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# EXISTENCE RESULTS FOR $\alpha$-NONSTANDARD EQUILIBRIUM TRIFUNCTION

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Abstract. This paper investigates an existence theory for a class of $\alpha-$nonstandard equilibrium problems defined by trifunctions in real reflexive Banach spaces $H$. We introduce a generalized framewor $\mathcal{K}$ that unifies several models including another types of equilibrium issues. The considered model involves a multivalued operator and a $\sigma-$monotone trifunction coupled with an auxiliary bifunction $\alpha$. By employing the Fan–KKM technique together with weak compactness args in reflexive Banach space, We provide sufficient requirements for the existence of solutions under hemicontinuity, convexity, and diagonal convexity assumptions.The proposed results extend and improve several known existence theorems for equilibrium-type problems involving generalized monotonicity structures.

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## 1. Introduction

This paper is primary goal is to determine whether there are solutions for $\alpha$-nonstandard equilibrium trifunctions. lies in providing a flexible mathematical framewor $\mathcal{K}$ for equilibrium Problems, which arise in various fields of physics and engineering.Scientific investigations of generalized research on generalized monotonicity into the concept of diversity includes monotonicity concepts have led to significant developments in solving equilibrium-type problems in Banach spaces, even in complex cases where sets are unbounded. In such cases, and the entire process is mathematically feasible. In 1961, [8] Blum and Oettli decided to work on the equilibrium problem (EP), where the idea was to find the solution point. In 2003 [9], YP Fang and NJ Huang used the KKM approach to develop two types of variational inequalities using generalized monotone mappings in Banach spaces. They were able to solve variational inequalities in reflexive Banach spaces by focusing on proof of existence rather than calculation, shifting the focus from "how to calculate" to proof of existence. Hashoosh

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introduced significant contributions in his research has addressed new forms of inequality systems and their possible solutions, as well as the fact that certain groups of these inequalities have solutions. Studies have also extended to include the use of KKM maps and he studied equilibrium problems that are vital topics in mathematical analysis, and his research has contributed to the development of this aspect by studying the results of equilibrium problems involving trifunctions of type Bh-monotone, Some studies have also focused on improving the formulation (well-posedness) of hemiequilibrium problems, Sobolev spaces and their applications in hemiequilibrium (see [1,3,4,11,13,14]). For many significant problems, including optimization, variational inequality, integration, minimum inequality, and fixed-point problems, equilibrium problems offer a cohesive framewor$\mathcal{K}$. It has been extensively used to investigate real-world applications in the domains of engineering, mechanics, and economics ( [2,5,6]). Numerous findings solutions to vector equilibrum issues and equilibrum issues have been made over the last few decades. Active circularity is used in the study of competitive issues in recent times, Various authors have put out a number of fundamental circularity generalizations, including relaxed monotonicity, quasimonotonicity, and $\alpha-$monotonicity (see [2,15,17]). Many issues of practical importance in the fields of economic,mechanics, and engineering include the concept of equilibrum according to their description.Due to their widespread applications., a number of researchers have studied and generalized equilibrium problems and mixed equilibrium problems (see [10,12,16,18].

## 2. Preliminaries

Consider $\mathcal{K}$ as a non-empty subset of a real reflexive Banach space H, and $T : \mathcal{K} \longrightarrow 2^{\mathcal{K}}$ be a multivalued compact operator. Let the function $\alpha : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a continuous value functions. For a given trifunction $\mathfrak{F} : \mathcal{K} \times \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ real-valued function is equilibrium function, we consider the problem of finding $\mu \in \mathcal{K}$, in which

$$\sup_{\propto \in T\mu} (\mu, \propto, \hbar) + \alpha(\mu, \hbar) \geq 0 \quad \forall \hbar \in \mathcal{K}. \tag{1}$$

that is $\alpha$-non standard equilibrium trifunction in short (NSET).

1-If $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \equiv \mathfrak{F}(\mu, \hbar)$, then, issue (1) is the same as finding $\mu \in \mathcal{K}$ in which:

$$\mathfrak{F}(\mu, \hbar) + \alpha(\mu, \hbar) \geq 0 \quad \forall \hbar \in \mathcal{K}. \tag{2}$$

which is referred to as a generalized equilibrium problem(for short $EP_\alpha$); [13].

