,n,d,m]}(y)\big)\, dy \\
&= \int_0^\infty \frac{\Omega_\delta\big(\phi_{[l,n,d,m;\theta]}(K_{Y_1}(y))\big)}{K_{Y_1}(y)}\, K_{Y_1}(y)\, dy \\
&\leq \int_0^\infty \frac{\Omega_\delta\Big(\phi_{[l,n,d,m;\theta]}\Big(K_{Y_2}\big(\chi_{Y_2}^{-1}(\chi_{Y_1}(y))\big)\Big)\Big)}{K_{Y_2}\big(\chi_{Y_2}^{-1}(\chi_{Y_1}(y))\big)}\, K_{Y_1}(y)\, dy.
\end{aligned} \tag{33}$$

The preceding inequality is a direct consequence of (32) together with the assumed monotonicity of $\Omega_\delta(\phi_{[l,n,d,m;\theta]}(t))/t$. Next, introduce the change of variable $u = \chi_{Y_2}^{-1}(\chi_{Y_1}(y))$. Differentiation gives

$$dy = \frac{K_{Y_2}(u)}{K_{Y_1}\big(\chi_{Y_1}^{-1}(\chi_{Y_2}(u))\big)}\, du.$$

Substituting this expression into the right-hand side of (33) leads to

$$\begin{aligned}
&\int_{\chi_{Y_2}^{-1}(\chi_{Y_1}(0))}^\infty \frac{\Omega_\delta\big(\phi_{[l,n,d,m;\theta]}(K_{Y_2}(u))\big)}{K_{Y_2}(u)} K_{Y_1}\big(\chi_{Y_1}^{-1}(\chi_{Y_2}(u))\big) \frac{K_{Y_2}(u)}{K_{Y_1}\big(\chi_{Y_1}^{-1}(\chi_{Y_2}(u))\big)}\, du \\
&= \int_{\chi_{Y_2}^{-1}(\chi_{Y_1}(0))}^\infty \Omega_\delta\big(\phi_{[l,n,d,m;\theta]}(K_{Y_2}(u))\big)\, du \\
&= \int_0^\infty \Omega_\delta\big(\overline{K}_{Y_2,[l,n,d,m]}(y)\big)\, dy \\
&= \int_0^\infty \big(\overline{K}_{Y_2,[l,n,d,m]}(y)\big)^\delta\, dy.
\end{aligned}$$

The second-to-last step follows from the representation

$$\overline{K}_{Y_2,[l,n,d,m]}(y) = \phi_{[l,n,d,m;\theta]}\big(K_{Y_2}(y)\big),$$

while the lower integration limit vanishes because $\chi_{Y_2}^{-1}(\chi_{Y_1}(0)) = 0$. Combining the above inequalities yields $RGI_\delta(Y_{1,[l,n,d,m]}) \leq RGI_\delta(Y_{2,[l,n,d,m]})$, for every $\delta \geq 1$. This completes the proof. $\square$

3. Applications involving nonparametric estimation

In this section, we develop two distribution-free estimators for RGI_δ associated with the concomitant $Y_{[l,n,d,m]}$. Consider a collection of independent and identically distributed nonnegative absolutely continuous random variables $Y_1, Y_2, \ldots, Y_N$ and denote their order statistics by $Y_{1:N} \leq Y_{2:N} \leq \cdots \leq Y_{N:N}$. Using (26) together with (31), the residual cumulative informational generating function of $Y_{[l,n,d,m]}$ may be expressed as

$$\begin{aligned} RGI_\delta(Y_{[l,n,d,m]}) &= \int_0^1 \frac{\Omega_\delta(\overline{K}_{Y,[l,n,d,m]}(u))}{k_Y(K_Y^{-1}(u))}\, du \\ &= \int_0^1 \Omega_\delta(\phi_{[l,n,d,m;\theta]}(u)) \left[\frac{d}{du} K_Y^{-1}(u)\right] du \\ &= \int_0^1 \left[(1-u) - \eta\, \Xi_{[l,n,d,m;\theta]} u \left(1 - u^\theta\right)\right]^\delta \left[\frac{d}{du} K_Y^{-1}(u)\right] du, \end{aligned} \tag{34}$$

where $\delta \geq 1$ and $\theta \geq 1$. Expression (34) serves as the basis for constructing the estimators of $\mathrm{RGI}_\delta(Y_{[l,n,d,m]})$.

