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GENERALIZED CONTRACTIONS WITH MAX-TYPE STRUCTURES AND NONLINEAR CONTROL FUNCTIONS IN MULTIPLICATIVE METRIC SPACES

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Abstract. In this paper, we establish new fixed point results for generalized contractive mappings in complete multiplicative metric spaces. A new Banach-type theorem under a max-type contractive structure is first obtained. We then develop a nonlinear control function approach for generalized Kannan-type contractions, extending recent results in the literature by replacing linear coefficients with nonlinear control functions. Examples are presented to illustrate the applicability of the obtained results.

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1. Introduction

The topic of multiplicative calculus was introduced by Grossman and Katz [8] in 1972. Inspired by [8], Bashirov et al. [3] introduced the notion of multiplicative metric spaces. We recall below some basic definitions related to such spaces.

Let M be a nonempty set. A mapping $D : M \times M \to [1, \infty)$ is called a multiplicative metric if for all $u, v, w \in M$, the following conditions hold:

(D1) $D(u, v) = 1$ if and only if $u = v$;

(D2) $D(u, v) = D(v, u)$;

(D3) $D(u, v) \leq D(u, w)D(w, v)$ (multiplicative triangle inequality).

Fixed point theory in multiplicative metric spaces has attracted considerable attention in recent years due to its natural connection with multiplicative calculus and its applications in various nonlinear problems. The concept of multiplicative metric spaces was introduced by Bashirov et al. [3], following the foundational work of Grossman and Katz [8]. Since then, several authors have established multiplicative analogues of classical fixed point results, including Banach, Kannan, Chatterjea, and

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Reich type contractions (see, for instance, [2, 4, 10, 12, 15]). Multiplicative metric techniques have also been applied to various problems involving integral equations, fuzzy mappings, and generalized compatibility conditions; see, for instance, [9, 14, 17].

Several variants and extensions of fixed point results in multiplicative metric spaces have been investigated, including common fixed points, rational contractions, cyclic contractions, and completeness characterizations; see, for example, [1, 6, 7, 16, 18].

Let D (resp. d) be a multiplicative metric (resp. metric) on M. We denote by d_D the mapping $d_D : M \times M \to [0, \infty)$ defined by

$$d_D(x, y) = \ln(D(x, y)), \quad \forall x, y \in M. \tag{1}$$

We denote by D_d the mapping $D_d : M \times M \to [1, \infty)$ defined by

$$D_d(x, y) = e^{d(x,y)}, \quad \forall x, y \in M. \tag{2}$$

Radenović and Samet [13] illustrated the following observations:

(i) If D is a multiplicative metric on M, then d_D is a metric on M;

(ii) If (M, D) is a complete multiplicative metric space, then (M, d_D) is a complete metric space;

(iii) If d is a metric on M, then D_d is a multiplicative metric on M;

(iv) If (M, d) is a complete metric space, then (M, D_d) is a complete multiplicative metric space.

Let M be an arbitrary nonempty set. For a given mapping $F : M \to M$, we denote by $\mathrm{Fix}(F)$ the set of fixed points of F.

For $u \in M$, the Picard sequence $\{F^n u\}$ is defined by

$$F^0 u = u, \quad F^{n+1} u = F(F^n u), \quad \forall n \geq 0.$$

Let Φ be the set of functions $\varphi : [1, \infty) \to [0, \infty)$ satisfying

$$\varphi(s) \geq c(\ln s)^p, \qquad \forall s > 1. \tag{3}$$

for some constants $c, p > 0$. From (3), we obtain that

$$\varphi(s) > 0, \quad \forall s > 1,$$

for more details see [13]. In this paper, we set

$$\Phi_0 = \{\varphi \in \Phi : \varphi(1) = 0\}.$$

A significant development in this direction in multiplicative metric spaces was made by Radenović and Samet [13], introduced new approaches to fixed point theory in multiplicative metric spaces via control functions. These results provide multiplicative analogues of classical Banach and Kannan type contraction principles under a φ-type control framework and play a central role in the development of fixed point theory in this setting, as follows:

Theorem 1.1 ([13, Theorem 2.5])**.** *Let (M, D) be a complete multiplicative metric space and let $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *There exist $\lambda \in (0, 1)$ and $\varphi \in \Phi$ such that*

$$\varphi(D(Fx, Fy)) \leq \lambda\varphi(D(x, y)), \tag{4}$$

for all $x, y \in M$ with $D(x, y) > 1$;

(ii) *For all $u, v \in M$, if*

$$\lim_{n\to\infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = 1.$$

Then F has a unique fixed point in M.