2-If $sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \equiv \mathfrak{F}(\mu, \hbar)$ and $\alpha(\mu, \hbar) \equiv 0$, then, issue (1) is the same as to find u $\in \mathcal{K}$ in which

$$\mathfrak{F}(\mu, \hbar) \geq 0. \tag{3}$$

It is referred to as the classical equilibrium problem; [7].

3- If $sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \equiv \mathfrak{F}(\mu, \hbar)$ and $\alpha(\mu, \hbar) \equiv \alpha(\hbar) - \alpha(\mu)$, that is known mixed equilibrium problem (MEP), which is to identify $\mu \in \mathcal{K}$ ; [3], is then reduced from problem (1).

**Definition 2.1.** [13] Assume that a Hausdorff topological vector space $H$ has a nonempty subset $\mathcal{K}$. A mapping $\zeta : \mathcal{K} \longrightarrow 2^H$ is considered a KKM$-$mapping, if for any $\{\mu_1, \mu_2, ..., \mu_p\} \subseteq \mathcal{K}$ ; then $Co\{\mu_1, \mu_2, ..., \mu_p\} \subset \bigcup_{\mathrm{i}=1}^{p} \zeta(\mu_i)$; such that the $Co\{\mu_1, \mu_2, ..., \mu_p\}$ represents the convex hull of the finite collection $\{\mu_1, \mu_2, ..., \mu_p\}$.

**Lemma 2.2.** [13] *Assume $H$ is a Hausdorff topological vector space, and $\mathcal{K}$ is a nonempty subset of $H$. Consider $\zeta : \mathcal{K} \longrightarrow 2^H$ as a* KKM$-$*mapping. if $\zeta(\mu)$ is compact for some $\mu_0 \in \mathcal{K}$ and closed $\forall \mu \in \mathcal{K}$, then $\bigcap_{\mu \in \mathcal{K}} \zeta(\mu) \neq \phi$.*

**Definition 2.3.** [13] A hemicontinuous (hco) real-value function g which is defined on a convex subset $\mathcal{K}$of a space $H$ if, for any $\mu, \hbar \in \mathcal{K}$ then $\lim_{t \to 0^+} \mathrm{g}(t\mu + (1-t)\hbar) = \mathrm{g}(\hbar)$.

**Definition 2.4.** [13] $\alpha : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ is a convex $0-$ diagonally $(0-dig)$ bifunction if for all subset $\{\mu_1, \mu_2, ..., \mu_P\}$ and $\lambda_i \geq 0 \ \ (i = 1, ..., \mathrm{P})$ with $\propto = \sum_{\mathrm{i}=1}^{\mathrm{P}} \lambda_i \mu_i$ and $\sum_{\mathrm{i}=1}^{P} \lambda_i = 1$ one has
$\sum_{\mathrm{i}=1}^{\mathrm{P}} \lambda_i \alpha(\propto, \propto_i) \geq 0$.

**Definition 2.5.** [13] A mapping $\zeta : H \longrightarrow \mathbb{R}$ from a Banach space $H$ is called;

(1) Lower semicontinuous (LSC) at $u_0$, if $\zeta(u_0) \leq \lim_n inf \zeta(x_n)$.

(2) Upper semicontinuous (USC) at $u_0$, if $\zeta(u_0) \geq \lim_n sup \zeta(u_n)$.

$\mathrm{u_n} \longrightarrow \mathrm{u_0}$ for any sequence $\mathrm{u_n}$ of $H$

**Definition 2.6.** [13] consider $\mathfrak{F} : H \longrightarrow R \bigcup \{+\infty\}$ as a proper function on a Banach space $H$. If $\mu^* \in H^*$ is $\sigma-$ subdifferential of $\mathfrak{F}$ in $\mu \in dom\mathfrak{F} = \{\mu : \mathfrak{F}(\mu) < \infty\}$, then

$$\partial_\sigma \mathfrak{F}(\mu) = \left\{ \mu^* \in H^* : \mathfrak{F}(\hbar) - \frac{\sigma(\hbar, \mu)}{2} \geq \mathfrak{F}(\mu) + \langle \mu^*, \hbar - \mu \rangle \ \ (\forall \hbar \in H) \right\}. \tag{4}$$

## 3. Existence Results For $\alpha$-non standard Equilibrium Trifunction

This section presents a novel concept of $\sigma$-monotonicity for trifunctions and formulate the corresponding $\alpha$-nonstandard equilibrium trifunction problem in real reflexive Banach spaces. Based on this framewor$\mathcal{K}$, We use the Fan–KKM approach under suitable convexity to establish existence results and hemicontinuity assumptions.

