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β_θ–$\mathfrak{I}$-OPEN SETS: A UNIFIED APPROACH TO COVERING PROPERTIES IN IDEAL TOPOLOGICAL SPACES

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Abstract. This study investigates covering properties in ideal topological spaces (ITP spaces). Central to our approach is the notion of a β_θ-$\mathfrak{I}$-open set, defined by $A \subseteq \mathrm{Cl}_\theta(\mathrm{Int}^*(\mathrm{Cl}_\theta(A)))$, where Cl_θ is the θ-closure and Int^* is the $*$-interior induced by the ideal $\mathfrak{I}$. We introduce β_θ-$\mathfrak{I}$-compactness and β_θ-$\mathfrak{I}$-paracompactness, which generalize both classical and existing ITP notions. Several characterizations are established via β_θ-$\mathfrak{I}$-open sets and locally finite refinements. We examine their behavior under unions, subspaces, and mappings including β_θ-irresolute and β_θ-$\mathfrak{I}$-open/closed functions, and explore their connections with β_θ-$\mathfrak{I}$-regularity and β_θ-$\mathfrak{I}$-normality. The results unify and extend classical compactness and paracompactness theories, enriching generalized topology with tools applicable to continuity, separation axioms, and ideal-based structural investigations.

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1. Introduction

Topology, originating as a branch of pure mathematics, has evolved into a foundational discipline with far-reaching applications across information science [35], biochemistry [6], and engineering design [25, 30]. Within this broad landscape, general topology provides essential frameworks for analyzing continuity, convergence, and covering properties—concepts that underpin both theoretical advances and practical tools in fields such as computational topology, geometric modeling, and system verification. Despite this maturity, the systematic unification of two fundamental covering properties—compactness and paracompactness—within enriched topological settings remains an ongoing challenge that motivates the present work.

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A central trend in modern topology is the enhancement of classical concepts through the introduction of additional structures. In this context, ideal theory has proven to be particularly fruitful. Ideals—families of subsets closed under finite unions and hereditary—provide a natural way to formalize concepts of "smallness" or "negligibility." Pioneering efforts by Vancin [37], Samuels [31], and Hamlett and Janković [17, 18] demonstrated the efficacy of topological ideals in extending classical concepts. By the 1960s, this framework had been systematically extended to investigate a broad range of topological properties, including compactness, paracompactness, connectedness, hyperconnectedness, submaximality, resolvability, and separation axioms.

The formal theory of ideal topological spaces—abbreviated as ITP spaces—traces its origins to Kuratowski [22] and was later advanced by Vaidyanathaswamy [36], who extended classical topology through the addition of an ideal-based framework. Recently, Boonpok [7] introduced the notion of $*$-hyperconnected spaces within this setting, whereas Özkoç et al. [28] investigated connectedness in primal topological spaces and utilized primals in connection with rough operators. In a separate direction, Al-Omari and Alqurashi [2, 3] developed a theory for soft ITP spaces, whereas Boonpok and Sama-Ae [10] introduced submaximality and connectedness in ITP spaces.

Compactness stands as one of the most fundamental concepts in topology, capturing the principle that every open cover admits a finite subcover. This notion extends the familiar Heine–Borel characterization of closed and bounded subsets of Euclidean space to arbitrary topological spaces and guarantees essential properties such as the existence of extreme values for continuous functions and the convergence of subsequences. Newcomb [26] was the first to examine compactness in relation to an ideal, a direction later pursued in depth by Rancin [29] and by Hamlett and Janković [17, 18], who developed a comprehensive theory of ideal-based compactness. More recently, Gupta and Kaur [15] introduced the notion of compact spaces relative to an ideal and systematically examined their characteristics.

Paracompactness, introduced by Dieudonné [12], provides a natural generalization that subsumes both compact and metrizable spaces. A topological space is paracompact if every open cover admits a locally finite open refinement. Within Hausdorff spaces, this property is equivalent to the existence of partitions of unity and guarantees normality [14], making it indispensable in analysis, geometry, and topology. Numerous variations have emerged, including S-paracompactness [4, 23], P_3-paracompactness [5], β-paracompactness [11], and C-paracompactness [24], each offering distinct perspectives on covering properties.

The investigation of paracompactness in ITP spaces was first undertaken by Zahid [39] and later extended by Hamlett et al. [16], Sathiyasundari and Renukadevi [33], and Sanabria et al. [32], the last of whom introduced the concept of $\mathfrak{I}$-S-paracompactness. Following Demir and Özbakir's [11] definition of β-paracompact spaces, their ideal-based counterparts were examined by Yildirim et al. [38].

Most recently, Boonpok and Sama-Ae [8,9] introduced the notions of b^*-$\mathfrak{J}$-paracompactness and b_1^*-$\mathfrak{J}$-paracompactness as extensions of classical paracompactness based on b^*-$\mathfrak{J}$-open sets and ideal-induced refinements. They also studied paracompactness in ideal topological spaces using $\delta\beta^*$-$\mathfrak{J}$-open sets.

Collectively, these developments reflect a sustained effort to refine covering properties in ITP spaces by employing increasingly sophisticated classes of generalized open sets. However, a critical examination reveals that many of these concepts have been developed in isolation, resulting in a fragmented landscape where interconnections between different generalizations remain underexplored. A general framework that both extends and integrates compactness and paracompactness within a unified setting has yet to be formulated. In particular, the role of a sufficiently adaptable class of generalized open sets in offering a unified characterization of these covering properties remains unexplored.

To address this gap, the present work introduces and systematically investigates the notions of β_θ-$\mathcal{I}$-compactness and β_θ-$\mathcal{I}$-paracompactness in ITP spaces. The choice of β_θ-$\mathcal{I}$-open sets is motivated by their inherent flexibility: they simultaneously capture topological, ideal-theoretic, and θ-type information, making them particularly well-suited for unifying existing generalizations. Through the development of fundamental characterizations and the establishment of key properties, this study aims to provide a cohesive framework for investigating covering properties in ideal topological spaces. The results obtained not only enrich the structural theory of ITP spaces but also furnish new tools for further exploration in areas such as generalized continuity, separation axioms, and related ideal-theoretic structures.

2. Preliminaries

In this work, let (T, ϕ) be a topological space. By an ideal $\mathfrak{J}$ on T, we mean a nonempty collection of subsets of T which is closed under finite unions and satisfies the hereditary property: (i) if $E_1, E_2 \in \mathfrak{J}$, then $E_1 \cup E_2 \in \mathfrak{J}$; (ii) if $E \in \mathfrak{J}$ and $F \subseteq E$, then $F \in \mathfrak{J}$. Take a topological space (T, ϕ) and add an ideal $\mathfrak{J}$ to it. The resulting triple $(\mathrm{T}, \phi, \mathfrak{J})$ is what we call an ideal topological space. The notation $\mathrm{Cl}(E)$ stands for the closure of $E \subseteq \mathrm{T}$, while $\mathrm{Int}(E)$ represents its interior.

The θ-closure operator is fundamental to the study of generalized open sets in topology. The θ-closure of a set B, denoted by $\mathrm{Cl}_\theta(B)$ [27], is an operator defined with respect to the original topology and is given by

$$\mathrm{Cl}_\theta(B) = \{z \in \mathrm{T} : \mathrm{Cl}(G) \cap B \neq \emptyset \text{ for every } G \in \phi(z)\},$$

where $\phi(z)$ denotes the family of all open neighborhoods of z. It is important to note that $\mathrm{Cl}_\theta(B)$ is always a closed set in the space (T, ϕ), and for every $B \subseteq \mathrm{T}$, the inclusion $\mathrm{Cl}(B) \subseteq \mathrm{Cl}_\theta(B)$ holds. Furthermore, if B is open, then $\mathrm{Cl}(B)$ coincides with $\mathrm{Cl}_\theta(B)$ [1].

A set operator $(\cdot)^* : \mathrm{P(T)} \to \mathrm{P(T)}$, called the *local function* [22], is defined in terms of the topology ϕ and the ideal $\mathfrak{I}$ by

$$B^*(\mathfrak{I}, \phi) = \{z \in \mathrm{T} : G \cap B \notin \mathfrak{I} \text{ for all } G \in \phi(z)\},$$

where $B \subseteq \mathrm{T}$, $\mathrm{P(T)}$ denotes the power set of T. Through the operator $(\cdot)^*$ we refine the topology of T to the $*$-topology $\phi^*(\mathfrak{I})$. Its closure operator is $\mathrm{Cl}^*(B) = B \cup B^*(\mathfrak{I}, \phi)$ and its interior operator is $\mathrm{Int}^*(B)$ [19]. A subbasis for this topology is $\{G - A : G \in \phi \text{ and } A \in \mathfrak{I}\}$.

Building upon this framework, we combine the θ-closure operator $\mathrm{Cl}_\theta(\cdot)$ from the topological space (T, ϕ) with the ideal-based interior operator $\mathrm{Int}^*(\cdot)$ in the **ITP** space $(\mathrm{T}, \phi, \mathcal{I})$ to introduce a more general notion of openness. This new concept extends the family of β^*-$\mathcal{I}$-open sets—widely recognized as one of the most general classes in the hierarchy of generalized open sets in ideal topological spaces—and further incorporates various other classes of open sets inherent to ideal topological structures. Notably, this unified approach subsumes both classical $\mathcal{I}$-open sets and θ-open sets as special cases, thereby establishing a more comprehensive framework for the study of separation axioms and covering properties.

Definition 2.1. [34] *Given an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$, *a subset* $B \subseteq \mathrm{T}$ *is said to be* β_θ-$\mathfrak{I}$-*open if it satisfies the inclusion* $B \subseteq \mathrm{Cl}_\theta(\mathrm{Int}^*(\mathrm{Cl}_\theta(B)))$. *We say a set is* β_θ-$\mathfrak{I}$-*closed precisely when its complement is* β_θ-$\mathfrak{I}$-*open.*

By Definition 2.1, in an ITP space $(\mathrm{T}, \phi, \mathfrak{I})$, openness in (T, ϕ) implies β_θ-$\mathfrak{I}$-openness. Thus, each closed set is β_θ-$\mathfrak{I}$-closed. Furthermore, arbitrary unions of β_θ-$\mathfrak{I}$-open sets remain β_θ-$\mathfrak{I}$-open, and arbitrary intersections of β_θ-$\mathfrak{I}$-closed sets remain β_θ-$\mathfrak{I}$-closed.

Example 2.1. *Let* ϕ *be the discrete topology on* T *and* $\mathfrak{I} = \{\emptyset\}$. *Then* $\mathrm{Int}^*(B) = \mathrm{Int}(B)$ *and* $\mathrm{Cl}_\theta(B) = \mathrm{Cl}(B)$ *for every subset* $B \subseteq \mathrm{T}$. *Hence, a subset* $B \subseteq \mathrm{T}$ *is* β_θ-$\mathfrak{I}$-*open in the* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *if and only if it is open in* (T, ϕ).

We provide an example of a β_θ-$\mathfrak{I}$-open set that fails to be open in the underlying topological space.

Example 2.2. *Let* $\mathrm{T} = \{1, 2, 3\}$ *and let* $\phi = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \mathrm{T}\}$ *be a topology on* T. *Consider the ideal* $\mathfrak{I} = \{\emptyset, \{3\}\}$. *Then the set* $\{3\}$ *is* β_θ-$\mathfrak{I}$-*open, although it is not open in the topological space* (T, ϕ).

The following example illustrates that β_θ–$\mathfrak{I}$-openness is not preserved under taking arbitrary subsets or finite intersections.

Example 2.3. *Consider* $\mathbb{R}$ *with the usual topology and the trivial ideal* $\mathfrak{I} = \{\emptyset\}$. *The set of integers* $\mathbb{Z}$ *satisfies* $\mathrm{Cl}_\theta(\mathbb{Z}) = \mathbb{Z}$ *and* $\mathrm{Int}^*(\mathrm{Cl}_\theta(\mathbb{Z})) = \emptyset$, *so* $\mathbb{Z} \nsubseteq \mathrm{Cl}_\theta(\mathrm{Int}^*(\mathrm{Cl}_\theta(\mathbb{Z})))$. *Hence* $\mathbb{Z}$ *is not* β_θ-$\mathfrak{I}$-*open.*

However, the half-lines $(-\infty, a]$ *and* $[a, \infty)$ *are* β_θ-$\mathfrak{I}$-*open, while their intersection* a *is not, showing that* β_θ-$\mathfrak{I}$-*open sets are not closed under finite intersections.*

Definition 2.2. [34] *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and a subset* $B \subseteq \mathrm{T}$. *We define:*

(1) *The* β_θ-$\mathfrak{I}$-closure *of* B *as* $\mathrm{Cl}^\beta_\theta(B) := \bigcap\{F \subseteq \mathrm{T} : F$ *is* β_θ-$\mathfrak{I}$*-closed and* $B \subseteq F\}$.