Theorem 1.2 ([13, Theorem 2.12])**.** *Let (M, D) be a complete multiplicative metric space and let $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *There exist $\lambda \in (0, \frac{1}{2})$ and $\varphi \in \Phi$ such that*

$$\varphi(D(Fx, Fy)) \leq \lambda\big(\varphi(D(x, Fx)) + \varphi(D(y, Fy))\big), \tag{5}$$

for all $x, y \in M$ with $D(x, y) > 1$;

(ii) *For all $u, v \in M$, if*

$$\lim_{n\to\infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = 1.$$

Then F has a fixed point in M.

Motivated by this line of research, several generalizations have been proposed by incorporating max-type structures, leading to broader classes of contractive mappings. These developments aim to weaken the contraction assumptions while preserving the existence and uniqueness of fixed points. Such extensions are closely related to several classical nonlinear contractive frameworks, including Meir–Keeler and Ćirić type contractions (see [5], [11]).

In this paper, we continue this line of investigation and further extend the framework introduced in [13]. Our main contribution is to establish new fixed point results for generalized Kannan-type contractions in complete multiplicative metric spaces by employing nonlinear control functions. Unlike the classical linear setting, where a constant coefficient is used, we replace it with a function ψ satisfying suitable conditions, allowing for greater flexibility in the contractive structure.

More precisely, we introduce a nonlinear inequality of the form

$$\varphi(D(Fx, Fy)) \leq \psi\big(\varphi(D(x, Fx)) + \varphi(D(y, Fy))\big),$$

for all $x, y \in M$ with $D(x, y) > 1$, which can be viewed as a functional extension of the multiplicative Kannan contraction. This approach enables us to cover a wider class of mappings that are not accessible through the existing results in [13]. Several examples are provided to illustrate the applicability of our results and to highlight the differences with the existing literature.

2. Main results

In this section, we develop new fixed point results for generalized contractions in multiplicative metric spaces. We first introduce a Banach-type theorem under a max-type structure, and subsequently extend the analysis to Kannan-type contractions with nonlinear control functions.

2.1. **A generalized Banach-type contraction via max-type structure.** We establish a new Banach-type fixed point theorem in complete multiplicative metric spaces under a max-type contractive condition. This result provides a strict extension of the multiplicative contraction principle in [13] by allowing the contractive inequality to depend on both the distance between points and the distances between points and their images.

Theorem 2.1. *Let (M, D) be a complete multiplicative metric space and $F : M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) *There exist $\lambda \in (0, 1)$ and $\varphi \in \Phi_0$ such that*

$$\varphi(D(Fx, Fy)) \leq \lambda \max\big\{\varphi(D(x, y)), \varphi(D(x, Fx)), \varphi(D(y, Fy))\big\},$$

for all $x, y \in M$ with $D(x, y) > 1$.

(ii) *For all $u, v \in M$, if*

$$\lim_{n\to\infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k} u\}$ of $\{F^n u\}$ such that

$$\lim_{k\to\infty} D(F^{n_k+1} u, Fv) = 1.$$

Then F has a unique fixed point in M.

Proof. Let $x_0 \in M$ be arbitrary and define the Picard sequence $\{x_n\}$ in M by

$$x_{n+1} = Fx_n, \qquad \forall n \geq 0.$$

If there exists $n_0 \geq 0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F. We now assume that $x_n \neq x_{n+1}, \forall n \geq 0$. By definition of the sequence $\{x_n\}$ and condition (i), we have

$$\begin{aligned}\varphi(D(x_n, x_{n+1})) &= \varphi(D(Fx_{n-1}, Fx_n)) \\ &\leq \lambda \max\Big\{\varphi(D(x_{n-1}, x_n)), \varphi(D(x_{n-1}, Fx_{n-1})), \varphi(D(x_n, Fx_n))\Big\} \\ &= \lambda \max\Big\{\varphi(D(x_{n-1}, x_n)), \varphi(D(x_n, x_{n+1}))\Big\}.\end{aligned}$$

If

$$\max\{\varphi(D(x_{n-1}, x_n)), \varphi(D(x_n, x_{n+1}))\Big\} = \varphi(D(x_n, x_{n+1})),$$

then

$$\varphi(D(x_n, x_{n+1})) \leq \lambda\varphi(D(x_n, x_{n+1})),$$

which is impossible because $\lambda \in (0, 1)$ and $\varphi(s) > 0$ for all $s > 1$. Therefore,

$$\varphi(D(x_n, x_{n+1})) \leq \lambda\varphi(D(x_{n-1}, x_n)), \qquad \forall n \geq 0.$$

