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CONNECTED GEODETIC DOMINATION IN GRAPHS UNDER SOME BINARY OPERATIONS

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Abstract. Let G be a simple connected graph. A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$, and a *geodetic set* of G if $I_G[S] = V(G)$. A set S is a *geodetic dominating set* of G if it is both a geodetic set and a dominating set. A *connected geodetic dominating set* of G is a geodetic dominating set S such that the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected geodetic dominating set of G is called the *connected geodetic domination number* of G and is denoted by $\gamma_{cg}(G)$. This paper characterizes connected geodetic dominating sets of some graphs and in graphs obtained from the join, corona, and edge corona of two graphs. Exact values or bounds for the corresponding connected geodetic domination numbers are also established.

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1. Introduction

The concept of domination in graphs was formally introduced by Ore in 1962 [10]; see also [8]. The study of distance in graphs was systematized by Buckley and Harary [1], while the geodetic number of a graph was studied by Harary, Loukakis, and Tsouros [7]. Also, Sampathkumar and Walikar published their research about the connected domination number of a graph in 1979 [11]. In 2009, Santhakumaran, Titus, and John published their study on the connected geodetic number of a graph [12]. On the other hand, the concept of geodetic domination was introduced by Hansberg and Volkmann in their 2011 paper entitled geodetic domination in graphs [5], combining the concept of geodetic and domination in graphs where they presented some results regarding the different

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bounds of the geodetic domination number of some families of graphs and its relationship in other parameters. Moreover, in 2017, Stalin et.al [14] published further study in edge geodetic domination in graphs where they investigated those subsets of vertices of graphs that are both edge geodetic and a dominating set. The concept of geodetic domination has branched out into several connected variations. For instance, Tejaswini, Goudar, and Venkatesha later published the geodetic connected domination number of a graph in 2014 [16]. Also, in 2021, Xaviour et.al [17] published their study in the connected geodetic global domination number of a graph where they extended the requirement of a geodetic and globally dominating set to induce a connected subgraph. Motivated by these developments, this paper studies the concept of connected geodetic domination in graphs that provides a mathematical framework for wireless sensor networks. Moreover, we characterized connected geodetic dominating sets of some graphs and graphs resulting from the join, corona, and edge corona of two graphs, and determine values or bounds of the corresponding connected geodetic domination number.

2. Preliminaries

Throughout this paper, all graphs considered are finite, simple, and undirected. For a graph G, the vertex-set and edge-set of G are denoted by $V(G)$ and $E(G)$, respectively. Let $S \subseteq V(G)$. The *induced subgraph* $\langle S \rangle$ of G is the graph with vertex-set S and edge-set $E(\langle S \rangle) = \{xy \in E(G) : x, y \in S\}$. Two vertices u and v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A vertex x of a graph G is called a *cut-vertex* if the removal of x increases the number of components of the graph G. A vertex v in a graph G is an *extreme vertex* if the subgraph induced by its neighborhood is complete. [6]

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \,\dot{\cup}\, V(H)$ and edge-set $E(G + H) = E(G) \,\dot{\cup}\, E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ [6]. The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the vertex v of G to every vertex of the copy of H. For every $v \in V(G)$, denoted by H^v, is the copy of H whose vertices are joined or attached to the vertex v. Denoted by $v + H^v$, is the subgraph of the corona $G \circ H$ corresponding the join $\langle \{v\} \rangle + H^v$ [4]. The *edge corona* $G \diamond H$ of graphs G and H, is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and then joining each of the end vertices u and v of each edge $uv \in E(G)$ to every vertex of the copy H^{uv} of H [9].