(2) *The* β_θ-$\mathfrak{I}$-interior *of* B *as* $\mathrm{Int}^\beta_\theta(B) := \bigcup\{V \subseteq \mathrm{T} : V$ *is* β_θ-$\mathfrak{I}$*-open and* $V \subseteq B\}$.

The definitions of the β_θ–$\mathfrak{I}$-closure and the β_θ–$\mathfrak{I}$-interior directly imply the following lemma.

Lemma 2.1. [34] *For* $B \subseteq \mathrm{T}$ *in an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$, *we have* $\mathrm{Cl}^\beta_\theta(B) = \mathrm{T} - \mathrm{Int}^\beta_\theta(\mathrm{T} - B)$.

The next two lemmas are crucial for establishing the paracompactness results presented in the next section.

Lemma 2.2. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and let* $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ *be a family of subsets of* T. *Suppose that the finite intersection of* β_θ-$\mathfrak{I}$*-open subsets of* T *is again* β_θ-$\mathfrak{I}$*-open. Then*

$$\mathrm{Cl}^\beta_\theta\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) = \bigcup_{i=1}^{n} \mathrm{Cl}^\beta_\theta(U_{\alpha_i}).$$

Proof. We prove both inclusions.

Let $z \in \mathrm{Cl}^\beta_\theta(\bigcup_{i=1}^n U_{\alpha_i})$. Suppose, for contradiction, that $z \notin \bigcup_{i=1}^n \mathrm{Cl}^\beta_\theta(U_{\alpha_i})$. That means, for every $i \in \{1, 2, \ldots, n\}$, $z \notin \mathrm{Cl}^\beta_\theta(U_{\alpha_i})$. By the definition of $\mathrm{Cl}^\beta_\theta(U_{\alpha_i})$, there exists a β_θ-$\mathfrak{I}$-open set W_i containing z such that $W_i \cap U_{\alpha_i} = \emptyset$. Thus we have a finite collection of β_θ-$\mathfrak{I}$-open sets $W_1, W_2, \ldots, W_n$, each containing z, with $W_i \cap U_{\alpha_i} = \emptyset$ for all i. Set $W = W_1 \cap W_2 \cap \cdots \cap W_n$. By assumption W is a β_θ-$\mathfrak{I}$-open set containing z. Then:

$$W \cap \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) = \bigcup_{i=1}^{n}(W \cap U_{\alpha_i}) \subseteq \bigcup_{i=1}^{n}(W_i \cap U_{\alpha_i}) = \bigcup_{i=1}^{n} \emptyset = \emptyset.$$

But this contradicts the assumption that $z \in \mathrm{Cl}^\beta_\theta(\bigcup_{i=1}^n U_{\alpha_i})$, which requires that every β_θ-$\mathfrak{I}$-open set containing z intersects $\bigcup_{i=1}^n U_{\alpha_i}$. Therefore our supposition is false, and we must have $z \in \bigcup_{i=1}^n \mathrm{Cl}^\beta_\theta(U_{\alpha_i})$.

Conversely, let $z \in \bigcup_{i=1}^n \mathrm{Cl}^\beta_\theta(U_{\alpha_i})$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that $z \in \mathrm{Cl}^\beta_\theta(U_{\alpha_j})$. Thus every open neighborhood W of z intersects U_{α_j}, and hence $W \cap (\bigcup_{i=1}^n U_{\alpha_i}) \neq \emptyset$. Therefore, $z \in \mathrm{Cl}^\beta_\theta(\bigcup_{i=1}^n U_{\alpha_i})$. □

The subsequent lemma establishes the preservation of β_θ-$\mathfrak{I}$-closure under unions of β_θ-$\mathfrak{I}$-locally finite β_θ-$\mathfrak{I}$-open set refinements.

Lemma 2.3. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$. *Suppose that the finite intersection of* β_θ-$\mathfrak{I}$*-open subsets of* T *is again* β_θ-$\mathfrak{I}$*-open. Then the* β_θ-$\mathfrak{I}$*-closure is preserved under unions of* β_θ-$\mathfrak{I}$*-locally finite* β_θ-$\mathfrak{I}$*-open set refinements; that is, for every such family* $\{U_\alpha : \alpha \in \Omega\}$, *the following equality holds:*

$$\bigcup_{\alpha \in \Omega} \mathrm{Cl}^\beta_\theta(U_\alpha) = \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha \in \Omega} U_\alpha\right).$$

Proof. For each $\alpha \in \Omega$, we clearly have $U_\alpha \subseteq \bigcup_{\beta\in\Omega} U_\beta$ and then $\mathrm{Cl}_\theta^\beta(U_\alpha) \subseteq \mathrm{Cl}_\theta^\beta\left(\bigcup_{\beta\in\Omega} U_\beta\right)$. Taking the union over all $\alpha \in \Omega$, we obtain $\bigcup_{\alpha\in\Omega} \mathrm{Cl}_\theta^\beta(U_\alpha) \subseteq \mathrm{Cl}_\theta^\beta\left(\bigcup_{\alpha\in\Omega} U_\alpha\right)$.

Conversely, let $z \in \mathrm{Cl}_\theta^\beta\left(\bigcup_{\alpha\in\Omega} U_\alpha\right)$. By the definition of β_θ-$\mathfrak{I}$-closure, every β_θ-$\mathfrak{I}$-open neighborhood W of z satisfies $W \cap \left(\bigcup_{\alpha\in\Omega} U_\alpha\right) \neq \emptyset$. Since the family $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite, one can find a β_θ-$\mathfrak{I}$-open neighborhood V of z and $\{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq \Omega$ such that

$$V \cap U_\alpha \neq \emptyset \quad \text{for all} \quad \alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \quad \text{and} \quad V \cap U_\alpha = \emptyset \quad \text{otherwise.}$$

Let W be a β_θ-$\mathfrak{I}$-open set containing z and set $S = W \cap V$. Therefore, S is a β_θ-$\mathfrak{I}$-open set that contains z. Since $z \in \mathrm{Cl}_\theta^\beta(\bigcup_{\alpha\in\Omega} U_\alpha)$, it follows that $S \cap \left(\bigcup_{\alpha\in\Omega} U_\alpha\right) \neq \emptyset$. By the choice of V, we get that $S \cap (\bigcup_{i=1}^n U_{\alpha_i}) \neq \emptyset$, and hence $W \cap (\bigcup_{i=1}^n U_{\alpha_i}) \neq \emptyset$. Therefore, $z \in \mathrm{Cl}_\theta^\beta(\bigcup_{i=1}^n U_{\alpha_i})$. Using Lemma 2.2, we have $\mathrm{Cl}_\theta^\beta(\bigcup_{i=1}^n U_{\alpha_i}) = \bigcup_{i=1}^n \mathrm{Cl}_\theta^\beta(U_{\alpha_i})$, and consequently, $z \in \bigcup_{i=1}^n \mathrm{Cl}_\theta^\beta(U_{\alpha_i}) \subseteq \bigcup_{\alpha\in\Omega} \mathrm{Cl}_\theta^\beta(U_\alpha)$. Hence, $\mathrm{Cl}_\theta^\beta\left(\bigcup_{\alpha\in\Omega} U_\alpha\right) \subseteq \bigcup_{\alpha\in\Omega} \mathrm{Cl}_\theta^\beta(U_\alpha)$.

This completes the proof. □

To examine how compactness and paracompactness are preserved between ITP spaces, we first develop generalized notions of open, closed and continuous functions.

Definition 2.3. *Consider* ITP *spaces* $(\mathrm{T}, \phi, \mathfrak{I})$ *and* $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. *A function* $g : (\mathrm{T}, \phi, \mathfrak{I}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ *is said to be:*

(1) β_θ-$\mathfrak{I}$-open (*resp.* β_θ-$\mathfrak{I}$-closed) *if the image of each* β_θ-$\mathfrak{I}$*-open* (*resp.* β_θ-$\mathfrak{I}$*-closed*) *set in* T *is* β_θ-$\mathfrak{J}$*-open* (*resp.* β_θ-$\mathfrak{J}$*-closed*) *in* $\widetilde{\mathrm{T}}$;

(2) β_θ-irresolute *if the preimage of every* β_θ-$\mathfrak{J}$*-open set in* $\widetilde{\mathrm{T}}$ *is* β_θ-$\mathfrak{I}$*-open in* T.

Remark 2.1. *Consider two topological spaces* (T, ϕ) *and* $(\widetilde{\mathrm{T}}, \rho)$, *and a function* $g\colon (\mathrm{T}, \phi) \to (\widetilde{\mathrm{T}}, \rho)$.

(1) *If* $\widetilde{\mathrm{T}}$ *carries an ideal* $\mathfrak{J}$, *then the family*

$$g^{-1}(\mathfrak{J}) = \{\, E \subseteq \mathrm{T} : g(E) \in \mathfrak{J} \,\}$$

is an ideal on T.

(2) *If* T *carries an ideal* $\mathfrak{I}$ *and* g *is surjective, then the family*

$$g(\mathfrak{I}) = \{\, g(F) : F \in \mathfrak{I} \,\}$$

constitutes an ideal on $\widetilde{\mathrm{T}}$.

Remark 2.2. *Consider* ITP *spaces* $(\mathrm{T}, \phi, \mathfrak{I})$ *and* $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$, *and a bijective function* $g\colon (\mathrm{T}, \phi, \mathfrak{I}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$.

(1) *If* g *is* β_θ*-irresolute, then its inverse* g^{-1} *is both* β_θ-$\mathfrak{J}$*-open and* β_θ-$\mathfrak{J}$*-closed.*

(2) *For the bijective* g, *the properties of being* β_θ-$\mathfrak{I}$*-open and* β_θ-$\mathfrak{I}$*-closed are equivalent.*

3. β_θ-$\mathfrak{I}$-Compactness and its Characterizations

Compactness is a fundamental property in topology that generalizes the notion of finiteness and extends the classical concept of closed and bounded sets in Euclidean spaces. We say a topological space is compact provided that for every open cover, there exists a finite subfamily that still covers the space. This concept is central to analysis, as it ensures the existence of limit points, guarantees that continuous functions attain their maximum and minimum values, and serves as a foundational tool for establishing many important results in topology, functional analysis, and differential geometry.

This section is devoted to the introduction of a generalized form of compactness in ITP spaces, together with several results that generalize known compactness theorems in both topological and ideal topological settings.

Definition 3.1. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and* $K \subseteq \mathrm{T}$. *The set* K *is called* β_θ-$\mathfrak{I}$-compact *if for every cover* $\{G_\alpha : \alpha \in \Omega\}$ *of* K *consisting of* β_θ-$\mathfrak{I}$-*open sets, has a finite subcollection* $\{G_\alpha : \alpha \in \Omega_1\}$ *with* $\Omega_1 \subseteq \Omega$ *that still covers* K.

The set K *is said to be* compatibly β_θ-$\mathfrak{I}$-compact *if for every* β_θ-$\mathfrak{I}$-*open cover* $\{G_\alpha : \alpha \in \Omega\}$ *of* K *has a finite subfamily* $\{G_\alpha : \alpha \in \Omega_1\}$ *with* $\Omega_1 \subseteq \Omega$ *such that* $K - \bigcup_{\alpha \in \Omega_1} G_\alpha \in \mathfrak{I}$.

From Definition 3.1, the following result follows immediately.

Theorem 3.1. *The* β_θ-$\mathfrak{I}$-*compactness of an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *implies that* (T, ϕ) *is compact.*

The implication in Theorem 3.1 cannot be inverted, as shown by Example 3.1.

Example 3.1. *Consider* $\mathrm{T} = \mathbb{N}$, *endowed with the indiscrete topology* $\phi = \{\emptyset, \mathrm{T}\}$, *and let* $\mathfrak{I} = \{\emptyset\}$ *be the trivial ideal. Then the topological space* (T, ϕ) *is compact.*

In the ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$, *every subset of* T *is* β_θ-$\mathfrak{I}$-*open. Consider the cover by singletons,* $\{\{z\} : z \in \mathrm{T}\}$, *which is a* β_θ-$\mathfrak{I}$-*open cover of* T. *This cover admits no finite subcover. Consequently,* $(\mathrm{T}, \phi, \mathfrak{I})$ *is not* β_θ-$\mathfrak{I}$-*compact. This shows that compactness of* (T, ϕ) *does not, in general, imply* β_θ-$\mathfrak{I}$-*compactness.*

By Definition 3.1 and considering the special case of the trivial ideal on T, we obtain the following results.