It follows that

$$\varphi(D(x_n, x_{n+1})) \leq \lambda^n\varphi(D(x_0, x_1)), \quad \forall n \geq 0. \tag{6}$$

Since $\varphi \in \Phi$, there exist constants $c > 0$ and $p > 0$ such that

$$\varphi(s) \geq c(\ln s)^p, \qquad \forall s > 1. \tag{7}$$

Using (6) and (7), we have

$$c\big(\ln D(x_n, x_{n+1})\big)^p \leq \lambda^n\varphi\big(D(x_0, x_1)\big), \qquad \forall n \geq 0,$$

which implies that

$$\ln D(x_n, x_{n+1}) \leq \left(\frac{\varphi(D(x_0, x_1))}{c}\right)^{1/p} \lambda^{n/p} = Cq^n, \ \forall n \geq 0, \tag{8}$$

where $C = \left(\frac{\varphi(D(x_0,x_1))}{c}\right)^{1/p}$ and $q = \lambda^{1/p} \in (0, 1)$. We next prove that $\{x_n\}$ is a multiplicative Cauchy sequence. Let $m > n$, by the multiplicative triangle inequality and from (8), we obtain that

$$\begin{aligned}D(x_n, x_m) &\leq D(x_n, x_{n+1})\, D(x_{n+1}, x_{n+2}) \cdots D(x_{m-1}, x_m) \\ &\leq e^{Cq^n}\, e^{Cq^{n+1}} \cdots e^{Cq^{m-1}} \\ &= e^{\left(Cq^n + Cq^{n+1} + \cdots + Cq^{m-1}\right)} \\ &\leq e^{\left(\frac{Cq^n}{1-q}\right)}.\end{aligned}$$

Letting $n \to \infty$, we can conclude that

$$D(x_n, x_m) \to 1.$$

Thus $\{x_n\}$ is a multiplicative Cauchy sequence. Since (M, D) is complete, there exists $z \in M$ such that

$$\lim_{n\to\infty} D(x_n, z) = 1.$$

Using (ii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} D(x_{n_k+1}, Fz) = 1.$$

By the uniqueness of the multiplicative limit implies that $z = Fz$. Hence z is a fixed point of F in M. We finally prove the uniqueness of a fixed point of F. Suppose that $u, v \in M$ are two distinct fixed points of F. By condition (i), we obtain that

$$\begin{aligned}\varphi(D(u,v)) &= \varphi(D(Fu,Fv)) \\ &\le \lambda \max\{\varphi(D(u,v)), \varphi(D(u,Fu)), \varphi(D(v,Fv))\} \\ &\le \lambda \max\{\varphi(D(u,v)), \varphi(1), \varphi(1)\} \\ &= \lambda\varphi(D(u,v)),\end{aligned}$$

which is a contradiction since $\lambda \in (0,1)$. Therefore, the fixed point of F is unique. □

Remark 2.2. Theorem 2.1 requires the additional assumption $\varphi(1) = 0$, which is not explicitly imposed in Theorem 2.5 of [13]. This difference stems from the structure of the contractive conditions used in the two results.

In Theorem 2.5 of [13], the contractive condition

$$\varphi(D(Fx,Fy)) \le \lambda\varphi(D(x,y))$$

is assumed only for $x, y \in M$ with $D(x,y) > 1$. As explained in Remark 2.6 of [13], this restriction avoids the case $x = y$, and consequently the condition $\varphi(1) = 0$ is not required.

In contrast, Theorem 2.1 employs a max-type contractive condition involving additional terms depending on the distances between points and their images. As a consequence, the value $\varphi(1)$ naturally appears when the contractive inequality is evaluated at a fixed point. Indeed, if $x = Fx$, then

$$D(x, Fx) = 1,$$

and the contractive condition yields

$$\varphi(D(Fx,Fx)) \le \lambda \max\{\varphi(D(x,x)), \varphi(D(x,Fx)), \varphi(D(x,Fx))\}.$$

Since

$$D(Fx,Fx) = D(x,x) = D(x,Fx) = 1,$$

we obtain that

$$\varphi(1) \le \lambda\varphi(1).$$

Because $0 < \lambda < 1$, it follows that $\varphi(1) = 0$.

Therefore, the condition $\varphi(1) = 0$ in Theorem 2.1 is not merely technical, but is intrinsically linked to the generalized contractive framework. Observe that the assumption $\varphi(1) = 0$ is essential in proving the uniqueness of the fixed point of F. This highlights a fundamental structural difference between Theorem 2.1 and Theorem 2.5 in [13].