The first estimator relies on approximating the derivative of the quantile function at the observed sample points. Following the idea proposed by Vasicek [24], the derivative $\frac{d}{du} K_Y^{-1}(u)$ is approximated by the finite difference

$$\frac{d}{du} K_Y^{-1}(u) \approx \frac{N\left(Y_{i+r:N} - Y_{i-r:N}\right)}{2r},$$

where the boundary values are taken as $Y_{i:N} = Y_{1:N}$ whenever $i < 1$ and $Y_{i:N} = Y_{N:N}$ whenever $i > N$. Here, N denotes the sample size and r is a positive integer satisfying $r \leq N/2$, commonly referred to as the window length. Consequently, an estimator of $RGI_\delta(Y_{[l,n,d,m]})$ is given by

$$\begin{aligned} \widehat{RGI}_{\delta,1}(Y_{[l,n,d,m]}) &= \frac{1}{N}\sum_{i=1}^N \Omega_\delta\left(\phi_{[l,n,d,m;\theta]}\left(\frac{i}{N+1}\right)\right)\left(\frac{N\left(Y_{i+r:N} - Y_{i-r:N}\right)}{2r}\right) \\ &= \frac{1}{N}\sum_{i=1}^N \left[\left(1 - \left(\frac{i}{N+1}\right)\right) - \eta\, \Xi_{[l,n,d,m;\theta]}\left(\frac{i}{N+1}\right)\left(1 - \left(\frac{i}{N+1}\right)^\theta\right)\right]^\delta \\ &\quad \times \left(\frac{N\left(Y_{i+r:N} - Y_{i-r:N}\right)}{2r}\right). \end{aligned} \tag{35}$$

An alternative estimator can be obtained by employing the empirical survival function associated with $\overline{K}_Y(y)$, namely,

$$\overline{K}_{Y_N}(y) = \sum_{i=1}^{N-1} \frac{N-i}{N}\, I_{[Y_{i:N},\, Y_{i+1:N})}(y), \qquad y \geq 0,$$

where

$$I_A(y) = \begin{cases} 1, & y \in A, \\ 0, & y \notin A. \end{cases}$$

By virtue of (15), the corresponding empirical residual cumulative informational generating function for $Y_{[l,n,d,m]}$ takes the form

$$\begin{aligned} \widehat{RGI}_{\delta,2}(Y_{[l,n,d,m]}) &= \int_0^{\infty} \left(\overline{K}_{Y,[l,n,d,m]}(y) \right)^{\delta} dy \\ &= \sum_{i=1}^{N-1} \int_{Y_{i:N}}^{Y_{i+1:N}} \Omega_\delta \left(\phi_{[l,n,d,m;\theta]} \left(\frac{i}{N+1} \right) \right) dy \qquad (36) \\ &= \sum_{i=1}^{N-1} \left[\left(1 - \left(\frac{i}{N+1} \right) \right) - \eta\, \Xi_{[l,n,d,m;\theta]} \left(\frac{i}{N+1} \right) \left(1 - \left(\frac{i}{N+1} \right)^{\theta} \right) \right]^{\delta} S_{i+1}, \end{aligned}$$

where

$$S_{i+1} = Y_{i+1:N} - Y_{i:N}, \qquad i = 1, 2, \ldots, N-1,$$

represents the successive sample spacings.

We use a simulation study using the standard exponential distribution data to examine the finite-sample behavior of the suggested estimators $\widehat{RGI}_{\delta,1}(Y_{[l,n,d,m]})$ and $\widehat{RGI}_{\delta,2}(Y_{[l,n,d,m]})$, defined in (35) and (36), respectively. We assess the bias, root mean square error (RM), and standard deviation (SD) of both estimators for various sample sizes ($N = 50, 100, 150$), different values of r and δ, based on the concomitant of the 20th order statistics, $Y_{[20,60,0,1]}$. Tables 2 and 3 shows the obtained results of the estimators $\widehat{RGI}_{\delta,1}(Y_{[l,n,d,m]})$ and $\widehat{RGI}_{\delta,2}(Y_{[l,n,d,m]})$, respectively. We can conclude that:

(1) The bias, RM, and SD of the two estimators decrease as sample size N increases for all values of r and δ.

(2) The bias, RM, and SD of the two estimators increases as the values of r and δ increases, for fixed sample size N.

Overall, we can see that the values of r, δ, and sample size N, affect the efficiency of the estimators.

3.1. **Real data application.** To illustrate the practical applicability of the proposed methodology, we consider a real dataset representing electricity sold to the commercial consumption sector across the administrative regions of Saudi Arabia during 2024, measured in gigawatt-hours (GWh). The data are publicly available from the Saudi General Authority for Statistics and can be accessed at `https://www.stats.gov.sa/en/statistics-tabs?tab=436312&category=124767` (accessed on 23 June 2026).