For $u, v \in V(G)$, a shortest path joining u and v is called a u-v *geodesic*. The set $I_G(u, v)$ refers to the set consisting of all vertices lying on any u-v geodesic of G. For $S \subseteq V(G)$, a 2-*path closure* $P_2[S]_G$ of a set S is the set $P_2[S]_G = S \cup \{w \in V(G) : w \in I_G(u, v)$ with $d_G(u, v) = 2$ for some $u, v \in S\}$. A set is called a 2-*path closure absorbing set* of G if $P_2[S]_G = V(G)$ [3]. A 2-path closure absorbing set S of G is a *connected* 2-*path closure absorbing set* of G if $\langle S\rangle$ is connected. The minimum cardinality of a 2-path closure absorbing set (resp. connected 2-path closure absorbing set) of G is called the 2-*path closure absorbing number* (resp. *connected* 2-*path closure absorbing number*) of G and is denoted by $\rho_2(G)$ (resp. ${\rho_2}^c(G)$). A 2-path closure absorbing set of G with cardinality $\rho_2(G)$ (resp. ${\rho_2}^c(G)$) is called a ρ_2-set (resp. ${\rho_2}^c$-set) of G. A set $D \subseteq V(G)$ is a *vertex cover* of G if, for each $uv \in E(G)$, either $u \in D$ or $v \in D$. A vertex cover D of G is a *connected vertex cover* if $\langle D\rangle$ is connected [13]. A connected vertex cover of a graph is a *connected geodetic vertex cover* if it is also a geodetic set. The minimum cardinality of a connected vertex cover (resp. connected geodetic vertex cover) of G, denoted by $\beta_c(G)$ (resp. $\beta_{cg}(G)$), is called the *connected vertex cover number* (resp. *connected geodetic vertex cover number*) of G. Any connected vertex cover (resp. connected geodetic vertex cover) of G with cardinality $\beta_c(G)$ (resp. $\beta_{cg}(G)$) is called a β_c-set (resp. β_{cg}-set) of G.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$ [8]. A dominating set S of G is said to be a *connected dominating set* if the subgraph $\langle S\rangle$ induced by S is connected [11]. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. For vertices $u, v \in V(G)$, $I_G[u, v]$ is the closed interval consisting of u, v, and all vertices lying on some u-v geodesic. A set $S \subseteq V(G)$ is a *geodetic set* of G if $I_G[S] = V(G)$, where $I_G[S] = \bigcup_{u,v\in S} I_G[u, v]$. The *geodetic number* of G, denoted by $g(G)$, is the smallest cardinality of a geodetic set of G [7]. A geodetic set S of G is called a *connected geodetic set* of G if the subgraph $\langle S\rangle$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is called the *connected geodetic number* of G and is denoted by $g_c(G)$ [12]. A set $S \subseteq V(G)$ is a *geodetic dominating set* of G if S is both a geodetic set and a dominating set of G. The minimum cardinality of a geodetic dominating set of G is called the *geodetic domination number* of G and is denoted by $\gamma_g(G)$ [5]. A *connected geodetic dominating set* $S \subseteq V(G)$ is a geodetic dominating set of G such that the subgraph $\langle S\rangle$ induced by S is connected. The minimum cardinality of a connected geodetic dominating set of G is called the *connected geodetic domination number* of G and is denoted by $\gamma_{cg}(G)$. A connected geodetic dominating set with cardinality $\gamma_{cg}(G)$ is called a γ_{cg}-set of G.

Remark 2.1. [5] If G is a connected graph of order $n \geq 2$, then
$2 \leq max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n$.

Theorem 2.2. [5] *Let G be a connected graph of order $n \geq 2$. Then:*

(*i*) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *of* G *such that* $d(u, v) \leq 3$,

(*ii*) $\gamma_g(G) = n$ *if and only if* G *is a complete graph of* n *vertices,*

Proposition 2.3. [16] *If* G *is a connected graph of order* $n \geq 2$, *then* $2 \leq max\{g(G), \gamma(G)\} \leq g_{\gamma c}(G) \leq n$.

Theorem 2.4. [12] *Every extreme vertex of a connected graph* G *belongs to every connected geodetic set of* G. *In particular, every end vertex of* G *belongs to every connected geodetic set of* G.

Theorem 2.5. [12] *Every cut vertex of a connected graph* G *belongs to every connected geodetic set of* G.

Theorem 2.6. [12] *Let* G *be a connected graph of order* p. *Then every vertex of* G *is either a cut vertex or an extreme vertex if and only if* $g_c(G) = p$.

3. Main Results

Given a graph G, we denote by $C(G)$ and $Ext(G)$, the sets of cut and extreme vertices of G, respectively.

Remark 3.1. For a connected graph G, $\max\{\beta_c(G), \gamma_{cg}(G)\} \leq \beta_{cg}(G)$.