**Definition 3.1.** Let $T$ be a multivalued compact operator, A trifunction $\mathfrak{F} : \mathcal{K} \times \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ is called $\sigma-$monotone w.r.t $T$ if there is a bifunction $\sigma : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ in which

$$\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) + \sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) + \sigma(\mu, \hbar) \leq 0 \quad \forall \mu, \hbar \in \mathcal{K}. \tag{5}$$

**Remark 3.2.** Through this paper we consider $\sup_{\propto\in T\mu}\mathfrak{F}(\mu,\propto,\mu)=0\quad\forall\mu\in\mathcal{K}$, and $\forall\hbar\in(0,1)$, $\sigma:\mathcal{K}\times\mathcal{K}\to\mathbb{R}$ be a convex bifunction in the second $(2^{nd})$ arg then:

$$\lim_{\hbar\to 0}\frac{\alpha(\propto,(1-\hbar)\propto+\hbar\kappa))}{\hbar}=0,\qquad \alpha(\propto,\kappa)\leq\lim_{\hbar\to 0}\frac{\hbar-1}{\hbar}[\sigma(\propto,\propto)+\alpha(\propto,\propto)] \tag{6}$$

Reflexive Banach spaces require the concept of a KKM map and Fan's intersection lemma [8]. In addition, For the case of unbounded sets, we present adequate requirements for the existence of a solution. The findings in this paper can be interpreted as a generalization of previously published research.

**Lemma 3.3.** *Assume that $H$ is a real reflexive Banach space and that $\mathcal{K}\subset H$ be a nonempty convex set. Let $T:\mathcal{K}\to 2^{\mathcal{K}}$ be a multivalued compact operator with nonempty compact values. Assume that $\mathfrak{F}:\mathcal{K}\times\mathcal{K}\times\mathcal{K}\to\mathbb{R}$ is continuous in the two first $(1^{st})$ args. If $\mu_{\mathrm{r}}\to\mu$ in $\mathcal{K}$ as $r\to 0^+$, then for any fixed $\hbar\in\mathcal{K}$,*

$$\lim_{r\to 0^+}\sup_{\propto\in T\mu_r}\mathfrak{F}(\mu,\propto,\hbar)=\sup_{\propto\in T\mu}\mathfrak{F}(\mu,\propto,\hbar). \tag{7}$$

*Proof.* Let $\phi(r)=\sup_{\propto\in T_{\mu_{\mathrm{r}}}}\mathfrak{F}(u_r,\propto,\hbar)$. We prove the result by establishing both the upper and lower limits.

**Claim 1.** Upper limit. Let $r_n\to 0^+$ and choose $\propto_n\in T\mu_{r_n}$ such that

$$\mathfrak{F}(\mu_{r_n},\propto_n,\hbar)\geq\sup_{\propto\in T\mu_{r_n}}\mathfrak{F}(\mu_{r_n},\propto,\hbar)-\epsilon_{\mathrm{n}}, \tag{8}$$

where $\epsilon_{\mathrm{n}}\to 0^+$.

Since $\mu_{r_n}\to\mu$ and $T$ is multivalued compact operator, the sequence $\{\propto_n\}$ has a convergent subsequence such that

$$\propto_n\to\propto\in T\mu. \tag{9}$$

By continuity of $\mathfrak{F}$ in the $1^{st}$ two variables,

$$\mathfrak{F}(\mu_{r_n},\propto_n,\hbar)\to\mathfrak{F}(\mu,\propto,\hbar). \tag{10}$$

Hence,

$$\limsup_{n\to\infty}\sup_{\propto\in T\mu_{r_n}}\mathfrak{F}(\mu_{r_n},\propto,\hbar)\leq\mathfrak{F}(\mu,\propto,\hbar)\leq\sup_{\kappa\in T\mu}\mathfrak{F}(\mu,\kappa,\hbar). \tag{11}$$