TABLE 2. The estimator $\widehat{RGI}_{\delta,1}(Y_{[20,60,0,1]})$ with standard exponential distribution component lifetimes for various values of r and δ with $\theta = 2$ and $\eta = 0.5$ yielded bias, RM, and SD results.

r	δ	$N = 50$			$N = 100$			$N = 150$		
		Bias	RM	SD	Bias	RM	SD	Bias	RM	SD
	2	-0.0366703	0.0703988	0.0601240	-0.0203250	0.0485933	0.0441606	-0.0146286	0.0400485	0.0372998
2	3	-0.0307124	0.0549918	0.0456391	-0.0158935	0.0381589	0.0347088	-0.0112538	0.0311468	0.0290572
	4	-0.0289510	0.0473820	0.0375274	-0.0162060	0.0328681	0.0286094	-0.0103025	0.0253796	0.0232060
	2	-0.0527871	0.0789744	0.0587702	-0.0267727	0.0500548	0.0423143	-0.0203875	0.0401593	0.0346167
3	3	-0.0399170	0.0583415	0.0425697	-0.0235654	0.0401515	0.0325249	-0.0148806	0.0320819	0.0284364
	4	-0.0343166	0.0496290	0.0358706	-0.0190908	0.0333232	0.0273263	-0.0126322	0.0267357	0.0235750
	2	-0.0612239	0.0809187	0.0529364	-0.0323852	0.0540642	0.0433130	-0.0233521	0.0428559	0.0359528
4	3	-0.0469049	0.0617018	0.0401080	-0.0237472	0.0408933	0.0333082	-0.0186290	0.0337949	0.0282109
	4	-0.0410929	0.0524815	0.0326612	-0.0228808	0.0353992	0.0270240	-0.0162250	0.0278767	0.0226798
	2	-0.0715766	0.0880628	0.0513273	-0.0401108	0.0575771	0.0413274	-0.0265742	0.0436982	0.0347066
5	3	-0.0546097	0.0674265	0.0395686	-0.0320207	0.0446307	0.0311054	-0.0209185	0.0346258	0.0276065
	4	-0.0461463	0.0556738	0.0311619	-0.0262280	0.0373869	0.0266567	-0.0180663	0.0296824	0.0235628
	2	-0.0792232	0.0937498	0.0501519	-0.0459828	0.0615061	0.0408688	-0.0314576	0.0473060	0.0353487
6	3	-0.0606581	0.0712881	0.0374698	-0.0357977	0.0473990	0.0310830	-0.0242813	0.0366251	0.0274330
	4	-0.0525286	0.0603943	0.0298180	-0.0310100	0.0396391	0.0247032	-0.0210406	0.0303697	0.0219110
	2	-0.0899564	0.1016683	0.0473976	-0.0526636	0.0671120	0.0416209	-0.0356898	0.0497407	0.0346637
7	3	-0.0676473	0.0767279	0.0362261	-0.0386507	0.0491284	0.0303421	-0.0271606	0.0384908	0.0272871
	4	-0.0577596	0.0644022	0.0285004	-0.0332810	0.0415829	0.0249426	-0.0229554	0.0318331	0.0220654
	2	-0.0988907	0.1096603	0.0474158	-0.0563876	0.0685314	0.0389681	-0.0398771	0.0526821	0.0344442
8	3	-0.0738612	0.0805790	0.0322263	-0.0422312	0.0519532	0.0302751	-0.0306237	0.0400762	0.0258642
	4	-0.0607125	0.0663278	0.0267223	-0.0362192	0.0434967	0.0240980	-0.0251975	0.0331680	0.0215794
	2	-0.1093320	0.1174474	0.0429215	-0.0629430	0.0738878	0.0387179	-0.0445455	0.0555390	0.0331870
9	3	-0.0785123	0.0841520	0.0303035	-0.0458117	0.0539007	0.0284145	-0.0345229	0.0425521	0.0248892
	4	-0.0648097	0.0696859	0.0256219	-0.0390672	0.0458954	0.0240981	-0.0283242	0.0357308	0.0217923
	2	-0.1150197	0.1230477	0.0437393	-0.0686909	0.0790935	0.0392287	-0.0488816	0.0584107	0.0319910
10	3	-0.0848636	0.0902342	0.0306809	-0.0495829	0.0579143	0.0299415	-0.0364288	0.0440835	0.0248377
	4	-0.0675074	0.0717124	0.0242076	-0.0405958	0.0465665	0.0228241	-0.0299763	0.0363198	0.0205177