Theorem 3.2. *Every* β_θ-$\mathfrak{I}$-*compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is automatically compatibly* β_θ-$\mathfrak{I}$-*compact.*

Theorem 3.3. *In an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *with the ideal* $\mathfrak{I} = \{\emptyset\}$, *a subset of* T *is compatibly* β_θ-$\mathfrak{I}$-*compact if and only if it is* β_θ-$\mathfrak{I}$-*compact.*

Theorem 3.2 cannot be reversed, as demonstrated by the following example.

Example 3.2. *Consider* $\mathrm{T} = \mathbb{N}$ *with the indiscrete topology* $\phi = \{\emptyset, \mathrm{T}\}$ *and the ideal* $\mathfrak{I} = \mathrm{P}(\mathrm{T})$*. One can readily verify that every subset of* T *is* β_θ*-*$\mathfrak{I}$*-open. Consequently, the family of singletons* $\{\{z\} : z \in \mathrm{T}\}$ *forms a* β_θ*-*$\mathfrak{I}$*-open cover of* T *that admits no finite subcover. Hence, the* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *does not satisfy* β_θ*-*$\mathfrak{I}$*-compactness.*

However, $(\mathrm{T}, \phi, \mathfrak{I})$ *is compatibly* β_θ*-*$\mathfrak{I}$*-compact. Indeed, for any* β_θ*-*$\mathfrak{I}$*-open cover* $\{G_\alpha : \alpha \in \Omega\}$ *of* T*, we may select an arbitrary element* G_{α_0}*, for which* $\mathrm{T} - G_{\alpha_0} \in \mathfrak{I}$*. Hence the subfamily* $\{G_{\alpha_0}\}$ *satisfies the compatibility condition.*

Therefore, compatible β_θ*-*$\mathfrak{I}$*-compactness does not, in general, imply* β_θ*-*$\mathfrak{I}$*-compactness.*

From a basic ideal property on T we immediately get:

Theorem 3.4. *Compatibly* β_θ*-*$\mathfrak{I}$*-compactness of* $(\mathrm{T}, \phi, \mathfrak{I})$ *implies compatibly* β_θ*-*$\mathfrak{J}$*-compactness of* $(\mathrm{T}, \phi, \mathfrak{J})$ *for every ideal* $\mathfrak{J}$ *such that* $\mathfrak{I} \subseteq \mathfrak{J}$*.*

The two theorems below provide characterizations of β_θ-$\mathfrak{I}$-compact spaces and compatibly β_θ-$\mathfrak{I}$-compact spaces.

Theorem 3.5. *An* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is* β_θ*-*$\mathfrak{I}$*-compact if and only if every family of* β_θ*-*$\mathfrak{I}$*-closed set whose total intersection is empty contains a finite subfamily with empty intersection*

Proof. The proof is a straightforward application of the definitions. □

Theorem 3.6. *An* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is compatibly* β_θ*-*$\mathfrak{I}$*-compact if and only if every family of* β_θ*-*$\mathfrak{I}$*-closed sets with empty total intersection contains a finite subfamily whose intersection belongs to* $\mathfrak{I}$*.*

Proof. Suppose that $(\mathrm{T}, \phi, \mathfrak{I})$ possesses the property of compatibly β_θ-$\mathfrak{I}$-compact. Consider a collection $\{F_\alpha : \alpha \in \Omega\}$ of β_θ-$\mathfrak{I}$-closed sets satisfying $\bigcap_{\alpha\in\Omega} F_\alpha = \emptyset$. Hence, $\{\mathrm{T} - F_\alpha : \alpha \in \Omega\}$ forms a β_θ-$\mathfrak{I}$-open cover of T, since $\bigcup_{\alpha\in\Omega}(\mathrm{T} - F_\alpha) = \mathrm{T} - \bigcap_{\alpha\in\Omega} F_\alpha = \mathrm{T}$. By the hypothesis, we can select a finite subset $\Omega_1 \subseteq \Omega$ for which $\mathrm{T} - \bigcup_{\alpha\in\Omega_1}(\mathrm{T} - F_\alpha) \in \mathfrak{I}$. Therefore, $\bigcap_{\alpha\in\Omega_1} F_\alpha = \mathrm{T} - \bigcup_{\alpha\in\Omega_1}(\mathrm{T} - F_\alpha) \in \mathfrak{I}$.

Conversely, suppose that whenever $\{F_\alpha : \alpha \in \Omega\}$ is a family of β_θ-$\mathfrak{I}$-closed subsets of T with $\bigcap_{\alpha\in\Omega} F_\alpha = \emptyset$, one can find a finite subset $\Omega_1 \subseteq \Omega$ such that $\bigcap_{\alpha\in\Omega_1} F_\alpha \in \mathfrak{I}$. Consider a family $\{G_\alpha : \alpha \in \Omega\}$ of β_θ-$\mathfrak{I}$-open sets covering T; that is, $\bigcup_{\alpha\in\Omega} G_\alpha = \mathrm{T}$. Then $\bigcap_{\alpha\in\Omega}(\mathrm{T} - G_\alpha) = \mathrm{T} - \bigcup_{\alpha\in\Omega} G_\alpha = \emptyset$, and each set $\mathrm{T} - G_\alpha$ is β_θ-$\mathfrak{I}$-closed. By the hypothesis, we can find a finite subset $\Omega_1 \subseteq \Omega$ satisfying $\bigcap_{\alpha\in\Omega_1}(\mathrm{T} - G_\alpha) \in \mathfrak{I}$, or equivalently, $\mathrm{T} - \bigcup_{\alpha\in\Omega_1} G_\alpha \in \mathfrak{I}$. This demonstrates that $(\mathrm{T}, \phi, \mathfrak{I})$ is compatibly β_θ-$\mathfrak{I}$-compact. □

These two theorems characterize the hereditary property of compactness.

Theorem 3.7. *Consider a* β_θ*-*$\mathfrak{I}$*-compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*. Then any* β_θ*-*$\mathfrak{I}$*-closed subset is* β_θ*-*$\mathfrak{I}$*-compact.*

Proof. Let $K \subseteq \mathrm{T}$ be a β_θ-$\mathfrak{I}$-closed subset and let $\{G_\alpha : \alpha \in \Omega\}$ be a β_θ-$\mathfrak{I}$-open cover such that $K = \bigcup_{\alpha\in\Omega} G_\alpha$. The β_θ-$\mathfrak{I}$-closedness of K implies that $\mathrm{T} - K$ is β_θ-$\mathfrak{I}$-open. Consequently, the family

$\{G_\alpha : \alpha \in \Omega\} \cup \{\mathrm{T} - K\}$, consisting of β_θ-$\mathfrak{I}$-open sets, covers T. From the β_θ-$\mathfrak{I}$-compactness of T, we can extract a finite subcollection $\{G_\alpha : \alpha \in \Omega_1\}$ such that $\mathrm{T} = \bigcup_{\alpha\in\Omega_1} G_\alpha \cup (\mathrm{T} - K)$. It implies that $K \subseteq \bigcup_{\alpha\in\Omega_1} G_\alpha$. Consequently, K possesses the property of β_θ-$\mathfrak{I}$-compactness. □

Theorem 3.8. *Consider a compatibly β_θ-$\mathfrak{I}$-compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*. Then any β_θ-$\mathfrak{I}$-closed subset is compatibly β_θ-$\mathfrak{I}$-compact.*

Proof. Let $K \subseteq \mathrm{T}$ be a β_θ-$\mathfrak{I}$-closed subset and let $\{G_\alpha : \alpha \in \Omega\}$ be a β_θ-$\mathfrak{I}$-open cover of K. Therefore, the collection $\{G_\alpha : \alpha \in \Omega\} \cup \{\mathrm{T} - K\}$ of β_θ-$\mathfrak{I}$-open sets covers T. Compatibly compactness gives a finite $\Omega_1 \subseteq \Omega$ satisfying:

$$\mathrm{T} - \bigcup_{\alpha\in\Omega_1} G_\alpha \in \mathfrak{I} \quad \text{or} \quad \mathrm{T} - \Big(\bigcup_{\alpha\in\Omega_1} G_\alpha \cup (\mathrm{T} - K)\Big) \in \mathfrak{I}.$$

First, if $\mathrm{T} - \bigcup_{\alpha\in\Omega_1} G_\alpha \in \mathfrak{I}$ is satisfied. Since $K - \bigcup_{\alpha\in\Omega_1} G_\alpha \subseteq \mathrm{T} - \bigcup_{\alpha\in\Omega_1} G_\alpha$, it follows from the property of the ideal $\mathfrak{I}$ that $K - \bigcup_{\alpha\in\Omega_1} G_\alpha \in \mathfrak{I}$.

Next, suppose that $\mathrm{T} - \Big(\bigcup_{\alpha\in\Omega_1} G_\alpha \cup (\mathrm{T} - K)\Big) \in \mathfrak{I}$. By the properties of ideals on T, we obtain

$$K \cap \left(\mathrm{T} - \Big(\bigcup_{\alpha\in\Omega_1} G_\alpha \cup (\mathrm{T} - K)\Big)\right) \in \mathfrak{I}.$$

Moreover,

$$K - \bigcup_{\alpha\in\Omega_1} G_\alpha = K \cap \left(\mathrm{T} - \bigcup_{\alpha\in\Omega_1} G_\alpha\right) = K \cap \left(\mathrm{T} - \Big(\bigcup_{\alpha\in\Omega_1} G_\alpha \cup (\mathrm{T} - K)\Big)\right).$$

Hence, $K - \bigcup_{\alpha\in\Omega_1} G_\alpha \in \mathfrak{I}$. This proves that K is compatibly β_θ-$\mathfrak{I}$-compact. □

As a consequence of the fact that every closed set is β_θ-$\mathfrak{I}$-closed, we have the next results.

Corollary 3.1. *Consider a β_θ-$\mathfrak{I}$-compact space $(\mathrm{T}, \phi, \mathfrak{I})$. Every closed subset is β_θ-$\mathfrak{I}$-compact. Furthermore, the intersection of finite closed subsets is also β_θ-$\mathfrak{I}$-compact.*

Corollary 3.2. *In a compatibly β_θ-$\mathfrak{I}$-compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*, every closed subset is compatibly β_θ-$\mathfrak{I}$-compact. Moreover, the intersection of finite closed subsets is also compatibly β_θ-$\mathfrak{I}$-compact.*

Theorem 3.9. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *where* (T, ϕ) *is Hausdorff. If a subset $K \subseteq \mathrm{T}$ is compatibly β_θ-$\mathfrak{I}$-compact, then it is closed in the space* $(\mathrm{T}, \phi^*(\mathfrak{I}))$.

Proof. Assume that $K \subseteq \mathrm{T}$ is a compatibly β_θ-$\mathfrak{I}$-compact subset, and let $p \in \mathrm{T} - K$. For each point $q \in K$, there exist neighborhoods V_q of p and W_q of q such that $V_q \cap W_q = \emptyset$. Hence $p \notin \mathrm{Cl}(W_q)$ for all $q \in K$. Clearly, $\{W_q : q \in K\}$ is the family of β_θ-$\mathfrak{I}$-open sets that cover of K. Because K is compatibly β_θ-$\mathfrak{I}$-compact, we can select a finite collection of points $K_1 \subseteq K$ for which $K - \bigcup_{q\in K_1} W_q \in \mathfrak{I}$. Moreover, because $p \notin \mathrm{Cl}(W_q)$ for each $q \in K_1$, it follows that $p \notin \bigcup_{q \in K_1} \mathrm{Cl}(W_q) = \mathrm{Cl}\left(\bigcup_{q \in K_1} W_q\right)$. Consider the sets

$$G = \mathrm{T} - \mathrm{Cl}\left(\bigcup_{q \in K_1} W_q\right) \quad \text{and} \quad E = K - \mathrm{Cl}\left(\bigcup_{q \in K_1} W_q\right).$$

By construction, G is an open neighborhood of p, and E is a member of the ideal $\mathfrak{I}$. Hence, $G - E$ is an open neighborhood of p in $(\mathrm{T}, \phi^*(\mathfrak{I}))$ that is disjoint from K. Therefore, p does not belong to K^*. It follows that $\mathrm{Cl}^*(K) \subseteq K$; consequently, K is a closed set in the topology $\phi^*(\mathfrak{I})$. □

Theorem 3.10. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$. *The finite union of* β_θ-$\mathfrak{I}$-*compact subsets of* T *is also* β_θ-$\mathfrak{I}$-*compact.*

Proof. Suppose that K_1 and K_2 are β_θ-$\mathfrak{I}$-compact subsets of T, and let $\{G_\alpha : \alpha \in \Omega\}$ be a β_θ-$\mathfrak{I}$-open cover of $K_1 \cup K_2$. Clearly, this family also covers both K_1 and K_2. Since K_1 and K_2 are β_θ-$\mathfrak{I}$-compact, there exist finite subsets $\Omega_1, \Omega_2 \subseteq \Omega$ such that

$$K_1 \subseteq \bigcup_{\alpha \in \Omega_1} G_\alpha \quad \text{and} \quad K_2 \subseteq \bigcup_{\lambda \in \Omega_2} G_\lambda.$$

Consequently, the finite subfamily indexed by $\Omega_1 \cup \Omega_2$ covers $K_1 \cup K_2$:

$$K_1 \cup K_2 \subseteq \left(\bigcup_{\alpha \in \Omega_1} G_\alpha\right) \cup \left(\bigcup_{\lambda \in \Omega_2} G_\lambda\right) = \bigcup_{\gamma \in \Omega_1 \cup \Omega_2} G_\gamma,$$

which shows that $K_1 \cup K_2$ is β_θ-$\mathfrak{I}$-compact. □

The next theorem shows that compatibly β_θ-$\mathfrak{I}$-compactness in ITP spaces is preserved under finite unions.