Taking the supremum over $\propto\in T\mu$, we have

$$\limsup_{r\to 0^+}\phi(r)\leq\sup_{\propto\in T\mu}\mathfrak{F}(\mu,\propto,\hbar). \tag{12}$$

**Claim 2.** Lower limit. Let $\propto\in T\mu$ be arbitrary. Since $T$ is multivalued compact operator and $\mu_{\mathrm{r}}\to\mu$, then there is a sequence $x_r\in T\mu_{\mathrm{r}}$ such that $x_r\to x$. By continuity of $\mathfrak{F}$,

$$\mathfrak{F}(\mu_{\mathrm{r}}, \propto_r, \hbar) \to \mathfrak{F}(\mu, \propto, \hbar). \tag{13}$$

Since

$$\sup_{\kappa \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, y, \hbar) \geq \mathfrak{F}(\mu_{\mathrm{r}}, x_r, \hbar) \tag{14}$$

we obtain

$$\liminf_{r \to 0^+} \sup_{\kappa \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, y, \hbar) \geq \mathfrak{F}(\mu, \propto, \hbar). \tag{15}$$

Taking the supremum over $\propto \in T\mu$ yields

$$\liminf_{r \to 0^+} \phi(r) \geq \sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar). \tag{16}$$

Combining the two estimates completes the proof. □

**Theorem 3.4.** *Suppose that a trifunction $\mathfrak{F} : \mathcal{K} \times \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ is $\sigma-$monotone, hco in the two $1^{st}$ argument (arg), and convex trifunction in the third arg. Consider $\alpha, \sigma : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ is convex bifunctions in the $(2^{nd})$arg; in addition $T$ be a multivalued compact hco operator.suppose that $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \mu) = 0 \quad \forall \mu \in \mathcal{K}$. Then the $\alpha$-non-standard equilibrium trifunction $\mathfrak{F}$ $(NSET)$ is equivalent to the problem described below.*
*Find $\mu \in \mathcal{K}$ in which*

$$\sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) + \sigma(\mu, \hbar) \leq \alpha(\mu, \hbar) \quad \forall\, \hbar \in \mathcal{K}. \tag{17}$$

*Proof.* Assume that there is a solution for $(NSET)$. Next, there is a u $\in \mathcal{K}$ in which

$$\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) + \alpha(\mu, \hbar) \geq, 0 \quad \forall\, \hbar \in \mathcal{K}. \tag{18}$$

Since $\mathfrak{F}$ is $\sigma - monotone$ trifunction

$$\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) + \sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) + \sigma(\mu, \hbar) \leq, 0 \quad \forall, \mu, , \hbar \in \mathcal{K}. \tag{19}$$

then

$$\sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) + \sigma(\mu, \hbar) \leq - \sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \leq \alpha(\mu, \hbar) \quad \forall \mu, \hbar \in \mathcal{K}. \tag{20}$$

Therefore, $\mu \in \mathcal{K}$ is an answer to the issue (17). In contrast, suppose that $\mu \in \mathcal{K}$ is a solution of the problem (17) Letting $\mu_{\mathrm{r}} = \mathrm{r}\hbar + (1 - \mathrm{r})\mu, \mathrm{r} \in (0, 1)$, then $\mu_{\mathrm{r}} \in \mathcal{K}$, since $\mathcal{K}$ is convex, then

$$\sup_{\kappa \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, \kappa, \mu) + \sigma(\mu, \mu_{\mathrm{r}}) \leq \alpha(\mu, \mu_{\mathrm{r}}). \tag{21}$$

then

$$\sup_{\kappa \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, \kappa, \mu) - \alpha(\mu, \mu_{\mathrm{r}}) \leq -\sigma(\mu, \mu_{\mathrm{r}}). \tag{22}$$

Given that the third arg's $\mathfrak{F}$ is convex

$$0 = \sup_{\propto \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, \propto, \mu_{\mathrm{r}}) \leq r \sup_{\propto \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, \propto, \hbar) + (1 - r) \sup_{\propto \in T\mu_{\mathrm{r}}} \mathfrak{F}(\mu_{\mathrm{r}}, \propto, \mu). \tag{23}$$

so

$$r[\sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},\propto,\mu) - \sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},\propto,\hbar)] \leq \sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},\propto,\mu). \tag{24}$$