	2	-0.1222148	0.1293741	0.0424619	-0.0728162	0.0824488	0.0386923	-0.0525465	0.0615005	0.0319719
11	3	-0.0878528	0.0928122	0.0299481	-0.0547714	0.0610753	0.0270373	-0.0379291	0.0454774	0.0251039
	4	-0.0711110	0.0749888	0.0238140	-0.0449628	0.0496329	0.0210289	-0.0316212	0.0374837	0.0201378
	2	-0.1289132	0.1345614	0.0385959	-0.0789567	0.0869631	0.0364657	-0.0556899	0.0639067	0.0313637
12	3	-0.0927866	0.0969505	0.0281216	-0.0567601	0.0625786	0.0263641	-0.0413942	0.0481135	0.0245363
	4	-0.0755284	0.0784865	0.0213554	-0.0474542	0.0517631	0.0206867	-0.0343297	0.0396223	0.0197937
	2	-0.1343871	0.1394762	0.0373510	-0.0818812	0.0892014	0.0354062	-0.0597101	0.0673033	0.0310710
13	3	-0.0961630	0.0997014	0.0263392	-0.0608062	0.0664935	0.0269205	-0.0446513	0.0509253	0.0244998
	4	-0.0780972	0.0807823	0.0206649	-0.0501758	0.0544469	0.0211495	-0.0362349	0.0413402	0.0199108
	2	-0.1411352	0.1457645	0.0364618	-0.0868964	0.0931910	0.0336854	-0.0632766	0.0709126	0.0320264
14	3	-0.1009195	0.1038002	0.0242966	-0.0634426	0.0683472	0.0254365	-0.0456024	0.0513762	0.0236748
	4	-0.0799065	0.0823046	0.0197332	-0.0513088	0.0551282	0.0201724	-0.0378627	0.0423483	0.0189778
	2	-0.1445624	0.1485270	0.0341048	-0.0910889	0.0970653	0.0335502	-0.0662402	0.0725238	0.0295433
15	3	-0.1036803	0.1067267	0.0253304	-0.0663257	0.0708942	0.0250504	-0.0503863	0.0551495	0.0224318
	4	-0.0832701	0.0853713	0.0188335	-0.0547913	0.0583052	0.0199451	-0.0403653	0.0442059	0.0180313
	2	-0.1518981	0.1556124	0.0338134	-0.0953724	0.1012333	0.0339626	-0.0698338	0.0758962	0.0297382
16	3	-0.1058364	0.1084895	0.0238578	-0.0698055	0.0736670	0.0235495	-0.0518877	0.0567872	0.0230864
	4	-0.0850338	0.0868428	0.0176419	-0.0567825	0.0599230	0.0191540	-0.0426387	0.0464651	0.0184741
	2	-0.1568182	0.1600112	0.0318219	-0.0998636	0.1047417	0.0316084	-0.0749414	0.0802666	0.0287636
17	3	-0.1090133	0.1113638	0.0227711	-0.0720766	0.0757663	0.0233675	-0.0534114	0.0581923	0.0231105
	4	-0.0869727	0.0886969	0.0174124	-0.0588925	0.0617138	0.0184556	-0.0447610	0.0484780	0.0186255
	2	-0.1614549	0.1642580	0.0302311	-0.1048581	0.1094506	0.0313880	-0.0779322	0.0831883	0.0291154
18	3	-0.1126241	0.1147465	0.0219784	-0.0753558	0.0790017	0.0237346	-0.0554158	0.0596563	0.0221010
	4	-0.0889880	0.0904462	0.0161838	-0.0606636	0.0633454	0.0182453	-0.0453343	0.0487635	0.0179723
	2	-0.1645574	0.1674232	0.0308600	-0.1081825	0.1123041	0.0301606	-0.0816361	0.0864191	0.0283658
19	3	-0.1153092	0.1171596	0.0207507	-0.0774545	0.0806427	0.0224620	-0.0585234	0.0622528	0.0212336
	4	-0.0906021	0.0919891	0.0159218	-0.0623477	0.0648679	0.0179146	-0.0468169	0.0501510	0.0179895
	2	-0.1671735	0.1697822	0.0296631	-0.1123058	0.1163722	0.0305093	-0.0817216	0.0862857	0.0277050
20	3	-0.1192740	0.1210549	0.0206983	-0.0791543	0.0822237	0.0222673	-0.0612332	0.0647589	0.0210868
	4	-0.0930788	0.0943190	0.0152526	-0.0654392	0.0674710	0.0164415	-0.0499127	0.0524798	0.0162209