Theorem 3.11. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$. *The finite union of compatibly* β_θ-$\mathfrak{I}$-*compact subsets of* T *is also compatibly* β_θ-$\mathfrak{I}$-*compact.*

Proof. Assume that K_1 and K_2 are compatibly β_θ-$\mathfrak{I}$-compact subsets of T, and let $\{G_\alpha : \alpha \in \Omega\}$ be a β_θ-$\mathfrak{I}$-open cover of $K_1 \cup K_2$. Clearly, this family also covers both K_1 and K_2. Since K_1 and K_2 are compatibly β_θ-$\mathfrak{I}$-compact, there exist finite subsets $\Omega_1, \Omega_2 \subseteq \Omega$ such that

$$K_1 - \bigcup_{\alpha \in \Omega_1} G_\alpha \in \mathfrak{I} \quad \text{and} \quad K_2 - \bigcup_{\alpha \in \Omega_2} G_\alpha \in \mathfrak{I}.$$

Equivalently,

$$K_1 \subseteq \bigcup_{\alpha \in \Omega_1} G_\alpha \cup A_1 \quad \text{and} \quad K_2 \subseteq \bigcup_{\alpha \in \Omega_2} G_\alpha \cup A_2.$$

for some $A_1, A_2 \in \mathfrak{I}$. Consequently,

$$K_1 \cup K_2 \subseteq \left(\bigcup_{\alpha \in \Omega_1} G_\alpha\right) \cup \left(\bigcup_{\alpha \in \Omega_2} G_\alpha\right) \cup (A_1 \cup A_2).$$

Since ideals are closed under finite unions, $A_1 \cup A_2 \in \mathfrak{J}$. Hence,

$$(K_1 \cup K_2) - \left(\bigcup_{\alpha \in \Omega_1} G_\alpha \cup \bigcup_{\alpha \in \Omega_2} G_\alpha \right) \in \mathfrak{J},$$

which shows that $K_1 \cup K_2$ is compatibly β_θ-$\mathfrak{J}$-compact. □

4. Preservation of β_θ-$\mathfrak{J}$-Compactness

In this section, we show that β_θ-$\mathfrak{J}$-compactness and compatible β_θ-$\mathfrak{J}$-compactness of ITP spaces are preserved under certain mappings. We give sufficient criteria and identify structural assumptions that ensure these properties are maintained.

Theorem 4.1. *Consider an β_θ-$\mathfrak{J}$-compact space $(\mathrm{T}, \phi, \mathfrak{J})$ and an* ITP *space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a surjective β_θ-irresolute function, then $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-compact.*

Proof. Given a collection $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ of β_θ-$\mathfrak{J}$-open sets that covers $\widetilde{\mathrm{T}}$, that is, $\widetilde{\mathrm{T}} = \bigcup_{\alpha \in \Omega}(M_\alpha)$. Because $g\colon \mathrm{T} \to \widetilde{\mathrm{T}}$ is surjective and β_θ-irresolute, we have $\mathrm{T} = g^{-1}(\widetilde{\mathrm{T}}) = \bigcup_{\alpha \in \Omega} g^{-1}(M_\alpha)$ is a β_θ-$\mathfrak{J}$-open cover. Utilizing the β_θ–$\mathfrak{J}$-compactness of $(\mathrm{T}, \phi, \mathfrak{J})$, we obtain a finite subset $\Omega_1 \subseteq \Omega$ such that $\mathrm{T} = \bigcup_{\alpha \in \Omega_1} g^{-1}(M_\alpha)$. Applying g and using the surjectivity of g, we obtain

$$\widetilde{\mathrm{T}} = g(\mathrm{T}) = g(\bigcup_{\alpha \in \Omega_1} g^{-1}(M_\alpha)) = \bigcup_{\alpha \in \Omega_1} M_\alpha.$$

Therefore, $\{M_\alpha : \alpha \in \Omega_1\}$ is a finite subcover of $\mathcal{M}$. Hence $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-compact. □

Theorem 4.2. *Given an* ITP *space $(\mathrm{T}, \phi, \mathfrak{J})$ and a β_θ-$\mathfrak{J}$-compact space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijective, β_θ-$\mathfrak{J}$-open function, then $(\mathrm{T}, \phi, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-compact.*

Proof. Suppose $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ is a covering of T by β_θ-$\mathfrak{J}$-open sets. Because g is a surjective β_θ-$\mathfrak{J}$-open function, the family $\{g(M_\alpha) : \alpha \in \Omega\}$ constitutes a cover of $\widetilde{\mathrm{T}}$ by β_θ-$\mathfrak{J}$-open sets. The β_θ-$\mathfrak{J}$-compactness of $\widetilde{\mathrm{T}}$ ensures the existence of a finite subcover $\{g(M_\alpha) : \alpha \in \Omega_1\}$ with $\Omega_1 \subseteq \Omega$ finite such that $\widetilde{\mathrm{T}} = \bigcup_{\alpha \in \Omega_1} g(M_\alpha)$.

Applying the preimage under g and using the properties of g, we obtain

$$\mathrm{T} = g^{-1}(\widetilde{\mathrm{T}}) = \bigcup_{\alpha \in \Omega_1} g^{-1}\big(g(M_\alpha)\big) = \bigcup_{\alpha \in \Omega_1} M_\alpha.$$

Thus $(\mathrm{T}, \phi, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-compact, as required. □

The fact that g is bijective implies that for g, β_θ-$\mathfrak{J}$-openness and β_θ-$\mathfrak{J}$-closedness are equivalent properties. Hence, we deduce:

Corollary 4.1. *Given an* ITP *space $(\mathrm{T}, \phi, \mathfrak{J})$ and a β_θ-$\mathfrak{J}$-compact space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijective β_θ-$\mathfrak{J}$-closed function, then $(\mathrm{T}, \phi, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-compact.*

Theorem 4.3. *Consider a compatibly β_θ-$\mathfrak{J}$-compact space $(\mathrm{T}, \phi, \mathfrak{J})$ and another* ITP *space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. The existence of a surjective β_θ-irresolute function $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ with $g(\mathfrak{J}) \subseteq \mathfrak{J}$ implies that $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is compatibly β_θ-$\mathfrak{J}$-compact.*

Proof. Assume that $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a surjective β_θ-irresolute function with $g(\mathfrak{J}) \subseteq \mathfrak{J}$ and $\widetilde{\mathrm{T}}$ is covered by a collection $\{M_\alpha : \alpha \in \Omega\}$ of subsets, each of which is β_θ-$\mathfrak{J}$-open. Since g is β_θ-irresolute, the family $\{g^{-1}(M_\alpha) : \alpha \in \Omega\}$ consists of β_θ-$\mathfrak{J}$-open subsets of T and covers T. Because T is compatibly β_θ-$\mathfrak{J}$-compact, there exists a finite subset $\Omega_1 \subseteq \Omega$ such that $\mathrm{T} - \bigcup_{\alpha\in\Omega_1} g^{-1}(M_\alpha) \in \mathfrak{J}$. Hence,

$$\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega_1} M_\alpha = g\left(\mathrm{T} - \bigcup_{\alpha\in\Omega_1} g^{-1}(M_\alpha)\right) \in g(\mathfrak{J}) \subseteq \mathfrak{J},$$

as required. □

Theorem 4.4. *Consider an* ITP *space $(\mathrm{T}, \phi, \mathfrak{J})$ and a compatibly β_θ-$\mathfrak{J}$-compact space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijective β_θ-$\mathfrak{J}$-open function satisfying $g(\mathfrak{J}) = \mathfrak{J}$, then $(\mathrm{T}, \phi, \mathfrak{J})$ is compatibly β_θ-$\mathfrak{J}$-compact.*

Proof. Let $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ be a bijective β_θ-$\mathfrak{J}$-open function satisfying $g(\mathfrak{J}) = \mathfrak{J}$, and let $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ be a collection of β_θ-$\mathfrak{J}$-open sets that covers T. Given that g is a surjective β_θ-$\mathfrak{J}$-open function, the family $\{g(M_\alpha) : \alpha \in \Omega\}$ consists of β_θ-$\mathfrak{J}$-open subsets and covers $\widetilde{\mathrm{T}}$. Being compatible β_θ-$\mathfrak{J}$-compact of $\widetilde{\mathrm{T}}$ guarantees a finite $\Omega_1 \subset \Omega$ with $\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega_1} g(M_\alpha) \in \mathfrak{J}$. Consequently,

$$\mathrm{T} - \bigcup_{\alpha\in\Omega_1} M_\alpha = g^{-1}\left(\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega_1} g(M_\alpha)\right) \in g^{-1}(\mathfrak{J}) = \mathfrak{J}.$$

□

The bijectivity of g ensures that β_θ-$\mathfrak{J}$-openness and β_θ-$\mathfrak{J}$-closedness coincide in this setting. Thus, we obtain:

Corollary 4.2. *Consider an* ITP *space $(\mathrm{T}, \phi, \mathfrak{J})$ and a compatibly β_θ-$\mathfrak{J}$-compact space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijective β_θ-$\mathfrak{J}$-closed function such that $g(\mathfrak{J}) = \mathfrak{J}$, then $(\mathrm{T}, \phi, \mathfrak{J})$ is compatibly β_θ-$\mathfrak{J}$-compact.*

5. β_θ-$\mathfrak{J}$-Paracompactness and its Characterizations

A space is considered paracompact if, given any open cover, one can always find a second open cover that is both a refinement of the first and locally finite (meaning every point has a neighborhood intersecting only finitely many sets from this refined cover). This property serves as a fruitful generalization of compactness, where the stricter "finite subcover" condition is relaxed to "locally finite refinement." Because of this, paracompact spaces support partitions of unity and preserve many desirable features of compact spaces, including useful separation axioms, making them central in modern topology and its applications.

This section is devoted to the study of β_θ-$\mathfrak{I}$-paracompactness, which can be regarded as a generalization of classical paracompactness and paracompactness with respect to an ideal, as introduced, for example, by Yıldırım et al. [38]. We provide a rigorous definition of this notion.

Let $\mathcal{M}$ and $\mathcal{N}$ be collections of subsets in a space T. We say $\mathcal{M}$ refines $\mathcal{N}$, or that $\mathcal{M}$ is a refinement of $\mathcal{N}$, whenever each member of $\mathcal{M}$ is contained in at least one member of $\mathcal{N}$.

Definition 5.1. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$. *A collection $\mathcal{M}$ of subsets of T is β_θ-$\mathfrak{I}$-locally finite if every point of T has a β_θ-$\mathfrak{I}$-open neighborhood that meets only finitely many sets from $\mathcal{M}$.*

By Definition 5.1, $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite if each point $z \in \mathrm{T}$ has β_θ-$\mathfrak{I}$-open neighborhood U_z such that $U_z \cap M_\alpha \neq \emptyset$ for only finitely many members of $\alpha \in \Omega$.