By the convexity of $\alpha$ and $\sigma$ in the $(2^{nd})$argumen

$$\begin{aligned} \sigma(\mu,\mu_{\rm r}) &\leq r\sigma(\mu,\hbar) + (1-r)\sigma(\mu,\mu). \\ \alpha(\mu,\mu_{\rm r}) &\leq r\alpha(\mu,\hbar) + (1-r)\alpha(\mu,\mu). \end{aligned} \tag{25}$$

then

$$\begin{aligned} r[\sigma(\mu,\mu) - \sigma(\mu,\hbar)] &\leq \sigma(\mu,\mu) - \sigma(\mu,\mu_{\rm r}). \\ r[\alpha(\mu,\mu) - \alpha(\mu,\hbar)] &\leq \alpha(\mu,\mu) - \alpha(\mu,\mu_{\rm r}). \end{aligned} \tag{26}$$

From (22), (24), and (26)

$$\begin{aligned} r[\sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},x,\mu) &- \sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},x,\hbar) + \sigma(\mu,\mu) - \sigma(\mu,\hbar) + \alpha(\mu,\mu) - \alpha(\mu,\hbar)] \\ &\leq \sup_{\propto\in T\mu_{\rm r}} \mathfrak{F}(\mu_{\rm r},\propto,\mu) - \sigma(\mu,\mu_{\rm r}) - \alpha(\mu,\mu_{\rm r}) + \sigma(\mu,\mu) + \alpha(\mu,\mu) \\ &\leq -2\sigma(\mu,\mu_{\rm r}) + \sigma(\mu,\mu) + \alpha(\mu,\mu). \end{aligned} \tag{27}$$

from lemma (3.3),

$$r[-\sup_{\propto\in T\mu} \mathfrak{F}(\mu,\propto,\hbar) - \alpha(\mu,\hbar)] \leq -2\sigma(\mu,\mu_{\rm r}) + r\sigma(\mu,\hbar) + (1-r)[\alpha(\mu,\mu) + \sigma(\mu,\mu)]. \tag{28}$$

so

$$\sup_{\propto\in T\mu} \mathfrak{F}(\mu,\propto,\hbar) + \alpha(\mu,\hbar) \geq \frac{2\sigma(\mu,\mu_{\rm r})}{r} - \sigma(\mu,\hbar) - \frac{(r-1)}{r}[\alpha(\mu,\mu) + \sigma(\mu,\mu)]. \tag{29}$$

From (6)

$$\sup_{\propto\in T\mu} \mathfrak{F}(\mu,\propto,\hbar) + \alpha(\mu,\hbar) \geq 0 \quad \forall\hbar\in\mathcal{K}. \tag{30}$$

$\square$

**Theorem 3.5.** *Consider $\mathcal{K}$ as a nonempty closed, convex and bounded subset of $H$, where $H$ is a real reflexive Banach space Assuming that;*

(1) *$\mathfrak{F}:\mathcal{K}\times\mathcal{K}\times\mathcal{K}\longrightarrow\mathbb{R}$ is 0- dig convex trifunction,$\sigma$-monotone, hco in two first $(1^{st})$ args, LSC, and convex in third arg.*

(2) *$\alpha:\mathcal{K}\times\mathcal{K}\longrightarrow\mathbb{R}$ is a USC in $1^{st}$ arg,convex 0-dig in $2^{nd}$arg.*

(3) *$\sigma:\mathcal{K}\times\mathcal{K}\longrightarrow\mathbb{R}$ is LSC in $1^{st}$ arg, and convex in $2^{nd}$arg.*

*Then, the $(NSET)$ has a minimum of one solution.*

*Proof.* Create two mappings with set values. $\eta,\Gamma:\mathcal{K}\longrightarrow 2^{\mathcal{K}}$ as follows:

$$\begin{aligned}\eta(\hbar) &= \left\{\mu \in \mathcal{K} : \sup_{\propto \in T\mu} \mathfrak{F}\left(\mu, \propto, \hbar\right) + \alpha\left(\mu, \hbar\right) \geq 0\right\}. \\ \Gamma(\hbar) &= \left\{\mu \in \mathcal{K} : \sup_{\kappa \in T\hbar} \mathfrak{F}\left(\hbar, \kappa, \mu\right) + \sigma\left(\mu, \hbar\right) \leq \alpha\left(\mu, \hbar\right)\right\}.\end{aligned} \tag{31}$$