Proposition 3.2. *Let* G *be a connected graph of order* $n \geq 2$. *Then*

$$2 \leq \max\{\gamma_c(G), \gamma_g(G)\} \leq \gamma_{cg}(G) \leq n.$$

Proof. Since $n \geq 2$, every geodetic set of G has cardinality at least 2. Since every geodetic dominating set is a geodetic set, it follows that $\gamma_g(G) \geq 2$. Hence, $2 \leq \max\{\gamma_c(G), \gamma_g(G)\}$. Next, let S be a γ_{cg}-set of G. Since every connected geodetic dominating set is both a connected dominating set and a geodetic dominating set, it follows that S is a connected dominating set and a geodetic dominating set of G. Hence, $\gamma_c(G) \leq |S| = \gamma_{cg}(G)$ and $\gamma_g(G) \leq |S| = \gamma_{cg}(G)$, respectively. Therefore, $\max\{\gamma_c(G), \gamma_g(G)\} \leq \gamma_{cg}(G)$.

Finally, since G is connected, $V(G)$ itself is a connected geodetic dominating set of G. Thus, $\gamma_{cg}(G) \leq |V(G)| = n$. Combining the inequalities, we obtain $2 \leq \max\{\gamma_c(G), \gamma_g(G)\} \leq \gamma_{cg}(G) \leq n$. □

Theorem 3.3. *Let* G *be a connected graph of order* $n \geq 2$, *then the following statements hold:*

(*i*) $\gamma_{cg}(G) = 2$ *if and only if* $G \cong K_2$.

(*ii*) $\gamma_{cg}(G) = 3$ *if and only if* $G \in \{K_3, P_3\}$ *or there exist distinct vertices* $x, y, z \in V(G)$ *such that* $\langle\{x, y, z\}\rangle = P_3 = [x, y, z]$ *and* $p \in N_G(x) \cap N_G(z)$ *for all* $p \in V(G) \setminus \{x, y, z\}$.

(*iii*) *If* S *is a connected geodetic dominating set of* G, *then* $C(G) \cup Ext(G) \subseteq S$.

(*iv*) $\gamma_{cg}(G) = n$ *if and only if* $V(G) = C(G) \cup Ext(G)$.

Proof. (*i*) Suppose $\gamma_{cg}(G) = 2$ and let $S = \{u, v\}$ be a γ_{cg}-set of G. Since $\langle S\rangle$ is connected $uv \in E(G)$. Hence, $I_G[u, v] = \{u, v\}$. This implies that $V(G) = \{u, v\}$. Therefore, $G \cong K_2$.

Conversely, suppose that $G \cong K_2$ and let $S = V(G)$. Clearly, S is a connected geodetic dominating set of G. Hence, $\gamma_{cg}(G) \leq |S| = 2$. Since $|V(G)| = 2$, $\gamma_{cg}(G) \geq 2$. Therefore, $\gamma_{cg}(G) = 2$.

(ii) Suppose $\gamma_{cg}(G) = 3$, say $D = \{x, y, z\}$ is a γ_{cg}-set of G. If $n = 3$, then $G \in \{K_3, P_3\}$. Suppose $n \geq 4$. Since $\langle D \rangle$ is connected, $\langle D \rangle = K_3$ or $\langle D \rangle = P_3$. Let $p \in V(G) \setminus \{x, y, z\}$. Since D is a geodetic set in G, it follows that $p \in I_G(s, t)$ where $s, t \in D$. We may assume that $s = x$ and $t = z$. It follows that $\langle D \rangle = P_3$ and $p \in N_G(x) \cap N_G(z)$.

For the converse, suppose first that $G \in \{K_3, P_3\}$. Since $V(K_3)$ and $V(P_3)$ are connected geodetic dominating sets in K_3 and P_3, respectively, it follows that $\gamma_{cg}(G) \leq 3$. Hence, by part (ii), $\gamma_{cg}(G) = 3$. Next, suppose there exist distinct vertices $x, y, z \in V(G)$ such that $\langle \{x, y, z\} \rangle = P_3 = [x, y, z]$ and $p \in N_G(x) \cap N_G(z)$ for all $p \in V(G) \setminus \{x, y, z\}$. With (i), it follows that $R = \{x, y, z\}$ is a γ_{cg}-set in G. Therefore. $\gamma_{cg}(G) = 3$.

(iii) Let S be a connected geodetic dominating set. Since S is a connected geodetic set, it follows from Theorem 2.4 and Theorem 2.5 that $C(G) \cup Ext(G) \subseteq S$.