Lemma 5.1. *Every locally finite family in an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is β_θ-$\mathfrak{I}$-locally finite.*

Proof. Assume $\mathcal{M}$ is locally finite. To prove $\mathcal{M}$ is β_θ-$\mathfrak{I}$-locally finite, let $z \in \mathrm{T}$ be given. By the local finiteness of $\mathcal{M}$, we can choose an open set G_z containing z for which the collection $\{M \in \mathcal{M} : G_z \cap M \neq \emptyset\}$ is finite. From the general principle that all open sets are β_θ-$\mathfrak{I}$-open, it follows in particular that the open neighborhood G_z of z is β_θ-$\mathfrak{I}$-open. Because G_z has non-empty intersection with only a finite subfamily of $\mathcal{M}$, we infer that $\mathcal{M}$ satisfies β_θ-$\mathfrak{I}$-local finiteness. □

We proceed to define a concept extending paracompactness to ideal spaces.

Definition 5.2. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$. *A subset* $\mathrm{P} \subseteq \mathrm{T}$ *is called* β_θ-$\mathfrak{I}$-paracompact *if any open cover of* P *admits a β_θ-$\mathfrak{I}$-open refinement $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ that is β_θ-$\mathfrak{I}$-locally finite and satisfies* $\mathrm{P} - \bigcup_{\alpha \in \Omega} M_\alpha \in \mathfrak{I}$. *Such a collection $\mathcal{M}$ is called an* $\mathfrak{I}$-cover *of* P.

The ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is called* β_θ-$\mathfrak{I}$-paracompact *precisely when* T *itself is a β_θ-$\mathfrak{I}$-paracompact subset.*

We observe that β_θ-$\mathfrak{I}$-paracompactness is hereditary with respect to enlargement of the ideal: if $(\mathrm{T}, \phi, \mathfrak{I})$ has this paracompactness property and $\mathfrak{I} \subseteq \mathfrak{J}$, then $(\mathrm{T}, \phi, \mathfrak{J})$ likewise exhibits β_θ-$\mathfrak{I}$-paracompactness.

Theorem 5.1. *For any ideal $\mathfrak{I}$ on T, the paracompactness of the topological space (T, ϕ) implies that the corresponding* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is β_θ-$\mathfrak{I}$-paracompact.*

Proof. Assume we have a paracompact space (T, ϕ) and an open cover $\mathcal{W}$ of T. Under this assumption, $\mathcal{W}$ possesses a locally finite open refinement, which we denote by $\mathcal{V}$. Given that all open sets are β_θ-$\mathfrak{I}$-open, the refinement $\mathcal{V}$ is consequently a β_θ-$\mathfrak{I}$-locally finite refinement of $\mathcal{W}$ by β_θ-$\mathfrak{I}$-open sets. Furthermore, $\mathrm{T} - \bigcup_{G \in \mathcal{V}} G$ is the empty set and is a member of $\mathfrak{I}$. Therefore $(\mathrm{T}, \phi, \mathfrak{I})$ is β_θ-$\mathfrak{I}$-paracompact. □

The implication in Theorem 5.1 cannot be reversed; a counterexample is provided below.

Example 5.1. *Consider a set* $\mathrm{T} = [0, 1)$ *and a topology* $\phi = \{U \subseteq \mathrm{T} : 0 \in U\} \cup \{\emptyset\}$. *Take* $\mathfrak{I} = \mathrm{P}(\mathrm{T})$. *It is easy to verify that the space* (T, ϕ) *lacks the property of paracompactness. Indeed, the collection* $\mathcal{Z} = \{\{0, z\} : z \in \mathrm{T}\}$ *consisting of open sets forms a cover of* T *which possesses no open refinement that is locally finite.*

In the space $(\mathrm{T}, \phi, \mathfrak{I})$, *we can show that every subset is* β_θ-$\mathfrak{I}$-*open. In fact, for any open cover* $\mathcal{U}$ *of* T, *the collection* $\mathcal{Z}$ *provides a* β_θ-$\mathfrak{I}$-*locally finite* β_θ-$\mathfrak{I}$-*open refinement of* $\mathcal{U}$ *and satisfies* $\mathrm{T} - \bigcup_{z \in \mathrm{T}}\{z\} = \emptyset \in \mathfrak{I}$. *Therefore,* $(\mathrm{T}, \phi, \mathfrak{I})$ *is* β_θ-$\mathfrak{I}$-*paracompact.*

Consequently, β_θ-$\mathfrak{I}$-*paracompactness does not imply paracompactness of the underlying space* (T, ϕ).

Theorem 5.2. *Every compatibly* β_θ-$\mathfrak{I}$-*compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is also* β_θ-$\mathfrak{I}$-*paracompact.*

Proof. Suppose $(\mathrm{T}, \phi, \mathfrak{I})$ is compatibly β_θ-$\mathfrak{I}$-compact and $\mathcal{W} = \{U_\alpha : \alpha \in \Omega\}$ is an open cover of T. Because each open set is β_θ-$\mathfrak{I}$-open, $\mathcal{W}$ forms a β_θ-$\mathfrak{I}$-open cover of T. By the compatibly β_θ-$\mathfrak{I}$-compactness of $(\mathrm{T}, \phi, \mathfrak{I})$, we are able to pick finitely many indices from Ω—call this set Ω_1—such that $\mathrm{T} - \bigcup_{\alpha \in \Omega_1} U_\alpha \in \mathfrak{I}$. We have therefore established that $\{U_\alpha : \alpha \in \Omega_1\}$ forms a finite refinement of $\mathcal{W}$ by β_θ-$\mathfrak{I}$-open sets, and this refinement is β_θ-$\mathfrak{I}$-locally finite. Consequently, $(\mathrm{T}, \phi, \mathfrak{I})$ satisfies β_θ-$\mathfrak{I}$-paracompactness. □

The following corollary follows directly upon applying Theorems 3.2 and Theorem 5.2.

Corollary 5.1. *Every* β_θ-$\mathfrak{I}$-*compact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is* β_θ-$\mathfrak{I}$-*paracompact.*

Next, we present an example of an ITP space that is β_θ-$\mathfrak{I}$-paracompact but not β_θ-$\mathfrak{I}$-compact.

Example 5.2. *Let* $\mathrm{T} = \mathbb{R}$ *with the usual topology* ϕ, *and let* $\mathfrak{I} = \{\emptyset\}$. *Theorem 5.1 allows us to infer from the paracompactness of* (T, ϕ) *that the* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *possesses* β_θ-$\mathfrak{I}$-*paracompactness. However,* $(\mathrm{T}, \phi, \mathfrak{I})$ *is not* β_θ-$\mathfrak{I}$-*compact. To see this, consider the cover* $\mathcal{U} = \{(-x, x) : x \in \mathbb{N}\}$ *of* $\mathbb{R}$ *by* β_θ-$\mathfrak{I}$-*open sets. This cover has no finite subcover because any finite subcollection* $\{(-n_1, n_1), \ldots, (-n_k, n_k)\}$ *has union* $(-N, N)$ *where* $N = \max\{n_1, n_2, \ldots, n_k\}$, *which does not cover* T. *Hence,* $(\mathrm{T}, \phi, \mathfrak{I})$ *is not* β_θ-$\mathfrak{I}$-*compact.*

The following theorem will be used in the proofs of several results and can be verified easily.

Theorem 5.3. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and* $B \subseteq \mathrm{T}$. *If* V *is* β_θ-$\mathfrak{I}$-*open then,* $V \cap \mathrm{Cl}^\beta_\theta(B) = \emptyset$ *if and only if* $V \cap B = \emptyset$.

Theorem 5.4. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and a* β_θ-$\mathfrak{I}$-*locally finite family* $\{M_\alpha : \alpha \in \Omega\}$ *of subsets of* T.

(1) $\{N_\alpha : \alpha \in \Omega\}$ *is* β_θ-$\mathfrak{I}$-*locally finite if* $N_\alpha \subseteq M_\alpha$ *for all* $\alpha \in \Omega$.
(2) $\{\mathrm{Cl}^\beta_\theta(M_\alpha) : \alpha \in \Omega\}$ *is* β_θ-$\mathfrak{I}$-*locally finite.*

Proof. (1): Take any $z \in \mathrm{T}$. By assumption, one can find a β_θ-$\mathfrak{I}$-open neighborhood W of z such that

$$\{\alpha \in \Omega : W \cap M_\alpha \neq \emptyset\} \text{ is finite.}$$

Given that $N_\alpha \subseteq M_\alpha$ for each $\alpha \in \Omega$, it follows that

$$\{\alpha \in \Omega : W \cap N_\alpha \neq \emptyset\} \text{ is also finite.}$$

Hence $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite.
(2): Given $z \in \mathrm{T}$. By assumption, one can pick a β_θ-$\mathfrak{I}$-open set G containing z such that

$$\{\alpha \in \Omega : G \cap M_\alpha \neq \emptyset\} \text{ is finite.}$$

By Theorem 5.3, we have

$$\{\alpha \in \Omega : G \cap \mathrm{Cl}^\beta_\theta(M_\alpha) \neq \emptyset\} \text{ is finite.}$$

Thus $\{\mathrm{Cl}^\beta_\theta(M_\alpha) : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite. □

Given an open cover $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ of an ITP space $(\mathrm{T}, \phi, \mathfrak{I})$, suppose we can find a refinement of β_θ-$\mathfrak{I}$-open sets $\mathcal{N} = \{N_\lambda : \lambda \in \Lambda\}$ which is β_θ-$\mathfrak{I}$-locally finite and satisfies $\mathrm{T} - \bigcup_{\lambda\in\Lambda} N_\lambda \in \mathfrak{I}$. Then one can obtain a β_θ-$\mathfrak{I}$-open refinement $\mathrm{P} = \{P_\alpha : \alpha \in \Omega\}$ of $\mathcal{M}$ that is β_θ-$\mathfrak{I}$-locally finite and fulfills $\mathrm{T} - \bigcup_{\alpha\in\Omega} P_\alpha \in \mathfrak{I}$. Furthermore, the refinement P is *precise*, in the sense that its index set coincides with that of the original cover $\mathcal{M}$.

Lemma 5.2. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *and an open cover* $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ *of* T*. The existence of a* β_θ*-*$\mathfrak{I}$*-locally finite refinement* $\mathcal{N} = \{N_\mu : \mu \in \Lambda\}$ *of* $\mathcal{M}$ *consisting of* β_θ*-*$\mathfrak{I}$*-open sets with* $\mathrm{T} - \bigcup_{\mu\in\Lambda} N_\mu \in \mathfrak{I}$ *implies the existence of a precise refinement* $\mathrm{P} = \{P_\alpha : \alpha \in \Omega\}$ *of* $\mathcal{M}$ *that is also* β_θ*-*$\mathfrak{I}$*-locally finite, consists of* β_θ*-*$\mathfrak{I}$*-open sets, and satisfies* $\mathrm{T} - \bigcup_{\alpha\in\Omega} P_\alpha \in \mathfrak{I}$*.*

Proof. A similar technique is employed as in the proof of Lemma 1.3 in [32]. □

In the following, we introduce a generalized notion of regularity in ideal spaces.

Definition 5.3. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*. The space* T *is called* β_θ*-*$\mathfrak{I}$*-regular if whenever* F *is a closed set and a point* $z \notin F$*, one can find disjoint* β_θ*-*$\mathfrak{I}$*-open subsets* V *and* W *satisfying* $z \in V$ *and* $F - W \in \mathfrak{I}$*.*

The subsequent result establishes two equivalent characterizations for spaces that are β_θ-$\mathfrak{I}$-regular.

Theorem 5.5. *For an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*, the following are equivalent:*

(1) *The space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is* β_θ*-*$\mathfrak{I}$*-regular;*
(2) *Given a closed set* $K \subseteq \mathrm{T}$ *and a point* $z \notin K$*, one can find a* β_θ*-*$\mathfrak{I}$*-open set* V *containing* z *together with a* β_θ*-*$\mathfrak{I}$*-open set* W *satisfying* $V \cap W = \emptyset$ *and* $K - W \in \mathfrak{I}$*; and*
(3) *Given an open set* G *and a point* $z \in G$*, one can find a* β_θ*-*$\mathfrak{I}$*-open set* V *containing* z *satisfying* $\mathrm{Cl}^\beta_\theta(V) - G \in \mathfrak{I}$*.*

Proof. $(1) \Rightarrow (2)$: It is immediate.

$(2) \Rightarrow (3)$: Suppose that condition (2) is satisfied. Suppose G is open that contains a point z. Thus, $\mathrm{T} - G$ is closed and does not contain z. By the given assumption, one can choose two disjoint sets V and W that are β_θ-$\mathfrak{I}$-open satisfying $V \cap W = \emptyset$, $z \in V$, and $(\mathrm{T} - G) - W \in \mathfrak{I}$. Applying Theorem 5.3 to $V \cap W = \emptyset$, we obtain $\mathrm{Cl}^\beta_\theta(V) \subseteq \mathrm{T} - W$. From this, $\mathrm{Cl}^\beta_\theta(V) - G \subseteq (\mathrm{T} - W) - G = (\mathrm{T} - G) - W$. Moreover, by a property of the ideal $\mathfrak{I}$, we have $\mathrm{Cl}^\beta_\theta(V) - G \in \mathfrak{I}$.