The following example demonstrates that Theorem 2.1 is strictly more general than Theorem 2.5 in [13]. In particular, the mapping satisfies the generalized max-type contractive condition introduced in this paper, while the classical multiplicative contraction principle in [13] is not applicable.

Example 2.3. Let $M = \{a_1, a_2, a_3\}$ and define a mapping $D : M \times M \to [1, \infty)$ by

$$D(a_i, a_i) = 1,\ D(a_i, a_j) = D(a_j, a_i)$$

and

$$D(a_1, a_2) = 2,\ D(a_1, a_3) = 5,\ D(a_2, a_3) = 6.$$

Define $F : M \to M$ by

$$Fa_1 = a_1,\ Fa_2 = a_3,\ \text{and } Fa_3 = a_1.$$

We first note that D is a multiplicative metric on M. Indeed, we have

$$D(a_1, a_3)D(a_3, a_2) = (5)(6) = 30 > 2 = D(a_1, a_2),$$

$$D(a_1, a_2)D(a_2, a_3) = (2)(6) = 12 > 5 = D(a_1, a_3),$$

and

$$D(a_2, a_1)D(a_1, a_3) = (2)(5) = 10 > 6 = D(a_2, a_3).$$

Since M is finite, we obtain that (M, D) is a complete multiplicative metric space. Next, define $\varphi : [1, \infty) \to [0, \infty)$ by $\varphi(s) = \ln s$. Clearly, $\varphi(1) = 0$ and $\varphi(s) > 0$ for all $s > 1$. Moreover,

$$\varphi(s) = \ln s = 1 \cdot (\ln s)^1, \quad \forall s > 1,$$

and hence $\varphi \in \Phi$ with $c = 1$ and $p = 1$. This implies that $\varphi \in \Phi_0$. We now verify the condition (i) of Theorem 2.1.

For (a_1, a_2), we have $D(Fa_1, Fa_2) = D(a_1, a_3) = 5$ and hence $\varphi(D(Fa_1, Fa_2)) = \ln 5$. Since

$$D(a_1, a_2) = 2, D(a_1, Fa_1) = D(a_1, a_1) = 1, D(a_2, Fa_2) = D(a_2, a_3) = 6,$$

we obtain that

$$\max\{\varphi(D(a_1, a_2)), \varphi(D(a_1, Fa_1)), \varphi(D(a_2, Fa_2))\} = \max\{\ln 2, 0, \ln 6\} = \ln 6.$$

This implies that

$$\varphi(D(Fa_1, Fa_2)) \leq \lambda \max\{\varphi(D(a_1, a_2)), \varphi(D(a_1, Fa_1)), \varphi(D(a_2, Fa_2))\},$$

whenever $\lambda \geq \frac{\ln 5}{\ln 6}$.

For (a_1, a_3), we have $D(Fa_1, Fa_3) = D(a_1, a_1) = 1$ and hence $\varphi(D(Fa_1, Fa_3)) = 0$. Therefore required inequality holds for every $\lambda \in (0, 1)$.

For (a_2, a_3), we have $D(Fa_2, Fa_3) = D(a_3, a_1) = 5$ and hence $\varphi(D(Fa_2, Fa_3)) = \ln 5$. Since

$$D(a_2, a_3) = 6, D(a_2, Fa_2) = D(a_2, a_3) = 6, D(a_3, Fa_3) = D(a_3, a_1) = 5,$$

we obtain that

$$\max\{\varphi(D(a_2, a_3)), \varphi(D(a_2, Fa_2)), \varphi(D(a_3, Fa_3))\} = \max\{\ln 6, \ln 6, \ln 5\} = \ln 6.$$

This implies that

$$\varphi(D(Fa_2, Fa_3)) \leq \lambda \max\{\varphi(D(a_2, a_3)), \varphi(D(a_2, Fa_2)), \varphi(D(a_3, Fa_3))\},$$

whenever $\lambda \geq \frac{\ln 5}{\ln 6}$.

Consequently, the condition (i) of Theorem 2.1 is satisfied for every $\lambda \in [\frac{\ln 5}{\ln 6}, 1)$.

We now verify the condition (ii). For every $u \in M$, we have $F^n u = a_1$ for all $n \geq 2$. Hence if $\lim_{n\to\infty} D(F^n u, v) = 1$, then $v = a_1$. Since $Fa_1 = a_1$, we obtain that

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = D(a_1, a_1) = 1.$$

for any subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$. Thus the condition (ii) holds.

Therefore, all assumptions in Theorem 2.1 are fulfilled and then F has a unique fixed point, namely a_1.