(iv) Suppose $\gamma_{cg}(G) = n$. Suppose further that there exists $v \in V(G)$ such that $v \notin C(G) \cup Ext(G)$. Let $D = V(G) \setminus \{v\}$. Since $v \notin C(G)$, it follows that $\langle D \rangle$ is connected. Moreover, since $v \notin Ext(G)$, there exist $x, y \in D$ such that $xy \notin E(G)$ and $v \in I_G(x, y)$. Thus, D is a connected geodetic dominating set. This implies that $\gamma_{cg}(G) \leq |D| = n - 1$, a contradiction. Therefore, no such $v \in V(G)$ exists. Accordingly, $V(G) = C(G) \cup Ext(G)$.

Conversely, suppose that $V(G) = C(G) \cup Ext(G)$. By part (iii), it follows that $V(G)$ is the unique γ_{cg}-set of G. Therefore, $\gamma_{cg}(G) = n$. □

Example 3.4. Consider the graph G in Figure 1 with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$. Then the sets $S_1 = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\}$, $S_2 = \{v_1, v_2, v_3, v_4, v_7, v_8, v_9\}$, and $S_3 = \{v_1, v_2, v_3, v_5, v_7, v_8, v_9\}$ are the γ_{cg}-sets of G. Therefore, $\gamma_{cg}(G) = 7$.

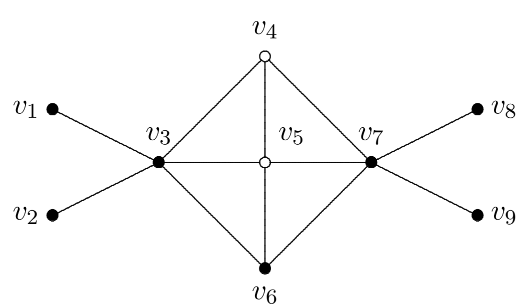

Figure 1. A graph G with $\gamma_{cg}(G) = 7$

3.1. **Connected Geodetic Domination in the Join of Graph**

Lemma 3.5. *Let G be a non-complete graph and let S be a 2-path closure absorbing set in G. Then S is a dominating set and induces a non-complete subgraph of G.*

Proof. Clearly, S is a dominating set in G. Next, suppose, on the contrary, that S induces a complete subgraph of G. Note that since G is non-complete, there exist $a, b \in V(G)$ such that $d_G(a, b) \neq 1$. This implies that $a \notin S$ or $b \notin S$. We may assume that $a \notin S$. Since $\langle S \rangle$ is complete, we have $a \notin P_2[S]$, a contradiction to the assumption that S is a 2-path closure absorbing in G. Therefore, S induces a non-complete subgraph of G. □

Theorem 3.6. *Let G and H be non-complete graphs. Then $S \subseteq V(G + H)$ is a connected geodetic dominating set of $G + H$ if and only if one of the following statements holds:*

- (i) *S is a connected 2-path closure absorbing set of G or H.*
- (ii) *$S = S_G \cup S_H$ and either S_G is a nonempty subset of $V(G)$ and S_H is a 2-path closure absorbing set of H or S_H is a nonempty subset of $V(H)$ and S_G is a 2-path closure absorbing set of G.*
- (iii) *$S = S_G \cup S_H$ and $\langle S_G \rangle$ and $\langle S_H \rangle$ are non-complete subgraphs of G and H, respectively.*

Proof. Suppose S is a connected geodetic dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. If $S_H = \varnothing$, then $S = S_G$. Since S is a connected geodetic dominating set of $G + H$, it follows that S is 2-path closure absorbing set of G. Similarly, S is 2-path closure absorbing set of H if $S_G = \varnothing$. Next, suppose that $S_G \neq \varnothing$ and $S_H \neq \varnothing$. If S_G is a 2-path closure absorbing set in G or S_H is a 2-path closure absorbing set in H, then (ii) holds. Suppose S_G and S_H are not 2-path closure absorbing sets. Then there exist $p \in V(G) \setminus S_G$ and $q \in V(H) \setminus S_H$ such that $p \notin P_2[S_G]$ and $q \notin P_2[S_H]$. Since $p \in V(G + H) \setminus S$ and S is a geodetic set in $G + H$, there exist $u, v \in S_H$ such that $[u, p, v]$ is a u-v geodesic in $G + H$. It follows that $d_H(u, v) \neq 1$, showing that $\langle S_H \rangle$ is not a complete subgraph of H. Similarly, $\langle S_G \rangle$ is not a complete subgraph of G. Thus, (iii) holds.