TABLE 3. The estimator $\widehat{RGI}_{\delta,2}(Y_{[20,60,0,1]})$ with standard exponential distribution component lifetimes for various values of r and δ with $\theta = 2$ and $\eta = 0.5$ yielded bias, RM, and SD results.

r	δ	$N=50$			$N=100$			$N=150$		
		Bias	RM	SD	Bias	RM	SD	Bias	RM	SD
	2	-0.0728344	0.0913109	0.0550987	-0.0395752	0.0578718	0.0422461	-0.0278047	0.0456403	0.0362111
2	3	-0.0596441	0.0722086	0.0407225	-0.0318635	0.0457792	0.0328867	-0.0219731	0.0355965	0.0280193
	4	-0.0543807	0.0633180	0.0324494	-0.0303687	0.0404632	0.0267532	-0.0201837	0.0299275	0.0221080
	2	-0.106426	0.118218	0.0514956	-0.0568311	0.0692572	0.0396025	-0.0411804	0.0529033	0.0332272
3	3	-0.0831189	0.0904682	0.0357356	-0.0478791	0.0563608	0.0297492	-0.0320217	0.0417010	0.0267264
	4	-0.0727989	0.0781162	0.0283419	-0.0411041	0.0478018	0.0244144	-0.0281234	0.0355928	0.0218266
	2	-0.131748	0.138767	0.0435960	-0.0729857	0.0829274	0.0393902	-0.0515844	0.0616362	0.0337523
4	3	-0.102766	0.107406	0.0312439	-0.0567205	0.0638235	0.0292760	-0.0415834	0.0490654	0.0260558
	4	-0.0893363	0.0924123	0.0236565	-0.0519133	0.0568860	0.0232715	-0.0369564	0.0422026	0.0203887
	2	-0.156373	0.161228	0.0392897	-0.0897484	0.0970509	0.0369522	-0.0619315	0.0696751	0.0319394
5	3	-0.121164	0.124405	0.0282252	-0.0719182	0.0767034	0.0266816	-0.0495722	0.0554351	0.0248248
	4	-0.103424	0.105498	0.0208286	-0.0618850	0.0656277	0.0218566	-0.0437293	0.0483273	0.0205841
	2	-0.176895	0.180714	0.0369758	-0.104901	0.110675	0.0352973	-0.0732255	0.0799638	0.0321446
6	3	-0.136811	0.139121	0.0252598	-0.0826221	0.0865391	0.0257539	-0.0583291	0.0631463	0.0242022
	4	-0.116396	0.117774	0.0179709	-0.0723073	0.0748112	0.0192026	-0.0512017	0.0544930	0.0186605
	2	-0.197895	0.200555	0.0325726	-0.119408	0.124353	0.0347356	-0.0841588	0.0895802	0.0307059
7	3	-0.151096	0.152662	0.0218188	-0.0927584	0.0958634	0.0242127	-0.0664161	0.0704055	0.0233749
	4	-0.126904	0.127804	0.0151474	-0.0801607	0.0823000	0.0186522	-0.0576649	0.0604944	0.0182942
	2	-0.215356	0.217659	0.0315972	-0.131020	0.134726	0.0313941	-0.0941350	0.0988650	0.0302294
8	3	-0.163156	0.164190	0.0184094	-0.101979	0.104576	0.0231715	-0.0745918	0.0776817	0.0217020
	4	-0.135162	0.135782	0.0129629	-0.0883311	0.0899684	0.0170944	-0.0643265	0.0666232	0.0173510
	2	-0.232918	0.234381	0.0261593	-0.144361	0.147675	0.0311238	-0.104855	0.108561	0.0281374
9	3	-0.173588	0.174354	0.0163330	-0.111463	0.113392	0.0208359	-0.0827492	0.0852389	0.0204611
	4	-0.142767	0.143170	0.0107353	-0.0956226	0.0969830	0.0161956	-0.0710325	0.0730034	0.0168573
	2	-0.246663	0.247823	0.0239689	-0.156760	0.159767	0.0308641	-0.114580	0.117761	0.0271974
10	3	-0.183373	0.183954	0.0146190	-0.120254	0.122120	0.0212740	-0.0891772	0.0914047	0.0200661
	4	-0.148995	0.149292	0.00942665	-0.101778	0.102864	0.0149165	-0.0765389	0.0781361	0.0157255
	2	-0.258973	0.259988	0.0229654	-0.167479	0.169931	0.0287774	-0.123296	0.126155	0.0267170
11	3	-0.191432	0.191851	0.0126913	-0.129406	0.130704	0.0183872	-0.0953256	0.0972677	0.0193493
	4	-0.154417	0.154612	0.00776139	-0.109281	0.110018	0.0127173	-0.0819135	0.0831898	0.0145231
	2	-0.270718	0.271378	0.0189294	-0.179271	0.181203	0.0264035	-0.131846	0.134291	0.0255184
12	3	-0.198448	0.198747	0.0109066	-0.136372	0.137462	0.0172885	-0.102300	0.104023	0.0188612
	4	-0.158984	0.159108	0.00629013	-0.115125	0.115763	0.0121491	-0.0874323	0.0885831	0.0142396
	2	-0.280363	0.280928	0.0178191	-0.188004	0.189660	0.0250185	-0.140288	0.142464	0.0248185
13	3	-0.204392	0.204597	0.00915343	-0.144068	0.145040	0.0167772	-0.109278	0.110740	0.0179430
	4	-0.163125	0.163208	0.00522102	-0.120436	0.121014	0.0118247	-0.0925299	0.0935368	0.0136941
	2	-0.289736	0.290159	0.0156770	-0.197725	0.199074	0.0231475	-0.148477	0.150586	0.0251284
14	3	-0.210086	0.210229	0.00776678	-0.150856	0.151631	0.0153224	-0.113909	0.115224	0.0173712
	4	-0.166273	0.166330	0.00433859	-0.125309	0.125772	0.0107864	-0.0972025	0.0980666	0.0129959
	2	-0.297644	0.297969	0.0139125	-0.206901	0.208082	0.0221528	-0.155830	0.157498	0.0228713
15	3	-0.214274	0.214388	0.00697171	-0.156544	0.157226	0.0146342	-0.121665	0.122656	0.0155686
	4	-0.168914	0.168952	0.00356867	-0.130357	0.130750	0.0101353	-0.102238	0.102895	0.0116218
	2	-0.305939	0.306182	0.0121880	-0.215596	0.216663	0.0214871	-0.163629	0.165155	0.0224135
16	3	-0.218408	0.218495	0.00615307	-0.163077	0.163600	0.0130774	-0.126541	0.127516	0.0157478
	4	-0.171138	0.171163	0.00296790	-0.134364	0.134666	0.00902952	-0.106828	0.107445	0.0115007
	2	-0.311955	0.312139	0.0107090	-0.223762	0.224671	0.0202013	-0.172197	0.173462	0.0209199
17	3	-0.221588	0.221647	0.00508629	-0.168152	0.168625	0.0126305	-0.131515	0.132428	0.0155328
	4	-0.173123	0.173140	0.00244666	-0.138364	0.138618	0.00838646	-0.110913	0.111495	0.0113815
	2	-0.318260	0.318395	0.00929444	-0.232238	0.233024	0.0191340	-0.179664	0.180880	0.0209436
18	3	-0.224688	0.224729	0.00426817	-0.173471	0.173893	0.0121047	-0.136074	0.136853	0.0145896
	4	-0.174580	0.174591	0.00195701	-0.142284	0.142488	0.00762029	-0.114364	0.114842	0.0104695
	2	-0.323236	0.323352	0.00867489	-0.239639	0.240327	0.0181926	-0.186443	0.187533	0.0202007
19	3	-0.227193	0.227223	0.00364257	-0.178502	0.178846	0.0110993	-0.141703	0.142325	0.0132983
	4	-0.175842	0.175849	0.00158459	-0.145410	0.145585	0.00714403	-0.117895	0.118337	0.0102171
	2	-0.327051	0.327150	0.00802516	-0.246321	0.246953	0.0176551	-0.190802	0.191746	0.0190151
20	3	-0.229484	0.229505	0.00311865	-0.182416	0.182706	0.0102924	-0.146633	0.147209	0.0130180
	4	-0.176888	0.176893	0.00130293	-0.148889	0.149017	0.00616565	-0.122230	0.122558	0.00896609