The issue $(NSET)$ has a solution iff $\bigcap_{\hbar\in\mathcal{K}} \eta(\hbar) \neq \phi, \ \ \forall \mu \in \mathcal{K}$.We will now demonstrate that $\eta$ is a KKM$-$ mapping. Conversely, $\eta$ it is not a KKM$-$mapping. Then there is $\{\mu_1, \mu_2, ..., \mu_{\mathrm{p}}\} \subseteq \mathcal{K}$ and $\lambda_i \geq 0 \ \ (\mathrm{i} = 1, ..., \mathrm{p})$ where $\sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}} = 1$ in which $\mu_0 = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}} \mu_{\mathrm{i}} \notin \bigcup_{\mathrm{i}=1}^{\mathrm{n}} \eta(\hbar_{\mathrm{i}})$. Then

$$\begin{aligned}\sup_{\propto \in T\mu_0} \mathfrak{F}(\mu_0, \propto, \mu_{\mathrm{i}}) + \alpha(\mu_0, \mu_{\mathrm{i}}) < 0 \quad \forall \mathrm{i} = 1, 2, ..., \mathrm{p}. \\ \sum\nolimits_{\mathrm{i}=1}^{\mathrm{p}} \lambda_{\mathrm{i}} [\sup_{\propto \in T\mu_0} \mathfrak{F}(\mu_0, \propto, \mu_{\mathrm{i}}) + \alpha(\mu_0, \mu_{\mathrm{i}})] < 0 \quad \forall \mathrm{i} = 1, 2, ..., \mathrm{p}.\end{aligned} \tag{32}$$

This contradiction the property 0-diagonally convex of $\mathfrak{F}$ and $\alpha$.Then $\eta$ is a KKM$-$mapping,to show that $\eta(\hbar) \subset \Gamma(\hbar)$ ,$\forall \hbar \in \mathcal{K}$.Let $\mu \in \eta(\hbar)$; then

$$\sup_{\propto \in T\mu} \mathfrak{F}\left(\mu, \propto, \hbar\right) + \alpha\left(\mu, \hbar\right) \geq 0 \quad \forall \hbar \in \mathcal{K}. \tag{33}$$

Sinc $\mathfrak{F}$ is $\sigma$-monotone trifunction

$$\sup_{\kappa \in T\hbar} \mathfrak{F}\left(\hbar, \kappa, \mu\right) + \sigma\left(\mu, \hbar\right) - \alpha\left(\mu, \hbar\right) \leq - \sup_{\propto \in T\mu} \mathfrak{F}\left(\mu, \propto, \hbar\right) - \alpha\left(\mu, \hbar\right) \leq 0. \tag{34}$$

This implies $\mu \in \Gamma(\hbar)$. Thus $\eta(\hbar) \subset \Gamma(\hbar)$ ,$\forall \hbar \in \mathcal{K}$.
So, $\Gamma(\hbar)$ is a KKM$-$mapping. Since $\alpha(.,\hbar)$ is USC, and $\sigma(.,\hbar), \sup_{\kappa\in T\hbar} \mathfrak{F}(\hbar, \kappa, .)$ are LSC and, then

$$\begin{aligned}\sup_{\kappa \in T\hbar} \mathfrak{F}\left(\hbar, \kappa, \mu\right) + \sigma\left(\mu, \hbar\right) &\leq \lim_{n} inf \left(\sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu_{\mathrm{n}}) + \sigma(\mu_{\mathrm{n}}, \hbar)\right) \\ &\leq \lim_{n} sup\, \alpha(\mu_{\mathrm{n}}, \hbar) \leq \alpha\left(\mu, \hbar\right).\end{aligned} \tag{35}$$