Finally, we show that Theorem 2.5 in [13] is not applicable. Indeed, if there exists $\lambda \in (0, 1)$ such that

$$\varphi(D(Fx, Fy)) \leq \lambda\varphi(D(x, y)), \text{ for all } x, y \in M \text{ with } D(x, y) > 1,$$

then we obtain that

$$\ln 5 = \varphi(D(Fa_1, Fa_2)) \leq \lambda\varphi(D(a_1, a_2)) = \lambda \ln 2.$$

This implies that $\lambda \geq \frac{\ln 5}{\ln 2} > 1$ which is a contradiction. Hence Theorem 2.5 in [13] cannot be used here, while Theorem 2.1 applies. This shows that Theorem 2.1 is an extension of Theorem 2.5 in [13].

The next example presents a more structured setting based on a max-type multiplicative metric and further illustrates the applicability of Theorem 2.1 beyond the classical contraction framework.

Example 2.4. Let $M = \{1, 2, 4\}$ and define a mapping $D : M \times M \to [1, \infty)$ by

$$D(x, y) = \max\left\{\frac{x}{y}, \frac{y}{x}\right\}, \quad \forall x, y \in M.$$

Define $F : M \to M$ by

$$F1 = 2, \quad F2 = 2, \quad F4 = 1.$$

We first verify that D is a multiplicative metric on M. Let $x, y \in M$. Since either $\frac{x}{y} \geq 1$ or $\frac{y}{x} \geq 1$, it follows that

$$D(x,y) = \max\left\{\frac{x}{y}, \frac{y}{x}\right\} \geq 1.$$

Moreover, $D(x,y) = 1$ if and only if

$$\max\left\{\frac{x}{y}, \frac{y}{x}\right\} = 1,$$

which implies $x = y$. Symmetry follows immediately from the definition, since

$$\max\left\{\frac{x}{y}, \frac{y}{x}\right\} = \max\left\{\frac{y}{x}, \frac{x}{y}\right\}, \quad \forall x, y \in M.$$

Let $x, y, z \in M$. Observe that

$$\frac{x}{y} = \frac{x}{z} \cdot \frac{z}{y} \leq \max\left\{\frac{x}{z}, \frac{z}{x}\right\} \max\left\{\frac{z}{y}, \frac{y}{z}\right\} = D(x,z)D(z,y),$$

and similarly,

$$\frac{y}{x} \leq D(x,z)D(z,y).$$

Taking the maximum of the above inequalities, we obtain that

$$D(x,y) \leq D(x,z)D(z,y).$$

Hence (M, D) is a multiplicative metric space. Since M is finite, every multiplicative Cauchy sequence in M is eventually constant and therefore is convergent. Thus (M, D) is a complete multiplicative metric space. Next, define $\varphi : [1, \infty) \to [0, \infty)$ by $\varphi(s) = \ln s$. Clearly, $\varphi(1) = 0$ and $\varphi \in \Phi$ with $c = 1$ and $p = 1$. Therefore $\varphi \in \Phi_0$. We will verify that F satisfies the conditions (i) and (ii) of Theorem 2.1. For $x, y \in M$ such that $D(x,y) > 1$, we consider all possible cases.

Case 1: $(x,y) = (1,2)$. We have $D(1,2) = 2$ and

$$D(F1, F2) = D(2,2) = 1, \quad \varphi(D(F1,F2)) = 0.$$

Since

$$\varphi(D(1,2)) = \ln 2, \quad \varphi(D(1,F1)) = \ln 2, \quad \varphi(D(2,F2)) = 0,$$

we obtain that,

$$\varphi(D(F1,F2)) \leq \lambda \max\{\varphi(D(1,2)), \varphi(D(1,F1)), \varphi(D(2,F2))\},$$

for any $\lambda \in (0,1)$.

Case 2: $(x,y) = (1,4)$. We have $D(1,4) = 4$ and

$$D(F1, F4) = D(2,1) = 2, \quad \varphi(D(F1,F4)) = \ln 2.$$

Since

$$\varphi(D(1,4)) = \ln 4, \quad \varphi(D(1,F1)) = \ln 2, \quad \varphi(D(4,F4)) = \ln 4,$$

we have

$$\varphi(D(F1,F4)) \leq \lambda \max\{\varphi(D(1,4)), \varphi(D(1,F1)), \varphi(D(4,F4))\} = \lambda \max\{\ln 4, \ln 2, \ln 4\},$$

for any $\lambda \in \left[\frac{1}{2}, 1\right)$.