For the converse, suppose first that (i) holds, say S is a connected 2-path closure absorbing set of G. By Lemma 3.5, we may choose any two distinct vertices c and d in S such that $d_G(c, d) \neq 1$. Then $[c, z, d]$ is a c-d geodesic in $G + H$ for any $z \in V(G + H) \setminus S$. This implies that S is a connected geodetic set in $G + H$. With Lemma 3.5, S must be a dominating set in $G + H$. Next, suppose (ii) holds. We may assume that S_G is a 2-path closure absorbing set in G. Then, by Lemma 3.5, $V(H) \subseteq P_2[S_G]$ and S_G is a dominating set in G. Therefore, because $S_H \neq \varnothing$, S is a connected geodetic dominating set in $G + H$. Clearly, we obtain the same conclusion for S if condition (iii) holds. □

Corollary 3.7. *There exist no non-complete graphs G and H such that $\gamma_{cg}(G + H) = 2$.*

Proof. Suppose there exist non-complete graphs G and H such that $\gamma_{cg}(G + H) = 2$, say $S = \{x, y\}$ is a γ_{cg}-set in $G + H$. Then $xy \in E(G + H)$. By Theorem 3.6, S is a connected 2-path closure absorbing set of G or H or $S = S_G \cup S_H$ and either S_G is a nonempty subset of $V(G)$ and S_H is a 2-path closure absorbing set of H or S_H is a nonempty subset of $V(H)$ and S_G is a 2-path closure absorbing set of G. If $S \subseteq V(G)$, then $V(G) = S$, i.e., $G = K_2$. Similarly, $H = K_2$ if $S \subseteq V(H)$. If $S = S_G \cup S_H$ and either S_G is a nonempty subset of $V(G)$ and S_H is a 2-path closure absorbing set of H or S_H is a nonempty subset of $V(H)$ and S_G is a 2-path closure absorbing set of G, then $G = K_1$ or $H = K_1$. Thus, both cases give a contradiction, showing that the assertion holds. □

Theorem 3.8. *If G and H are non-complete graphs, then $3 \leq \gamma_{cg}(G+H) \leq 4$.*

Proof. Since G and H are non-complete graphs, then by Corollary 3.7, $3 \leq \gamma_{cg}(G+H)$. Let $u, v \in V(G)$ and $p, q \in V(H)$ such that $uv \notin E(G)$ and $pq \notin E(H)$, respectively. Then by Theorem 3.6 (*iii*), $S = \{u, v, p, q\}$ is a connected geodetic dominating set of $G+H$. Hence, $\gamma_{cg}(G+H) \leq |S| = 4$. Therefore, $3 \leq \gamma_{cg}(G+H) \leq 4$. □

Theorem 3.9. *Let G and H be non-complete graphs, then $\gamma_{cg}(G+H) = 3$ if and only if at least one of the following statements holds:*

(i) $\rho_2^c(G) = 3$ *or* $\rho_2^c(H) = 3$.

(ii) $\rho_2(G) = 2$ *or* $\rho_2(H) = 2$.

Proof. Suppose $\gamma_{cg}(G+H) = 3$. Since G and H are non-complete graphs, then $\rho_2(G) \geq 2$, $\rho_2(H) \geq 2$, $\rho_2^c(G) \geq 3$, and $\rho_2^c(H) \geq 3$. Let $\{x, y, z\}$ be a γ_{cg}-set of $G+H$. WLOG, suppose $\{x, y, z\} \subseteq V(G)$. Then by Theorem 3.6 (i), $\{x, y, z\}$ is a connected 2-path closure absorbing set of G. Hence ${\rho_2}^c(G) \leq 3$. Since $\rho_2^c(G) \geq 3$, it follows that $\rho_2^c(G) = 3$. Thus, (i) holds. Now suppose that $\{x, y, z\} \cap V(G) \neq \varnothing$. Let $S_G = \{x\} \subseteq V(G)$ and $S_H = \{y, z\} \subseteq V(H)$. By Theorem 3.6 (ii), S_H is a 2-path closure absorbing set of H. Therefore $\rho_2(H) \leq 2$. Since $\rho_2(H) \geq 2$, we have $\rho_2(H) = 2$. Hence (ii) holds.