Thus, $\Gamma(\hbar)$ is a weakly closed set $\forall \hbar \in \mathcal{K}$. Since $\mathcal{K} \neq \emptyset$, closed,and bounded in real reflexive $H$, this implies that $\mathcal{K}$ has a weak compactness. Thus, $\Gamma(\hbar)$ is weakly compact $\forall \hbar \in \mathcal{K}$. From Theorem (3.4), and Lemma (2.2) and

$$\bigcap_{\hbar\in\mathcal{K}} \eta(\hbar) = \bigcap_{\hbar\in\mathcal{K}} \Gamma(\hbar) \neq \phi. \tag{36}$$

As a result, there is a solution to the problem $(NSET)$. □

**Corollary 3.6.** *Assuming that $\mathcal{K}$ is a convex nonempty subset of $H$, such that $0 \in \mathcal{K}$ and that Theorem (3.5) is assumptions (1-3) are true, assume that:*

(1) $\alpha(\mu, 0) = 0$.

(2) *The sets $\psi_1 = \{\mu \in \mathcal{K} : \sigma(\mu, 0) \leq 0\}$ and $\psi_2 = \{\mu \in \mathcal{K} : \sup_{\kappa\in T\hbar} \mathfrak{F}(0, \kappa, \mu) \leq 0\}$ are relatively compact sets.*

*Then the $\alpha-$nonstandard equilibrium trifunction $(NSET)$ has a solution.*

*Proof.* For some $\hbar \in \mathcal{K}$, it is sufficient to show that $\Gamma(y)$ is compact. Since:

$$\Gamma(\hbar) = \{\mu \in \mathcal{K} : \sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) + \sigma(\mu, \hbar) \leq \alpha(\mu, \hbar), \forall \hbar \in \mathcal{K}\}. \tag{37}$$

So it follows that:

$$\Gamma(\hbar) \subset \left\{\mu \in \mathcal{K} : \sup_{\kappa \in T\hbar} \mathfrak{F}(\hbar, \kappa, \mu) \leq \frac{\alpha(\mu, \hbar)}{2}\right\} \cup \left\{\mu \in \mathcal{K} : \sigma(\mu, \hbar) \leq \frac{\alpha(\mu, \hbar)}{2}\right\}. \tag{38}$$

From $0 \in \mathcal{K}$, we have,

$$\Gamma(0) \subset \left\{\mu \in \mathcal{K} : \sup_{\kappa \in T\hbar} \mathfrak{F}(0, \kappa, \mu) \leq \frac{\alpha(\mu, 0)}{2}\right\} \cup \left\{\mu \in \mathcal{K} : \sigma(\mu, 0) \leq \frac{\alpha(\propto, 0)}{2}\right\}. \tag{39}$$

From conditions 1. and 2. $\Gamma(0)$ represents a subset of a compact set either $\psi_1$ or $\psi_2$. That is follows, for some $\hbar \in \mathcal{K}$, $\Gamma(\kappa)$ is compact. □

## 4. Application to Differential Inclusion Problems

In this parts, we shall apply the existence results obtained in this research to class of nonstandard differential inclusion problem involving a trifunction and multivalued operato in Sobolev space as $S = H^2(\Omega, \mathbb{R}^m)$. let us consider a partial differential inclusion to Find $u \in \mathcal{K}$ such that.

$$-\triangle \mu \in \partial_{-2\sigma}\phi(\mu), \quad \text{in } \Omega. \tag{40}$$

where $\Omega$ be a subset open bounded of $\mathbb{R}^n$. Let $\phi : S \to \mathbb{R}$ be a convex scalar functional defined on $S$ in which $\sup_{\kappa \in T\mu} \mathfrak{F}(\mu, \kappa, \hbar) = \phi(\hbar) - \phi(\mu)$.let K $\neq \emptyset$, bounded, closed, and convex subset of the Sobolev space $S$.