The observed data were modeled using the Weibull distribution, whose probability density function is given by

$$k_Y(y) = \frac{\gamma_1}{\gamma_2}\left(\frac{y}{\gamma_2}\right)^{\gamma_1-1} \exp\left[-\left(\frac{y}{\gamma_2}\right)^{\gamma_1}\right], \qquad y > 0, \gamma_1, \gamma_2 > 0,$$

where γ_1 and γ_2 denote the shape and scale parameters, respectively.

The unknown parameters were estimated using the maximum likelihood method. The resulting estimates were $\hat{\gamma_1} = 0.98366$ and $\hat{\gamma_2} = 2861.585$. To assess the adequacy of the fitted model, several goodness-of-fit tests were conducted. The obtained results are summarized as follows:

(1) Kolmogorov–Smirnov statistic = 0.22164 with p-value = 0.4785;
(2) Anderson–Darling statistic = 0.54579 with p-value = 0.6973;
(3) Cram'er–von Mises statistic = 0.0941 with p-value = 0.622.

The relatively large p-values associated with these tests indicate that there is no evidence against the Weibull assumption, suggesting that the fitted model provides an adequate representation of the observed data. Furthermore, the graphical diagnostics displayed in Figure 3 demonstrate a satisfactory agreement between the fitted Weibull distribution and the empirical observations, with only slight discrepancies in the tail regions.