For the converse, suppose first that (i) holds. Here, we may assume that $\rho_2^c(G) = 3$. Let S be a connected 2-path closure absorbing set of G. By Theorem 3.6 (i), S is a connected geodetic dominating set of $G+H$. Hence, $\gamma_{cg}(G+H) \leq |S| = 3$. By Corollary 3.7, $\gamma_{cg}(G+H) = 3$. Next, suppose that (ii) holds. Again, we may assume without loss of generality that $\rho_2(G) = 2$. Let S be a 2-path closure absorbing set of G. Pick any $h \in V(H)$. Then $D = S \cup \{h\}$ is a connected geodetic dominating set of $G+H$ by Theorem 3.6 (ii). By Corollary 3.7, $\gamma_{cg}(G+H) = |D| = 3$. □

Corollary 3.10. *For integers $m, n \geq 2$, $r, s \geq 4$, and $p, q \geq 3$, the following hold:*

$$\gamma_{cg}(K_{m,n}) = \begin{cases} 3, & \text{if } \min\{m, n\} = 2, \\ 4, & \text{if } m, n \geq 3, \end{cases}$$

and

$$\gamma_{cg}(P_p + P_q) = \begin{cases} 3, & \text{if } \min\{p,q\} = 3, \\ 4, & \text{if } p, q \geq 4, \end{cases}$$

and

$$\gamma_{cg}(C_r + C_s) = \begin{cases} 3, & \text{if } \min\{r,s\} = 4, \\ 4, & \text{if } r, s \geq 5. \end{cases}$$

The next result is easy to prove.

Theorem 3.11. *Let H be a non-complete graph and let n be a positive integer. Then $S \subseteq V(K_n + H)$ is a connected geodetic dominating set of $K_n + H$ if and only if one of the following statements holds:*

(i) *S is a connected 2-path closure absorbing set of H.*

(ii) *$S = S_n \cup S_H$, where S_n is a nonempty subset of $V(K_n)$ and S_H is a 2-path closure absorbing set of H .*

The next result is immediate from Theorem 3.11.

Corollary 3.12. *Let n be a positive integer and let H be any non-complete graph. Then*

$$\gamma_{cg}(K_n + H) = \min\{\rho_2(H) + 1, \rho_2^c(H)\}.$$

Corollary 3.13. *Let n and m be positive integers such that $n \geq 4$ and $m \geq 2$. Then*

$$\gamma_{cg}(W_n) = \gamma_{cg}(K_1 + C_n) = \begin{cases} \dfrac{n+2}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n+3}{2}, & \text{if } n \text{ is odd}, \end{cases}$$

and

$$\gamma_{cg}(F_m) = \gamma_{cg}(K_1 + P_m) = \begin{cases} \dfrac{m+4}{2}, & \text{if } m \text{ is even}, \\ \dfrac{m+3}{2}, & \text{if } m \text{ is odd}. \end{cases}$$

3.2. **Connected Geodetic Domination in the Corona of Graph**

Theorem 3.14. *Let G and H be connected non-trivial graphs. A set $S \subseteq V(G \circ H)$ is a connected geodetic dominating set of $G \circ H$ if and only if*

$$S = V(G) \cup \bigcup_{v \in V(G)} S_v,$$

where S_v is a 2-path closure absorbing set in H^v for each $v \in V(G)$.

Proof. Let $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup \bigcup_{v \in V(G)} S_v$. Suppose $A \neq V(G)$, say $v \in V(G) \setminus A$. Since S is a dominating set in $G \circ H$ and $v \in S$, it follows that $S_v \neq \varnothing$. This implies that $\langle S_v \rangle$ is a (connected) component of $\langle S \rangle$. Since G is non-trivial, it follows that $\langle S \rangle$ has at least two components, a contradiction. Thus, $A = V(G)$. Next , let $v \in V(G)$. If $S_v = V(H^v)$, then S_v

is a 2-path closure absorbing set in H^v. Suppose $S_v \neq V(H^v)$. Let $p \in V(H^v) \setminus S_v$. Since S is a geodetic set in $G \circ H$, there exist $a, b \in S$ such that $p \in I_{G\circ H}(a,b)$. This implies that $a, b \in S_v$, $d_{H^v}(a,b) = 2$, and $p \in I_{H^v}(a,b)$. Therefore, S_v is a 2-path closure absorbing set in H^v for each $v \in V(G)$.