**Definition 4.1.** If the following inclusion is true, then $\mu \in \mathcal{K}$ is a weakly solution to the problem40:

$$\langle -\triangle \mu, T(\mu - \hbar)\rangle \leq \phi(\mu) - \phi(\hbar) - \sigma(\mu, \hbar) \quad \forall \hbar \in \mathcal{K}. \tag{41}$$

The integral from $\langle ., . \rangle$ can be used to obtain that

$$\int_\Omega \nabla\mu \cdot \nabla(\mu - \hbar)\, dx + \sigma(\mu, \hbar) \leq \phi(u) - \phi(\hbar) \qquad \forall \hbar \in \mathcal{K}. \tag{42}$$

set $\alpha(\mu, \hbar) = \int_\Omega \nabla\mu \cdot \nabla(\hbar - \mu)\, dx$. Consequently, we get

$$\sup_{\kappa \in T\mu} \mathfrak{F}(\mu, \kappa, \hbar) + \sigma(\mu, \hbar) \leq \alpha(\mu, \hbar). \tag{43}$$

Considering $\sigma(\mu, \hbar) = \beta\|\hbar - \mu\|_S^2, \;\; \beta > 0$. then $\mathfrak{F}$ is $\sigma$-monotone trifunction In Theorem 3.4 We showed that, under some conditions, 3.5 and 3.6 are equivalent.Therefore, we need to demonstrate that $\mathfrak{F}$ and $\alpha$ satisfy each condition of Theorem 3.5. Clearly, the bifunction $\sigma(\mu, \hbar) = \beta\|\hbar - \mu\|_S^2, \qquad \beta > 0$. satisfies all assumption in Theorem 3.5.

**Claim 1.** $\alpha$ in the $(2^{nd})$arg is convex. Let $\mathrm{z} = t\hbar_1 + (1-t)\hbar_2 \in \mathcal{K}$ for $t \in (0, 1)$. Then:

$$\alpha(\mu, \mathrm{z}) = \int_\Omega \nabla\mu \cdot \nabla((t\hbar_1 + (1-t)\hbar_2) - \mu)dx$$

$$= t\int_{\Omega}\nabla\mu\cdot\nabla(\hbar_1-\mu)dx + (1-t)\int_{\Omega}\nabla\mu\cdot\nabla(\hbar_2-\mu)dx$$
$$= t\alpha(\mu,\hbar_1) + (1-t)\alpha(\mu,\hbar_2).$$

**Claim 2.** $\alpha$ is 0-diagonally convex. For $\mu = \sum_{\text{i}=1}^{p}\lambda_i\mu_i$ with $\sum_{\text{i}=1}^{p}\lambda_i = 1$, then:

$$\begin{aligned}\sum_{\text{i}=1}^{p}\lambda_i\alpha(\mu,\mu_i) &= \sum_{\text{i}=1}^{p}\lambda_i\int_{\Omega}\nabla\mu\nabla(\mu_i-\mu)dx\\ &= \int_{\Omega}\nabla\mu\left(\sum_{\text{i}=1}^{p}\lambda_i\nabla\mu_i\right)dx - \int_{\Omega}\left(\sum_{\text{i}=1}^{p}\lambda_i\right)\nabla\mu\nabla\mu dx\\ &= \int_{\Omega}\nabla\mu\cdot\nabla\mu dx - \int_{\Omega}\nabla\mu\cdot\nabla\mu dx = \mathbf{0}.\end{aligned}$$

Similarly, we show that $sup_{\propto\in T\mu}\mathfrak{F}(\mu,\propto,\mu_i)$ is 0-diagonally convex:

$$\sum_{\text{i}=1}^{\text{p}}\lambda_i sup_{\propto\in T\mu}\mathfrak{F}(\mu,\propto,\mu_i) = \sum_{\text{i}=1}^{\text{p}}\lambda_i(\phi(\mu_i)-\phi(\mu)) = \sum_{\text{i}=1}^{\text{p}}\lambda_i\phi(\mu_i) - \sum_{\text{i}=1}^{\text{p}}\lambda_i\phi(\mu) \geq 0.$$

## 5. Conclusion

In this research, we present $\alpha-$nonstandard equilibrium trifunction, a new class of equilibrium problems with respect to multivalued operator $T$. and $\sigma-$monotone function, and we investigated an existence theory for it in real reflexive Banach spaces. We introduce new models including another types of equilibrium issues. By employing the Fan–KKM technique with weak compactness args in Banach spaces, We determine sufficient requirements under hemicontinuity for the existence of solutions, convexity, and 0-diagonal convexity assumptions. The proposed results extend and improve several known existence theorems for equilibrium-type problems involving generalized monotonicity structures.

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