$(3) \Rightarrow (1)$: Suppose that condition (3) holds. Let K be closed and $z \notin K$. Thus $\mathrm{T} - K$ is open that contains z. Using the assumption, we can pick a β_θ-$\mathfrak{I}$-open set G_1 containing z such that $\mathrm{Cl}^\beta_\theta(G_1) - (\mathrm{T} - K) \in \mathfrak{I}$. Define $G_2 = \mathrm{T} - \mathrm{Cl}^\beta_\theta(G_1)$. Then G_2 is β_θ-$\mathfrak{I}$-open and does not meet G_1. Moreover, $K - G_2 = K - \big(\mathrm{T} - \mathrm{Cl}^\beta_\theta(G_1)\big) = \mathrm{Cl}^\beta_\theta(G_1) - (\mathrm{T} - K) \in \mathfrak{I}$. Therefore, the space $(\mathrm{T}, \phi, \mathfrak{I})$ satisfies the condition of β_θ-$\mathfrak{I}$-regularity. □

Theorem 5.6 shows how local finiteness and β_θ-$\mathfrak{I}$-open sets ensure regularity, and it connects para-compactness, closure behavior, and separation axioms in ITP spaces.

Theorem 5.6. *Consider a β_θ-$\mathfrak{I}$-paracompact* ITP *space $(\mathrm{T}, \phi, \mathfrak{I})$ where (T, ϕ) is Hausdorff. Assume that $(\mathrm{T}, \phi, \mathfrak{I})$ possesses the following property: for any refinement $\{A_\alpha : \alpha \in \Omega\}$ of a family of β_θ-$\mathfrak{I}$-open sets that is β_θ-$\mathfrak{I}$-locally finite, we have $\bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(A_\alpha) = \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha\in\Omega} A_\alpha\right)$. Then the space $(\mathrm{T}, \phi, \mathfrak{I})$ satisfies β_θ-$\mathfrak{I}$-regularity.*

Proof. Suppose K is closed and $u \notin K$. Since T is Hausdorff, for each $v \in K$, one can find disjoint open neighborhoods V_u of u and W_{uv} of v; consequently, $v \notin \mathrm{Cl}(V_u)$. The cover $\mathcal{M} = \{W_{uv} : v \in K\} \cup \{\mathrm{T} - K\}$ of T admits a β_θ-$\mathfrak{I}$-locally finite β_θ-$\mathfrak{I}$-open refinement $\mathcal{N} = \{V_{uv} : v \in K\} \cup \{W\}$, with $W \subseteq \mathrm{T} - K$ and $V_{uv} \subseteq W_{uv}$ for all $v \in K$, and satisfying $\mathrm{T} - \bigcup_{v\in K}(V_{uv} \cup W) \in \mathfrak{I}$, by β_θ-$\mathfrak{I}$-paracompactness of T.

Set $O_1 = \bigcup_{v\in K} V_{uv}$ and $O_2 = \mathrm{T} - \mathrm{Cl}^\beta_\theta(\bigcup_{v\in K} V_{uv})$. We show that O_1 and O_2 are disjoint β_θ-$\mathfrak{I}$-open subsets of T. Since $V_{uv} \subseteq W_{uv}$, we have $\mathrm{Cl}^\beta_\theta(V_{uv}) \subseteq \mathrm{Cl}(V_{uv}) \subseteq \mathrm{Cl}(W_{uv})$. Because $u \notin \mathrm{Cl}(W_{uv})$, it follows that $u \notin \mathrm{Cl}^\beta_\theta(V_{uv})$. Consequently, $u \notin \bigcup_{v\in K} \mathrm{Cl}^\beta_\theta(V_{uv}) = \mathrm{Cl}^\beta_\theta\left(\bigcup_{v\in K} V_{uv}\right)$, which implies that $u \in O_2$. Furthermore, we obtain $K - O_1 = K - \bigcup_{v\in K} V_{uv} \subseteq \mathrm{T} - \left(\bigcup_{v\in K} V_{uv} \cup W\right) \in \mathfrak{I}$. This demonstrates that $(\mathrm{T}, \phi, \mathfrak{I})$ satisfies the condition of β_θ-$\mathfrak{I}$-regularity. □

Combining Lemma 2.3 with Theorem 5.6 yields the next corollary.

Corollary 5.2. *Consider a β_θ-$\mathfrak{I}$-paracompact* ITP *space $(\mathrm{T}, \phi, \mathfrak{I})$ with (T, ϕ) is Hausdorff. Given that the class of β_θ-$\mathfrak{I}$-open subsets is closed under finite intersections. Then the space $(\mathrm{T}, \phi, \mathfrak{I})$ exhibits β_θ-$\mathfrak{I}$-regularity.*

Theorem 5.7. *Consider a β_θ-$\mathfrak{I}$-paracompact* ITP *space $(\mathrm{T}, \phi, \mathfrak{I})$ where (T, ϕ) is regular. In such a space, any open cover can be refined to a β_θ-$\mathfrak{I}$-locally finite family consisting of β_θ-$\mathfrak{I}$-closed sets.*

Proof. Let $\mathcal{M}$ be an open cover of T. The β_θ-$\mathfrak{I}$-regularity of the space T implies that for any $z \in \mathrm{T}$ and any $U_z \in \mathcal{M}$ with $z \in U_z$, one can choose an open neighborhood G_z of z whose closure lies inside

U_z. We then define $\mathcal{M}_1 = \{G_z : z \in \mathrm{T}\}$, which forms a cover of T. By β_θ-$\mathfrak{I}$-paracompactness of T, one can find a β_θ-$\mathfrak{I}$-open refinement $\mathcal{N}_1 = \{N_\alpha : \alpha \in \Omega\}$ of $\mathcal{M}_1$ that is β_θ-$\mathfrak{I}$-locally finite and satisfies $\mathrm{T} - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{I}$. Since $N_\alpha \subseteq \mathrm{Cl}^\beta_\theta(N_\alpha)$ for each $\alpha \in \Omega$, we have $\mathrm{T} - \bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(N_\alpha) \subseteq \mathrm{T} - \bigcup_{\alpha\in\Omega} N_\alpha$, and therefore $\mathrm{T} - \bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(N_\alpha) \in \mathfrak{I}$. By Theorem 5.4, the collection $\mathcal{N} = \{\mathrm{Cl}^\beta_\theta(N_\alpha) : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite.

Because each $N_\alpha \in \mathcal{N}_1$ satisfies $N_\alpha \subseteq G_z$ for some $G_z \in \mathcal{M}_1$, we obtain

$$\mathrm{Cl}^\beta_\theta(N_\alpha) \subseteq \mathrm{Cl}(N_\alpha) \subseteq \mathrm{Cl}(G_z) \subseteq U_z.$$

This shows that $\mathcal{N}$ refines $\mathcal{M}$. Consequently, the family $\mathcal{N}$ is an $\mathfrak{I}$-cover of T and a β_θ-$\mathfrak{I}$-locally finite refinement consisting of β_θ-$\mathfrak{I}$-closed sets. □

Theorem 5.8. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *where* (T, ϕ) *is Hausdorff. Assume that* $(\mathrm{T}, \phi, \mathfrak{I})$ *possesses the following property: for any refinement* $\{A_\alpha : \alpha \in \Omega\}$ *of a family of* β_θ*-*$\mathfrak{I}$*-open sets that is* β_θ*-*$\mathfrak{I}$*-locally finite, we have* $\bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(A_\alpha) = \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha\in\Omega} A_\alpha\right)$. *Then the* β_θ*-*$\mathfrak{I}$*-paracompactness of a subset* M *implies its closedness in* $(\mathrm{T}, \phi^*(\mathfrak{I}))$.

Proof. We now show that $\mathrm{Cl}^*(M) \subseteq M$. Suppose $z \notin M$. By the Hausdorff property of T, each $y \in M$ admits an open neighborhood G_y that does not contain z. Hence $\mathcal{M} = \{G_y : y \in M\}$ is a class of open sets that covers M. The β_θ-$\mathfrak{I}$-paracompactness of M ensures the existence of a refinement consisting of β_θ-$\mathfrak{I}$-open sets $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\}$ that is a β_θ-$\mathfrak{I}$-locally finite and such that $M - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{I}$. The condition $z \notin G_y$ for all $y \in M$ implies that $z \notin \mathrm{Cl}(N_\alpha)$ for each $\alpha \in \Omega$. Therefore z is not in $\bigcup_{\alpha\in\Omega} \mathrm{Cl}(N_\alpha)$, and consequently, $z \notin \bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(N_\alpha)$. Together with the assumed equality $\bigcup_{\alpha\in\Omega} \mathrm{Cl}^\beta_\theta(N_\alpha) = \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha\in\Omega} N_\alpha\right)$, this gives $z \notin \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha\in\Omega} N_\alpha\right)$.

Now we set

$$U = \mathrm{T} - \mathrm{Cl}\Big(\bigcup_{\alpha\in\Omega} N_\alpha\Big) \quad \text{and} \quad H = M - \mathrm{Cl}^\beta_\theta\Big(\bigcup_{\alpha\in\Omega} N_\alpha\Big).$$

It follows that U is open and $U \subseteq \mathrm{T} - \mathrm{Cl}^\beta_\theta\left(\bigcup_{\alpha\in\Omega} N_\alpha\right)$ and $H \subseteq M - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{I}$. Observe that $U - H$ is open in the topology $\phi^*(\mathfrak{I})$ and satisfies

$$(U - H) \cap M \subseteq \bigg(\Big(\mathrm{T} - \mathrm{Cl}^\beta_\theta\Big(\bigcup_{\alpha\in\Omega} N_\alpha\Big)\Big) - \Big(M - \mathrm{Cl}^\beta_\theta\Big(\bigcup_{\alpha\in\Omega} N_\alpha\Big)\Big)\bigg) \cap M = \emptyset,$$

which implies $z \notin \mathrm{Cl}^*(M)$. Hence $\mathrm{Cl}^*(M) \subseteq M$, and we can conclude that M is closed in $(\mathrm{T}, \phi^*(\mathfrak{I}))$. □

Combining Lemma 2.3 with Theorem 5.8 yields the following special case as a corollary.

Corollary 5.3. *Suppose* $(\mathrm{T}, \phi, \mathfrak{I})$ *is an* ITP *space for which* (T, ϕ) *is Hausdorff and finite intersections of* β_θ*-*$\mathfrak{I}$*-open sets remain* β_θ*-*$\mathfrak{I}$*-open. Under these conditions, every* β_θ*-*$\mathfrak{I}$*-paracompact subset of* T *is necessarily closed in* $(\mathrm{T}, \phi^*(\mathfrak{I}))$.

We now introduce β_θ-$\mathfrak{I}$-normality, which generalizes the concept of normality to ITP spaces using their ideal structure.

Definition 5.4. *An* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is said to be* β_θ-$\mathfrak{I}$*-normal if, given any two disjoint closed subsets* F *and* K*, there exist disjoint* β_θ-$\mathfrak{I}$*-open sets* V *and* W *such that* $F - V \in \mathfrak{I}$ *and* $K - W \in \mathfrak{I}$.

The theorem that follows shows various conditions that are equivalent to β_θ-$\mathfrak{I}$-normality.

Theorem 5.9. *For an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*, the following are equivalent:*

(1) *The space* $(\mathrm{T}, \phi, \mathfrak{I})$ *is* β_θ-$\mathfrak{I}$*-normal;*
(2) *Given a closed set* K_1 *and an open set* G_1 *with* $K_1 \subseteq G_1$*, one can choose a* β_θ-$\mathfrak{I}$*-open set* G_2 *satisfying* $K_1 - G_2 \in \mathfrak{I}$ *and* $\mathrm{Cl}_\theta^\beta(G_2) - G_1 \in \mathfrak{I}$*.; and*
(3) *Given disjoint closed sets* K_1 *and* K_2*, one can find a* β_θ-$\mathfrak{I}$*-open set* U *that satisfies* $K_1 - U \in \mathfrak{I}$ *and* $\mathrm{Cl}_\theta^\beta(U) \cap K_2 \in \mathfrak{I}$.