Case 3: $(x,y) = (2,4)$. We have $D(2,4) = 2$ and

$$D(F2,F4) = D(2,1) = 2, \quad \varphi(D(F2,F4)) = \ln 2.$$

Since

$$\varphi(D(2,4)) = \ln 2, \varphi(D(2,F2)) = 0, \varphi(D(4,F4)) = \ln 4$$

we obtain that

$$\varphi(D(F2,F4)) \leq \lambda \max\{\varphi(D(2,4)), \varphi(D(2,F2)), \varphi(D(4,F4))\} = \lambda \max\{\ln 2, 0, \ln 4\},$$

for any $\lambda \in \left[\frac{1}{2}, 1\right)$.

From the above cases, there exists $\lambda \in (0,1)$ such that condition (i) of Theorem 2.1 is satisfied. For every $u \in M$, we have $F^n u = 2$ for all $n \geq 2$. Hence if $\lim_{n\to\infty} D(F^n u, v) = 1$, then $v = 2$. Since $F2 = 2$, we obtain that

$$\lim_{k\to\infty} D(F^{n_k+1}u, Fv) = D(2,2) = 1,$$

for any subsequence $\{F^{n_k}u\}$ of $\{F^n u\}$. Thus the condition (ii) holds. Consequently, all assumptions in Theorem 2.1 hold, and hence F admits a unique fixed point in M, namely 2. On the other hand, assume on the contrary that there exists $\lambda \in (0,1)$ such that

$$\varphi(D(Fx,Fy)) \leq \lambda \varphi(D(x,y)), \ \forall x,y \in M \text{ with } D(x,y) > 1.$$

Taking $x = 2$ and $y = 4$, we obtain that

$$\ln 2 = \varphi(D(F2,F4)) \leq \lambda \varphi(D(2,4)) = \lambda \varphi(2) = \ln 2.$$

This yields $\lambda \geq 1$ which is a contradiction. Hence Theorem 2.5 in [13] cannot be applied. This shows that Theorem 2.1 is an extension of Theorem 2.5 in [13].

2.2. **A multiplicative version of Kannan's contraction principle.** We now establish a generalized Kannan-type fixed point theorem in multiplicative metric spaces by employing a nonlinear control function. Unlike Theorem 2.12 in [13] where the contraction is governed by a linear coefficient, our approach replaces the constant parameter with a nonlinear function ψ, leading to a more flexible contractive framework.

Theorem 2.5. *Let (M, D) be a complete multiplicative metric space and let $F : M \to M$ be a given mapping. Suppose that there exists a function $\psi : [0, \infty) \to [0, \infty)$ such that*

$$\psi(0) = 0, \qquad \lim_{t \to 0^+} \psi(t) = 0,$$

$$\psi(t) < \frac{t}{2} \quad \text{and} \quad \limsup_{s \to t^+} \psi(s) < \frac{t}{2}, \quad \forall t > 0.$$

Assume that the following conditions hold:

(*i*) *There exists $\varphi \in \Phi_0$ such that*

$$\varphi(D(Fx, Fy)) \leq \psi\big(\varphi(D(x, Fx)) + \varphi(D(y, Fy))\big),$$

for all $x, y \in M$ with $D(x, y) > 1$;

(*ii*) *For all $u, v \in M$, if*

$$\lim_{n \to \infty} D(F^n u, v) = 1,$$

then there exists a subsequence $\{F^{n_k} u\}$ of $\{F^n u\}$ such that

$$\lim_{k \to \infty} D(F^{n_k + 1} u, Fv) = 1.$$

Then F has a unique fixed point in M.

Proof. Let $x_0 \in M$ be arbitrary and define the Picard sequence $\{x_n\}$ by

$$x_{n+1} = Fx_n, \quad \forall n \geq 0.$$

If there exists $n_0 \geq 0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F, and the existence follows. We may assume that

$$x_n \neq x_{n+1}, \quad \forall n \geq 0,$$

and therefore $D(x_n, x_{n+1}) > 1$ for all $n \geq 0$. Put

$$u_n = \varphi(D(x_n, x_{n+1})), \quad \forall n \geq 0.$$

Applying (i), we obtain that

$$\begin{aligned} u_{n+1} &= \varphi(D(Fx_n, Fx_{n+1})) \\ &\leq \psi\big(\varphi(D(x_n, Fx_n)) + \varphi(D(x_{n+1}, Fx_{n+1}))\big). \\ &= \psi(u_n + u_{n+1}). \end{aligned}$$