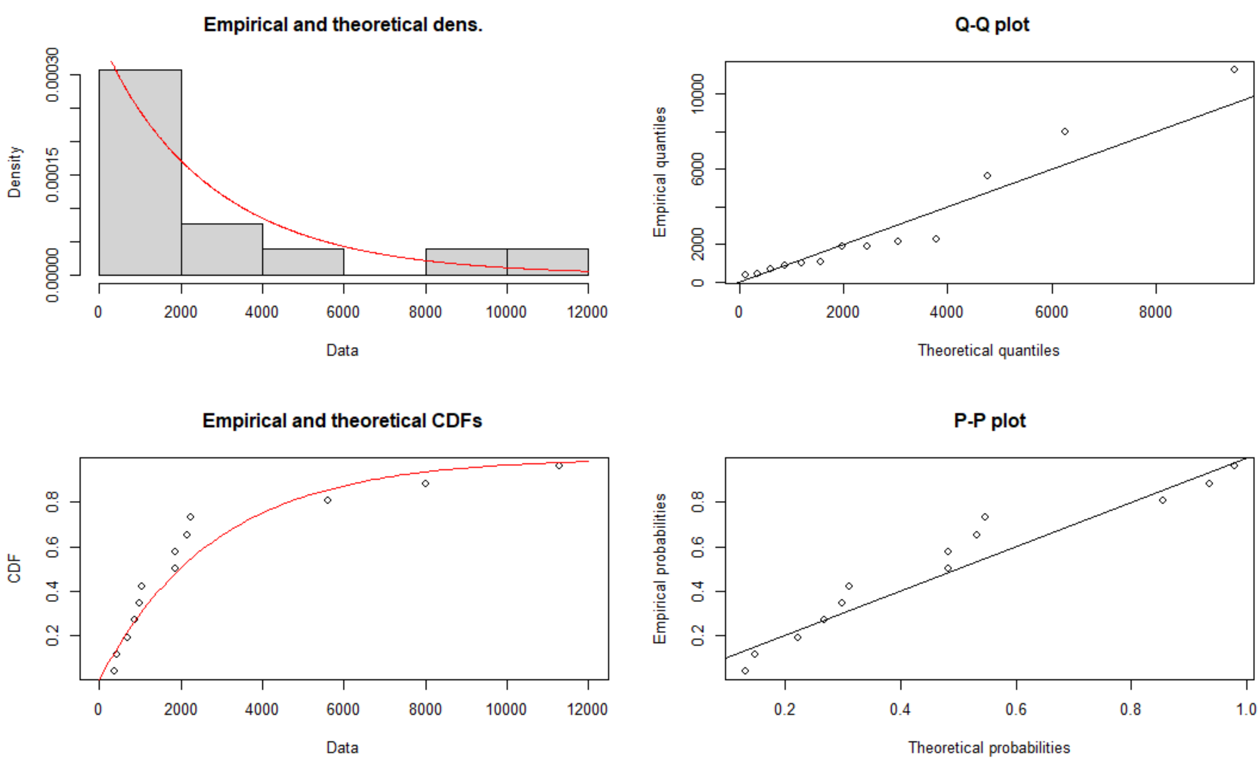


FIGURE 3. Diagnostic graphs for the fitted Weibull distribution: density plot, Q–Q plot, CDF plot, and P–P plot.

Based on the concomitant of the 20th order statistic, $Y_{[20,60,0,1]}$, Figure 4 illustrates the absolute biases associated with the theoretical value and the proposed estimators $\widehat{RGI}_{\delta,1}(Y_{[20,60,0,1]})$ and $\widehat{RGI}_{\delta,2}(Y_{[20,60,0,1]})$, for $\theta = 4$ and $\eta = 0.25$. The figure highlights the influence of the parameters δ and r on the estimation accuracy. Specifically, larger values of δ lead to smaller absolute biases, whereas increasing r results in larger absolute biases.