For the converse, suppose S is as described and satisfies the given property. Since $V(G)$ is a dominating set in $G \circ H$, G is connected, and each S_v is a 2-path closure absorbing set in H^v, it follows that S is a connected geodetic dominating set in $G \circ H$. □

Corollary 3.15. *If G and H are connected non-trivial graphs of orders m and n, respectively, then $\gamma_{cg}(G \circ H) = m + m\,\rho_2(H)$.*

Proof. For each $v \in V(G)$, let S_v be a ρ_2-set in H^v. Then by Theorem 3.14, $D = V(G) \cup (\bigcup_{v\in V(G)} S_v)$ is a connected geodetic dominating set of $G \circ H$. Hence,

$$\begin{aligned}\gamma_{cg}(G\circ H) \leq |D| &= m + \sum_{v\in V(G)} |S_v| \\ &= m + m\,\rho_2(H).\end{aligned}$$

Next, let S be a γ_{cg}-set of $G \circ H$. By Theorem 3.14, $S = V(G) \cup (\bigcup_{v\in V(G)} R_v)$ where R_v is a 2-path closure absorbing set in H^v for each $v \in V(G)$. Hence,

$$\begin{aligned}\gamma_{cg}(G\circ H) = |S| &= m + \sum_{v\in V(G)} |R_v| \\ &\geq m + m\,\rho_2(H).\end{aligned}$$

Therefore, $\gamma_{cg}(G \circ H) = m + m\rho_2(H)$. □

The next result is a consequence of Corollary 3.15.

Corollary 3.16. *For all positive integers m and n,*

(i) $\gamma_{cg}(P_n \circ P_m) = n + n\lceil \frac{m+1}{2} \rceil$, for $n \geq 2$, $m \geq 3$;
(ii) $\gamma_{cg}(P_n \circ C_m) = n + n\lceil \frac{m}{2} \rceil$, for $n \geq 2$, $m \geq 4$;
(iii) $\gamma_{cg}(C_n \circ C_m) = n + n\lceil \frac{m}{2} \rceil$, for $n \geq 3$, $m \geq 4$;
(iv) $\gamma_{cg}(P_n \circ K_m) = n + nm$, for $n, m \geq 2$;
(v) $\gamma_{cg}(C_n \circ K_m) = n + nm$, for $n \geq 3$, $m \geq 2$.

3.3. **Connected Geodetic Domination in the Edge Corona of Graph**

Theorem 3.17. *Let G be a connected graph such that $G \notin \{K_1, K_2\}$ and let H be any non-complete graph. Then $S \subseteq V(G \diamond H)$ is a connected geodetic dominating set of $G \diamond H$ if and only if*

$$S = A \cup [\cup_{uv\in E(G)} S_{uv}]$$

and satisfies the following conditions:

(*i*) *S_{uv} is a 2-path closure absorbing set in H^{uv} for every $uv \in E(G)$; and*

(*ii*) *A is a connected vertex cover of G.*

Proof. Let $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then $S = A \cup [\cup_{uv \in E(G)} S_{uv}]$. Let $uv \in E(G)$ and let $q \in V(H^{uv}) \setminus S_{uv}$. Since S is a geodetic set in $G \diamond H$, there exist $s, t \in S$ such that $q \in I_{G \diamond H}(s,t)$. This implies that $s, t \in S_{uv}$, $d_{G\diamond H}(s,t) = d_{H^{uv}}(s,t) = 2$, and $q \in I_{H^{uv}}(s,t)$. Thus, S_{uv} is a 2-path closure absorbing set in H^{uv}. Hence, (i) holds. Next, suppose there exists $uv \in E(G)$ such that $\{u,v\} \cap A = \varnothing$. Since $\langle S \rangle$ is connected in $G \diamond H$, it follows that $\langle S_{uv} \rangle$ is a component of $\langle S \rangle$. Since $G \neq K_2$, this implies that $\langle S \rangle$ is disconnected, a contradiction. Therefore, A is a vertex cover of G. Suppose now that $\langle A \rangle$ is not connected. Then there exist distinct vertices $v, w \in A$ that are not connected by a path in $\langle A \rangle$. Let $vx, wy \in E(G)$. Then $S_{vx} \cup \{v\} \subseteq V(C_1)$ and $S_{wy} \cup \{w\} \subseteq V(C_2)$, where C_1 and C_2 are two distinct components of $\langle S \rangle$. This contradicts the assumption that $\langle S \rangle$ is connected. Thus, $\langle A \rangle$ must be connected. This shows that (ii) holds.