Proof. (1)⇒(2): We assume that the space $(\mathrm{T}, \phi, \mathfrak{I})$ is β_θ-$\mathfrak{I}$-normal. Consider a closed set K_1 inside an open set G_1. So we have that $\mathrm{T} - G_1$ is closed and disjoint from K_1. By β_θ-$\mathfrak{I}$-normality of T, We can find disjoint β_θ-$\mathfrak{I}$-open sets U and G_2 for which $(\mathrm{T} - G_1) - U \in \mathfrak{I}$ and $K_1 - G_2 \in \mathfrak{I}$. Because U and G_2 are disjoint, Theorem 5.3 gives $\mathrm{Cl}_\theta^\beta(G_2) \subseteq \mathrm{T} - U$. Consequently, $\mathrm{Cl}_\theta^\beta(G_2) - G_1 \subseteq (\mathrm{T} - U) - G_1 = (\mathrm{T} - G_1) - U$. Since $(\mathrm{T} - G_1) - U \in \mathfrak{I}$, we obtain $\mathrm{Cl}_\theta^\beta(G_2) - G_1 \in \mathfrak{I}$.

(2)⇒(3): Now suppose condition (2) holds. Take two disjoint closed sets $K_1, K_2 \subseteq \mathrm{T}$. Since K_2 is closed, $\mathrm{T} - K_2$ is open, and because $K_1 \cap K_2 = \emptyset$, we have $K_1 \subseteq \mathrm{T} - K_2$. From the assumption, we obtain a β_θ-$\mathfrak{I}$-open set U such that $K_1 - U \in \mathfrak{I}$ and $\mathrm{Cl}_\theta^\beta(U) - (\mathrm{T} - K_2) \in \mathfrak{I}$. Observing that $\mathrm{Cl}_\theta^\beta(U) - (\mathrm{T} - K_2) = \mathrm{Cl}_\theta^\beta(U) \cap K_2$, it follows that $K_1 - U \in \mathfrak{I}$ and $\mathrm{Cl}_\theta^\beta(U) \cap K_2 \in \mathfrak{I}$.

(3)⇒(1): Assume condition (3) holds. Given two disjoint closed sets K_1 and K_2, we can find a β_θ-$\mathfrak{I}$-open set U such that $K_1 - U \in \mathfrak{I}$ and $\mathrm{Cl}_\theta^\beta(U) \cap K_2 \in \mathfrak{I}$, which is equivalent to $K_2 - \big(\mathrm{T} - \mathrm{Cl}_\theta^\beta(U)\big) \in \mathfrak{I}$.

Take $V = \mathrm{T} - \mathrm{Cl}_\theta^\beta(U)$. Then V is β_θ-$\mathfrak{I}$-open and disjoint from U. Moreover, we have $K_1 - U \in \mathfrak{I}$ and $K_2 - V \in \mathfrak{I}$. These conditions verify that $(\mathrm{T}, \phi, \mathfrak{I})$ is β_θ-$\mathfrak{I}$-normal. □

Theorem 5.10 establishes that β_θ-$\mathfrak{I}$-paracompactness together with suitable properties of $\mathfrak{I}$ and the operator Cl_θ^β ensures β_θ-$\mathfrak{I}$-normality.

Theorem 5.10. *Consider a* β_θ-$\mathfrak{I}$*-paracompact* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$ *where* (T, ϕ) *is Hausdorff. Assume that* $\mathfrak{I}$ *is closed under arbitrary unions. Moreover, suppose that for every* β_θ-$\mathfrak{I}$*-locally finite* β_θ-$\mathfrak{I}$*-open refinement* $\{A_\alpha : \alpha \in \Lambda\}$*, the equality* $\bigcup_{\alpha\in\Lambda} \mathrm{Cl}_\theta^\beta(A_\alpha) = \mathrm{Cl}_\theta^\beta\big(\bigcup_{\alpha\in\Lambda} A_\alpha\big)$ *holds. Then the space* $(\mathrm{T}, \phi, \mathfrak{I})$ *satisfies* β_θ-$\mathfrak{I}$*-normality.*

Proof. Suppose H and K are closed subsets of T with $H \cap K = \emptyset$. It follows from the assumptions and Theorem 5.6 that the space $(\mathrm{T}, \phi, \mathfrak{I})$ satisfies β_θ-$\mathfrak{I}$-regularity. Consequently, given any $x \in H$, we can choose disjoint open sets U_x and V_x with $x \in U_x$ and $K - V_x \in \mathfrak{I}$. Now observe that $\mathcal{U} = \{U_x : x \in H\} \cup \{\mathrm{T} - H\}$ is a collection of open sets that covers T. Using β_θ-$\mathfrak{I}$-paracompactness of T, $\mathcal{U}$ admits a β_θ-$\mathfrak{I}$-locally finite β_θ-$\mathfrak{I}$-open refinement $\mathcal{Q} = \{Q_x : x \in H\} \cup \{G\}$ with $Q_x \subseteq U_x$, $G \subseteq \mathrm{T} - H$, and $\mathrm{T} - \left(\bigcup_{x\in H} Q_x \cup G\right) \in \mathfrak{I}$.

Set $W = \bigcup_{x\in H} Q_x$. Then W is β_θ-$\mathfrak{I}$-open. Moreover,

$$H - W = H \cap \left(\mathrm{T} - \bigcup_{x\in H} Q_x\right) \subseteq \mathrm{T} - \left(\bigcup_{x\in H} Q_x \cup G\right) \in \mathfrak{I}.$$

Hence $H - W \in \mathfrak{I}$.

Since U_x and V_x are disjoint for every $x \in H$, the β_θ-closure of U_x is contained in the complement of V_x; that is, $\mathrm{Cl}_\theta^\beta(U_x) \subseteq \mathrm{T} - V_x$. Thus, $\mathrm{Cl}_\theta^\beta(Q_x) \subseteq \mathrm{T} - V_x$. By the assumption, we obtain

$$\mathrm{Cl}_\theta^\beta\left(\bigcup_{x\in H} Q_x\right) = \bigcup_{x\in H} \mathrm{Cl}_\theta^\beta(Q_x) \subseteq \bigcup_{x\in H} (\mathrm{T} - V_x).$$

Consequently,

$$K \cap \mathrm{Cl}_\theta^\beta(W) = K \cap \mathrm{Cl}_\theta^\beta\left(\bigcup_{x\in H} Q_x\right) \subseteq K \cap \bigcup_{x\in H} \mathrm{Cl}_\theta^\beta(Q_x) \subseteq \bigcup_{x\in H} (K - V_x) \in \mathfrak{I},$$

since $\mathfrak{I}$ is closed under unions. Theorem 5.9 gives β_θ-$\mathfrak{I}$-normality of $(\mathrm{T}, \phi, \mathfrak{I})$. □

From Lemma 2.2 and Theorem 5.10 we deduce the next corollary.

Corollary 5.4. *Consider a β_θ-$\mathfrak{I}$-paracompact* ITP *space $(\mathrm{T}, \phi, \mathfrak{I})$ where (T, ϕ) is Hausdorff. Suppose that the finite intersection of β_θ-$\mathfrak{I}$-open subsets of* T *is again β_θ-$\mathfrak{I}$-open and $\mathfrak{I}$ is closed under arbitrary unions. Then the space $(\mathrm{T}, \phi, \mathfrak{I})$ is β_θ-$\mathfrak{I}$-normal.*

The following results demonstrate that β_θ-$\mathfrak{I}$-paracompactness is inherited by the union and the intersection of certain sets.

Theorem 5.11. *For an* ITP *space $(\mathrm{T}, \phi, \mathfrak{I})$, β_θ-$\mathfrak{I}$-paracompactness is preserved under taking finite unions of subsets.*

Proof. Let M and N be β_θ-$\mathfrak{I}$-paracompact subsets of T. We show that $M \cup N$ is β_θ-$\mathfrak{I}$-paracompact. Suppose that $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ is any open cover of $M \cup N$. Then $\mathcal{M}$ automatically covers each of M and N.

Using the given assumption, we can find families of β_θ-$\mathfrak{I}$-open sets which are β_θ-$\mathfrak{I}$-locally finite, namely $\mathcal{N} = \{N_\alpha : \alpha \in \Omega_1\}$ covering M and $\mathcal{O} = \{O_\lambda : \lambda \in \Omega_2\}$ covering N, such that each family refines $\mathcal{M}$ and meets these conditions: $M - \bigcup_{\alpha\in\Omega_1} N_\alpha \in \mathfrak{I}$ and $N - \bigcup_{\lambda\in\Omega_2} O_\lambda \in \mathfrak{I}$. From this we obtain: $M \subseteq \bigcup_{\alpha\in\Omega_1} N_\alpha \cup E_1$ and $N \subseteq \bigcup_{\lambda\in\Omega_2} O_\lambda \cup E_2$, where $E_1, E_2 \in \mathfrak{I}$. Therefore, we have:

$$M \cup N \subseteq \left(\bigcup_{\alpha\in\Omega_1} N_\alpha\right) \cup \left(\bigcup_{\lambda\in\Omega_2} O_\lambda\right) \cup (E_1 \cup E_2).$$

It follows that:

$$M \cup N - \bigcup_{\alpha\in\Omega_1,\lambda\in\Omega_2} (N_\alpha \cup O_\lambda) \subseteq E_1 \cup E_2 \in \mathfrak{I}.$$

Therefore $\mathrm{P} = \{N_\alpha \cup O_\lambda : \alpha \in \Omega_1, \lambda \in \Omega_2\}$ forms a β_θ-$\mathfrak{I}$-locally finite β_θ-$\mathfrak{I}$-open refinement of $\mathcal{M}$, proving that $M \cup N$ is β_θ-$\mathfrak{I}$-paracompact. □

The next theorem demonstrates that β_θ-$\mathfrak{I}$-paracompactness is inherited by intersections with closed subsets.

Theorem 5.12. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*, and suppose that* N *is a closed subset and* M *is a* β_θ-$\mathfrak{I}$-*paracompact subset of* T*. Then* $M \cap N$ *satisfies the* β_θ-$\mathfrak{I}$-*paracompactness property.*

Proof. Suppose we have an open cover $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ for $M \cap N$. Since $\mathrm{T} - N$ is open, the family of open sets $\mathcal{M}_1 = \{M_\alpha : \alpha \in \Omega\} \cup \{\mathrm{T} - N\}$ covers M. By our hypothesis and Lemma 5.2, we can find β_θ-$\mathfrak{I}$-locally finite β_θ-$\mathfrak{I}$-open refinement $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\} \cup \{V\}$ of $\mathcal{M}_1$ such that

$$N_\alpha \subseteq M_\alpha \quad \text{for all } \alpha \in \Omega, \qquad V \subseteq \mathrm{T} - N, \qquad \text{and} \qquad M - \left(\bigcup_{\alpha\in\Omega} N_\alpha \cup V\right) \in \mathfrak{I}.$$

Now, observe that:

$$M \cap N - \bigcup_{\alpha\in\Omega} N_\alpha = M \cap N - \left(\bigcup_{\alpha\in\Omega} N_\alpha \cup V\right) \subseteq M - \left(\bigcup_{\alpha\in\Omega} N_\alpha \cup V\right).$$

Thus, $M \cap N - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{I}$. The collection $\mathcal{N}_1 = \{N_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{I}$-locally finite, its members are β_θ-$\mathfrak{I}$-open, and it refines $\mathcal{M}$. It follows that $M \cap N$ is a β_θ-$\mathfrak{I}$-paracompact subset. □

Combining Theorem 5.11 with Theorem 5.12 yields the corollary below.

Corollary 5.5. *Consider an* ITP *space* $(\mathrm{T}, \phi, \mathfrak{I})$*. Then the following statements are true:*

(1) *All closed sets in* (T, ϕ) *are* β_θ-$\mathfrak{I}$-*paracompact.*
(2) *Finite unions of closed sets in* (T, ϕ) *are* β_θ-$\mathfrak{I}$-*paracompact.*
(3) *For a* β_θ-$\mathfrak{I}$-*paracompact set* M *and an open set* $N \subseteq M$*,* $M - N$ *is* β_θ-$\mathfrak{I}$-*paracompact.*

6. Preservation of β_θ-$\mathfrak{I}$-Paracompactness

In this section, we investigate functions between ITP spaces and study how specific topological properties are preserved under these mappings. We present several theorems that characterize when β_θ-$\mathfrak{I}$-paracompactness transfers from a space to its image under a mapping, or conversely, when it lifts from the codomain to the domain. These results provide a comprehensive understanding of stability for this generalized paracompactness property.

We first state a fundamental lemma that is essential for the proof of Theorem 6.1.