Since $u_n + u_{n+1} > 0$ and $\psi(t) < t/2$ for all $t > 0$, we have

$$u_{n+1} < \frac{u_n + u_{n+1}}{2},$$

which implies

$$u_{n+1} < u_n, \quad \forall n \geq 0.$$

Thus $\{u_n\}$ is decreasing and bounded below by 0. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} u_n = r.$$

We claim that $r = 0$. Suppose, to the contrary that $r > 0$. Since $\{u_n\}$ is decreasing and converges to r, we have

$$u_n + u_{n+1} \geq 2r, \;\forall n \geq 0,$$

and hence $u_n + u_{n+1} \to 2r^+$. This implies that

$$r = \lim_{n\to\infty} u_{n+1} \leq \limsup_{n\to\infty} \psi(u_n + u_{n+1}) \leq \limsup_{s\to(2r)^+} \psi(s) < \frac{2r}{2} = r,$$

which is a contradiction. Hence $r = 0$ and therefore

$$\lim_{n\to\infty} \varphi(D(x_n, x_{n+1})) = \lim_{n\to\infty} u_n = 0.$$

We next prove that $\{x_n\}$ is a multiplicative Cauchy sequence. Suppose not, then there exist $\varepsilon > 1$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $m_k > n_k \geq 1$ such that

$$D(x_{n_k}, x_{m_k}) \geq \varepsilon, \quad \forall k \geq 1.$$

Since $\varphi \in \Phi$, there exist constants $c > 0$ and $p > 0$ such that

$$\varphi(s) \geq c(\ln s)^p, \quad \forall s > 1.$$

Hence

$$\varphi(D(x_{n_k}, x_{m_k})) \geq c(\ln \varepsilon)^p > 0.$$

We now justify that the contractive condition can be applied to the pair (x_{n_k-1}, x_{m_k-1}). Since $D(x_{n_k}, x_{m_k}) \geq \varepsilon > 1$, we have $x_{n_k} \neq x_{m_k}$. If $x_{n_k-1} = x_{m_k-1}$, then applying F would give $x_{n_k} = x_{m_k}$, a contradiction. Thus

$$x_{n_k-1} \neq x_{m_k-1} \text{ and hence } D(x_{n_k-1}, x_{m_k-1}) > 1.$$

Applying the condition (i), we obtain that

$$\varphi(D(x_{n_k}, x_{m_k})) \leq \psi(u_{n_k-1} + u_{m_k-1}).$$

Since $u_n \to 0$, we have

$$u_{n_k-1} + u_{m_k-1} \to 0.$$

By $\lim_{t\to 0^+} \psi(t) = 0$, it follows that

$$\psi(u_{n_k-1} + u_{m_k-1}) \to 0,$$

which implies

$$\varphi(D(x_{n_k}, x_{m_k})) \to 0,$$

a contradiction. Hence $\{x_n\}$ is a multiplicative Cauchy sequence.

By the completeness of M, there exists $z \in M$ such that

$$\lim_{n\to\infty} D(x_n, z) = 1.$$

By the condition (ii), there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} D(x_{n_k+1}, Fz) = 1.$$

By the uniqueness of the multiplicative limit, we conclude that

$$Fz = z.$$

Finally, let $u, v \in M$ be two fixed points of F with $u \neq v$. Then $D(u, v) > 1$. Applying the condition (i), we get that

$$\varphi(D(u,v)) \leq \psi(\varphi(1) + \varphi(1)) = \psi(0) = 0,$$

which contradicts $\varphi(D(u, v)) > 0$. Hence the fixed point of F is unique. □

The following example illustrates the applicability of Theorem 2.5 and shows that the nonlinear control framework introduced in this paper covers cases that cannot be treated by Theorem 2.12 in [13].

Example 2.6. Let

$$M = \{0\} \cup \{x_n : n \in \mathbb{N}\}.$$

Define a metric d on M by

$$d(0, x_n) = \frac{1}{n}, \qquad d(x_m, x_n) = \frac{1}{m} + \frac{1}{n} \quad (m \neq n),$$

and define a mapping $D : M \times M \to [1, \infty)$ by

$$D(x, y) = e^{d(x,y)}, \quad \forall x, y \in M.$$

Then (M, D) is a complete multiplicative metric space.

Define $F : M \to M$ by

$$F0 = 0, \qquad Fx_n = x_{n+1}, \quad \forall n \in \mathbb{N}$$

and define $\varphi : [1, \infty) \to [0, \infty)$ by

$$\varphi(s) = \ln s, \quad \forall s \geq 1.$$

Then $\varphi \in \Phi_0$. Define $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} \dfrac{t}{2} - \dfrac{t^2}{100}, & 0 \leq t \leq 3, \\ \dfrac{t}{4}, & t > 3. \end{cases}$$

Then we obtain that

$$\psi(0) = 0, \qquad \lim_{t\to 0^+} \psi(t) = 0, \qquad \psi(t) < \frac{t}{2}, \quad \forall t > 0,$$

and

$$\limsup_{s\to t^+}\psi(s)<\frac{t}{2},\quad \forall t>0.$$

We now verify the condition (i) of Theorem 2.5.