For the converse, suppose $S = A \cup [\cup_{uv \in E(G)} S_{uv}$ and satisfies (i) and (ii). By (ii), it follows that S is a connected dominating set in $G \diamond H$. Next, let $x \in V(G \diamond H) \setminus S$. If $x \in V(G)$, then $x \in V(G) \setminus A$. Let $y \in N_G(x)$. By (i) and Lemma 3.5, there exist $p, q \in S_{xy}$ such that $d_{H^{xy}}(p,q) = 2$. It follows that $x \in I_{G \diamond H}(p,q)$. Finally, suppose $x \notin V(G)$. Then there exists $z \in N_G(x)$ such that $x \in V(H^{xz}) \setminus S_{xz}$. By (i), there exist $s, t \in S_{xz}$ such that $x \in I_{H^{xz}}(s,t) = I_{G \diamond H}(s,t)$. Thus, S is a geodetic set in $G \diamond H$. □

The next result follows from Theorem 3.17.

Corollary 3.18. *Let G be a connected graph of size q such that $G \notin \{K_1, K_2\}$ and let H be any non-complete graph. Then*

$$\gamma_{cg}(G \diamond H) = \beta_c(G) + q\rho_2(H).$$

Corollary 3.19. *Let m and n be positive integers. Then*

(*i*) $\gamma_{cg}(P_n \diamond P_m) = (n-2) + (n-1)\lceil \frac{m+1}{2} \rceil$*, for $n, m \geq 3$;*

(*ii*) $\gamma_{cg}(P_n \diamond C_m) = (n-2) + (n-1)\lceil \frac{m}{2} \rceil$*, for $n \geq 3$, $m \geq 4$;*

(*iii*) $\gamma_{cg}(S_n \diamond P_m) = 1 + (n-1)\lceil \frac{m+1}{2} \rceil$*, for $n, m \geq 3$, where $S_n = K_{1,n-1}$;*

(*iv*) $\gamma_{cg}(S_n \diamond C_m) = 1 + (n-1)\lceil \frac{m}{2} \rceil$*, for $n \geq 3$, $m \geq 4$;*

(*v*) $\gamma_{cg}(S_n \diamond S_m) = 1 + (n-1)(m-1)$*, for $n, m \geq 3$; and*

(*vi*) $\gamma_{cg}(C_n \diamond P_m) = (n-1) + n\lceil \frac{m+1}{2} \rceil$*, for $n, m \geq 3$.*

Theorem 3.20. *Let G be a connected graph of size q such that $G \notin \{K_1, K_2\}$ and let H be a complete graph. Then $S \subseteq V(G \diamond H)$ is a connected geodetic dominating set of $G \diamond H$ if and only if*

$$S = D \cup [\cup_{uv \in E(G)} V(H^{uv})],$$

where D is a connected geodetic vertex cover of G.

Proof. Let $D = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Suppose that $S_{uv} \neq V(H^{uv})$, say $w \in V(H^{uv}) \setminus S_{uv}$. Since S is a geodetic set in $G \diamond H$, it follows that there exist $p, q \in S_{uv}$ such that $w \in I_{H^{u,v}}(p,q)$. This, however, is not possible because H^{uv} is complete. Thus, $S_{uv} = V(H^{uv})$ for each $uv \in E(G)$. It is routine to show that D is a connected vertex cover of G. Now, let $z \in V(G) \setminus D$. Since S is a geodetic set in $G \diamond H$ and H is complete, D must be a geodetic set in G.

For the converse, suppose $S = D \cup [\cup_{uv \in E(G)} V(H^{uv})]$, where D is a connected geodetic vertex cover of G. Then clearly, S is a connected geodetic dominating set in $G \diamond H$. □

The next result is a consequence of Theorem 3.20.

Corollary 3.21. *Let G be a connected graph of size q such that $G \notin \{K_1, K_2\}$ and let H be a complete graph of order n. Then*

$$\gamma_{cg}(G \diamond H) = \beta_{cg}(G) + qn.$$

4. Conclusion

In this paper, we investigated connected geodetic dominating sets in graphs and established key structural properties and bounds for the connected geodetic domination number. We further examined this parameter in the graphs resulting from the join, corona, and edge corona of two graphs, using the concept of 2-path closure absorbing set, connected vertex cover, and connected geodetic vertex cover to characterize the sets and derive exact values or bounds for the corresponding connected geodetic domination numbers. In particular, we showed that for joins of non-complete graphs, the connected geodetic domination number lies between 3 and 4, while for corona and edge corona in graphs, explicit formulas were obtained in terms of these sets. These results extend existing work on geodetic domination by incorporating the connectivity requirement into the parameter. Future work may consider other graph operations and computational aspects of connected geodetic domination.

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