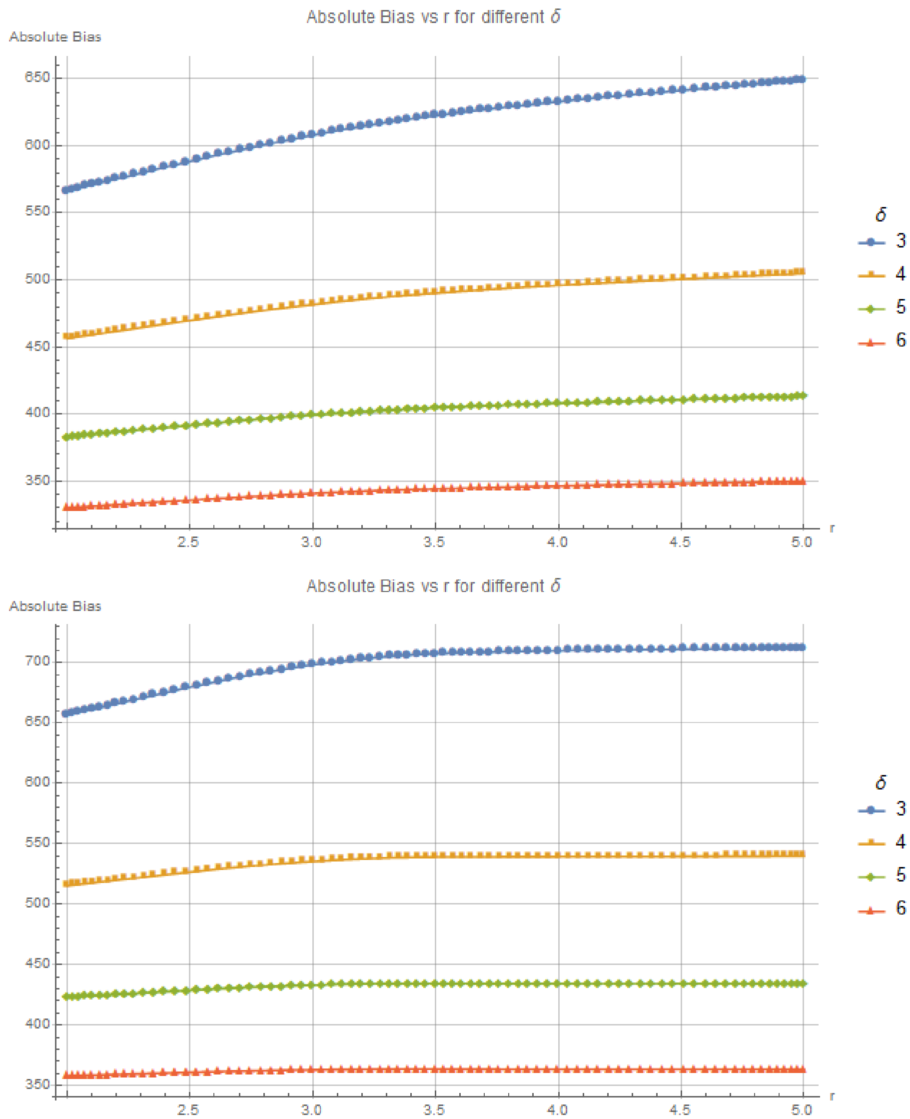


FIGURE 4. The absolute bias of the theoretical value and the proposed estimators $\widehat{RGI}_{\delta,1}(Y_{[6,20,0,1]})$ (up) and $\widehat{RGI}_{\delta,1}(Y_{[6,20,0,1]})$ (dawn), with $\theta = 4$ and $\eta = 0.25$.

Conclusion

In this paper, the residual cumulative informational generating function associated with the concomitant of d-generalized order statistics from the Huang–Kotz family has been investigated under the conditions $\delta \geq 1$ and $\theta \geq 1$. Several analytical properties of the proposed measure were established, including series expansions and illustrative examples for some lifetime distributions. In addition, bounds and stochastic comparisons based on classical stochastic orderings were derived, providing further insight into the behavior of the proposed measure.

The obtained results reveal that the residual cumulative informational generating function of the concomitant corresponding to d-generalized order statistics is smaller than that of the original random variable, indicating a reduction in uncertainty after conditioning on the associated order structure. Furthermore, two nonparametric estimation procedures were developed and studied through extensive simulation experiments and real data applications. The numerical investigations demonstrated that the performance of the estimators is influenced by the values of r, δ, and the sample size N. In particular, larger sample sizes generally improve estimation accuracy, whereas the parameters r and δ play a significant role in determining the efficiency of the proposed estimators.

Overall, the theoretical developments and numerical studies confirm the applicability of the proposed residual cumulative informational generating function as a useful tool for analyzing uncertainty in systems involving concomitants of d-generalized order statistics. The methodology introduced in this work may be extended to other distributional families and alternative estimation techniques, which constitute promising directions for future research.

Author Contributions. Conceptualization, Formal Analysis, Investigation, Methodology, Software, Visualization, Resources, Writing—Original Draft, Validation, and Writing—Review & Editing: H.H.S; M.S.M.

Data Availability. The dataset analyzed during this study is publicly available from the source cited in the Real Data Application subsection.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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