For $x=x_n$ and $y=0$, we have

$$\varphi(D(Fx_n,F0))=d(x_{n+1},0)=\frac{1}{n+1}.$$

Moreover, we obtain that

$$\varphi(D(x_n,Fx_n))+\varphi(D(0,F0))=d(x_n,x_{n+1})=\frac{1}{n}+\frac{1}{n+1}.$$

Put

$$T_n=\frac{1}{n}+\frac{1}{n+1}.$$

Then

$$\frac{1}{n+1}\le\frac{T_n}{2}-\frac{T_n^2}{100}=\psi(T_n).$$

Next, let $m\neq n$. It follows that

$$\varphi(D(Fx_m,Fx_n))=d(x_{m+1},x_{n+1})=\frac{1}{m+1}+\frac{1}{n+1}.$$

Also,

$$\varphi(D(x_m,Fx_m))+\varphi(D(x_n,Fx_n))=\left(\frac{1}{m}+\frac{1}{m+1}\right)+\left(\frac{1}{n}+\frac{1}{n+1}\right).$$

Put

$$T_{m,n}=\left(\frac{1}{m}+\frac{1}{m+1}\right)+\left(\frac{1}{n}+\frac{1}{n+1}\right).$$

A direct computation gives

$$\frac{1}{m+1}+\frac{1}{n+1}\le\frac{T_{m,n}}{2}-\frac{T_{m,n}^2}{100}=\psi(T_{m,n}).$$

Hence the condition (i) of Theorem 2.5 holds.

Moreover, for every $u\in M$, the Picard sequence $\{F^nu\}$ converges to 0. Hence if $\lim_{n\to\infty}D(F^nu,v)=1$, then $v=0$. Since $F0=0$, we obtain that

$$\lim_{k\to\infty}D(F^{n_k+1}u,F0)=1,$$

for any subsequence $\{F^{n_k}u\}$ of $\{F^nu\}$. Thus the condition (ii) of Theorem 2.5 is satisfied. Therefore F has a unique fixed point, namely 0.

We now show that Theorem 2.12 in [13] is not applicable. Suppose that there exists $\lambda\in(0,\frac{1}{2})$ such that

$$\varphi(D(Fx,Fy))\le\lambda\big(\varphi(D(x,Fx))+\varphi(D(y,Fy))\big),$$

for all $x,y\in M$ with $D(x,y)>1$. Taking $x=x_n$ and $y=0$, we obtain

$$\frac{1}{n+1}\le\lambda\left(\frac{1}{n}+\frac{1}{n+1}\right).$$

Thus

$$\lambda \geq \frac{n}{2n+1}.$$

Letting $n \to \infty$, we get

$$\lambda \geq \frac{1}{2},$$

which contradicts $\lambda < \frac{1}{2}$. Hence Theorem 2.12 in [13] cannot be applied.

Remark 2.7. Theorem 2.5 can be viewed as a nonlinear extension of Theorem 2.12 in [13]. Indeed, if one chooses

$$\psi(t) = \lambda t, \qquad 0 < \lambda < \frac{1}{2},$$

then

$$\psi(0) = 0, \qquad \lim_{t \to 0^+} \psi(t) = 0, \qquad \psi(t) < \frac{t}{2}$$

and

$$\limsup_{s \to t^+} \psi(s) = \lambda t < \frac{t}{2},$$

for all $t > 0$. In this case, the contractive condition (i) in Theorem 2.5 is reduced to

$$\varphi(D(Fx, Fy)) \leq \lambda\big(\varphi(D(x, Fx)) + \varphi(D(y, Fy))\big),$$

which is precisely the multiplicative Kannan-type contraction considered in Theorem 2.12 of [13]. Hence Theorem 2.5 strictly extends Theorem 2.12 by replacing the linear coefficient λ with a nonlinear control function ψ.

Conclusion

In this paper, we established new fixed point results for generalized contractions in complete multiplicative metric spaces. By employing max-type structures and nonlinear control functions, we obtained extensions of recent Banach-type and Kannan-type contraction principles in the literature. Several examples were presented to illustrate the applicability of the proposed framework and to demonstrate that the new results genuinely extend existing theorems.

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