Lemma 6.1. *Consider two* ITP *spaces* $(\mathrm{T}, \phi, \mathfrak{J})$ *and* $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ *with a surjective mapping* $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. *The map* g *is* β_θ*-*$\mathfrak{J}$*-closed if and only if for each* $w \in \widetilde{\mathrm{T}}$ *and every* β_θ*-*$\mathfrak{J}$*-open subset* V *with* $g^{-1}(\{w\}) \subseteq V$*, one can find a* β_θ*-*$\mathfrak{J}$*-open neighborhood* W *of* w *satisfying* $g^{-1}(W) \subseteq V$.

Proof. Given $w \in \widetilde{\mathrm{T}}$, choose a β_θ-$\mathfrak{J}$-open set V in T that contains $g^{-1}(\{w\})$. Defining $W = \widetilde{\mathrm{T}} - g(\mathrm{T} - V)$, we see that W is β_θ-$\mathfrak{J}$-open in $\widetilde{\mathrm{T}}$, $w \in W$, and $g^{-1}(W) \subseteq V$. This establishes necessity.

Consider a β_θ-$\mathfrak{J}$-closed set $K \subseteq \mathrm{T}$. For any $w \in \widetilde{\mathrm{T}} - g(K)$, we have $g^{-1}(\{w\}) \subseteq \mathrm{T} - K$. The hypothesis gives a β_θ-$\mathfrak{J}$-open neighborhood W_w of w with $g^{-1}(W_w) \subseteq \mathrm{T} - K$, implying $W_w \subseteq \widetilde{\mathrm{T}} - g(K)$. We then obtain that $\widetilde{\mathrm{T}} - g(K) = \bigcup\{ W_w : w \in \widetilde{\mathrm{T}} - g(K) \}$ is β_θ-$\mathfrak{J}$-open. Consequently, $g(K)$ is β_θ-$\mathfrak{J}$-closed. □

Theorem 6.1. *Consider a* β_θ*-*$\mathfrak{J}$*-paracompact* $(\mathrm{T}, \phi, \mathfrak{J})$ *and another* ITP *space* $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. *Suppose* $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ *is a bijective, continuous,* β_θ*-*$\mathfrak{J}$*-open function satisfying* $g(\mathfrak{J}) \subseteq \mathfrak{J}$ *and* $\{g^{-1}(w)\}$ *is* β_θ*-*$\mathfrak{J}$*-compact for each* $w \in \widetilde{\mathrm{T}}$. *Then* $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ *is* β_θ*-*$\mathfrak{J}$*-paracompact.*

Proof. Suppose $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$ is an open cover of $\widetilde{\mathrm{T}}$. The family $\mathcal{M}_1 = \{g^{-1}(M_\alpha) : \alpha \in \Omega\}$ is an open cover of T, because g is continuous. Employing β_θ-$\mathfrak{J}$-paracompactness of $(\mathrm{T}, \phi, \mathfrak{J})$, $\mathcal{M}_1$ has a refinement $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\}$ of β_θ-$\mathfrak{J}$-open sets which is β_θ-$\mathfrak{J}$-locally finite and satisfies $\mathrm{T} - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{J}$. Since g is β_θ-$\mathfrak{J}$-open, the family $g(\mathcal{N}) = \{g(N_\alpha) : \alpha \in \Omega\}$, which consists of β_θ-$\mathfrak{J}$-open subsets of $\widetilde{\mathrm{T}}$, refines $\mathcal{M}$. The bijectivity of g together with the condition $g(\mathfrak{J}) \subseteq \mathfrak{J}$ guarantees $\widetilde{\mathrm{T}} - \bigcup_\alpha g(N_\alpha) \in \mathfrak{J}$.

To see that $g(\mathcal{N})$ is β_θ-$\mathfrak{J}$-locally finite, fix $w \in \widetilde{\mathrm{T}}$. By the β_θ-$\mathfrak{J}$-local finiteness of $\mathcal{N}$, for every $z \in g^{-1}(\{w\})$ there is a β_θ-$\mathfrak{J}$-open set G_z containing z that intersects only a finite subfamily $\mathcal{N}$. The collection $\mathcal{G} = \{G_z : z \in g^{-1}(\{w\})\}$ then covers $g^{-1}(\{w\})$. As $g^{-1}(\{w\})$ is β_θ-$\mathfrak{J}$-compact, we can extract a finite subcover from the open cover $\mathcal{G}$; that is, there exist points $x_1, x_2, \ldots, x_k$ such that

$$g^{-1}(\{w\}) \subseteq G_{x_1} \cup G_{x_2} \cup \cdots \cup G_{x_k},$$

where each G_{x_i} is one of the G_z. Because each G_{x_i} meets only finitely many members of $\mathcal{N}$, their finite union also meets only finitely many members of $\mathcal{N}$. The map g being both bijective and β_θ-$\mathfrak{J}$-open implies that, g is β_θ-$\mathfrak{J}$-closed. From Lemma 6.1 we deduce the existence of a β_θ-$\mathfrak{J}$-open set U_w that is a neighborhood of w satisfying

$$g^{-1}(U_w) \subseteq G_{x_1} \cup \cdots \cup G_{x_k}.$$

Consequently, $g^{-1}(U_w)$ meets only finitely many sets from $\mathcal{N}$, implying that U_w itself meets only finitely many members of $g(\mathcal{N})$. It follows that $g(\mathcal{N})$ is β_θ-$\mathfrak{J}$-locally finite, establishing the β_θ-$\mathfrak{J}$-paracompactness of $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. □

For a β_θ-$\mathfrak{J}$-paracompact ITP space $(\mathrm{T}, \phi, \mathfrak{J})$ and an ITP space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$, our aim is to describe functions mapping $(\mathrm{T}, \phi, \mathfrak{J})$ to $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ that preserve structural properties. Specifically, a map g is said to preserve β_θ-$\mathfrak{J}$-local finiteness if $g(\mathcal{U})$ is β_θ-$\mathfrak{J}$-locally finite in $\widetilde{\mathrm{T}}$ for every β_θ-$\mathfrak{J}$-locally finite family $\mathcal{U}$ in T.

Theorem 6.2. *Consider a β_θ-$\mathfrak{J}$-paracompact space $(\mathrm{T}, \phi, \mathfrak{J})$ and an* ITP *space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. Assume that $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijection which is continuous and β_θ-irresolute, satisfies $g(\mathfrak{J}) \subseteq \mathfrak{J}$, and preserves β_θ-$\mathfrak{J}$-local finiteness. Then $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-paracompact.*

Proof. Suppose $\widetilde{\mathrm{T}}$ is covered by an open cover $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$. As g is continuous, it follows that the set $\mathcal{M}_1 = \{g^{-1}(M_\alpha) : \alpha \in \Omega\}$ is open family that covers T. Using that T is β_θ-$\mathfrak{J}$-paracompact, there exists a refinement $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\}$ of $\mathcal{M}_1$ that is β_θ-$\mathfrak{J}$-locally finite, consists of β_θ-$\mathfrak{J}$-open sets, and satisfies $\mathrm{T} - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{J}$. Since

$$\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega} g(N_\alpha) = g\left(\mathrm{T} - \bigcup_{\alpha\in\Omega} N_\alpha\right) \in g(\mathfrak{J}) \subseteq \mathfrak{J}.$$

A property of the ideal $\mathfrak{J}$ implies that $\widetilde{\mathrm{T}} - \bigcup_\alpha g(N_\alpha)$ belongs to $\mathfrak{J}$. By given assumption, $g(\mathcal{N}) = \{g(N_\alpha) : \alpha \in \Omega\}$ forms a β_θ-$\mathfrak{J}$-locally finite consisting β_θ-$\mathfrak{J}$-open sets in $\widetilde{\mathrm{T}}$.

We will certify that $g(\mathcal{N})$ refines $\mathcal{M}$. For a given $g(N_\alpha)$, the corresponding N_α belongs to $\mathcal{N}$. By refinement of $\mathcal{M}_1$ by $\mathcal{N}$, there exists $M_\alpha \in \mathcal{M}$ with $N_\alpha \subseteq g^{-1}(M_\alpha)$. Applying g yields $g(N_\alpha) \subseteq M_\alpha$, establishing the refinement. Consequently, $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ possesses the property of β_θ-$\mathfrak{J}$-paracompactness. □

Lemma 6.2. *Consider* ITP *spaces $(\mathrm{T}, \phi, \mathfrak{J})$ and $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$, and a bijection β_θ-$\mathfrak{J}$-open function $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. If $\{V_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{J}$-locally finite in T, then $\{g(V_\alpha) : \alpha \in \Omega\}$ is β_θ-$\mathfrak{J}$-locally finite in $\widetilde{\mathrm{T}}$.*

Proof. Assume $\{V_\alpha : \alpha \in \Omega\}$ is β_θ-$\mathfrak{J}$-locally finite. Fix $z \in \widetilde{\mathrm{T}}$ and put $w = g^{-1}(z)$. By the β_θ-$\mathfrak{J}$-local finiteness of $\{V_\alpha : \alpha \in \Omega\}$, we can choose a β_θ-$\mathfrak{J}$-open neighbourhood U of w such that U intersects only finitely many V_α. As g is β_θ-$\mathfrak{J}$-open, $g(U)$ is β_θ-$\mathfrak{J}$-open. Because g is bijective,

$$g(U) \cap g(V_\alpha) \neq \emptyset \iff U \cap V_\alpha \neq \emptyset,$$

so $g(U)$ also meets only finitely many $g(V_\alpha)$. Hence $g(\mathcal{V})$ is β_θ-$\mathfrak{J}$-locally finite. □

Combining Theorem 6.2 with Lemma 6.2 yields the next theorem.

Theorem 6.3. *Consider a β_θ-$\mathfrak{J}$-paracompact space $(\mathrm{T}, \phi, \mathfrak{J})$ and another space $(\widetilde{\mathrm{T}}, \rho)$. Suppose a function $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho)$ is bijective continuous map that is both β_θ-$\mathfrak{J}$-open and β_θ-irresolute. Then $(\widetilde{\mathrm{T}}, \rho, g(\mathfrak{J}))$ satisfies β_θ-$g(\mathfrak{J})$-paracompactness.*

We now state a theorem that characterizes when β_θ-$\mathfrak{J}$-paracompactness can be pulled back from $\widetilde{\mathrm{T}}$ to T by a suitable map.

Theorem 6.4. *Consider an* ITP *space $(\mathrm{T}, \phi, \mathfrak{J})$ and a β_θ-$\mathfrak{J}$-paracompact space $(\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$. Suppose $g\colon (\mathrm{T}, \phi, \mathfrak{J}) \to (\widetilde{\mathrm{T}}, \rho, \mathfrak{J})$ is a bijective β_θ-irresolute function that satisfies $g(\mathfrak{J}) = \mathfrak{J}$. Then $(\mathrm{T}, \phi, \mathfrak{J})$ is β_θ-$\mathfrak{J}$-paracompact.*

Proof. Assume T is covered by open collection $\mathcal{M} = \{M_\alpha : \alpha \in \Omega\}$. The collection of open sets $g(\mathcal{M}) = \{g(M_\alpha) : \alpha \in \Omega\}$ covers $\widetilde{\mathrm{T}}$, since g is open, .Using that $\widetilde{\mathrm{T}}$ is β_θ-$\mathfrak{J}$-paracompact, one can find a refinement $\mathcal{N} = \{N_\alpha : \alpha \in \Omega\}$ of $g(\mathcal{M})$ that is β_θ-$\mathfrak{J}$-locally finite, consists of β_θ-$\mathfrak{J}$-open sets, and satisfies $\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega} N_\alpha \in \mathfrak{J}$. This implies that $\widetilde{\mathrm{T}} - \bigcup_{\alpha\in\Omega} N_\alpha = E$ for some $E \in \mathfrak{J}$, which means

$$\mathrm{T} - \bigcup_{\alpha\in\Omega} g^{-1}(N_\alpha) = g^{-1}(\widetilde{\mathrm{T}}) - g^{-1}\left(\bigcup_{\alpha\in\Omega} N_\alpha\right) = g^{-1}(E) \in \mathfrak{J},$$

as g is a bijection. Since g is β_θ-irresolute, the collection $\mathcal{N}_1 = \{g^{-1}(N_\alpha) : \alpha \in \Omega\}$ is of β_θ-$\mathfrak{J}$-open sets and forms a β_θ-$\mathfrak{J}$-locally finite. For every $g^{-1}(N_\alpha) \in \mathcal{N}_1$, the refinement property of $\mathcal{N}$ gives an $M_\alpha \in \mathcal{M}$ with $N_\alpha \subseteq g(M_\alpha)$. Consequently,

$$g^{-1}(N_\alpha) \subseteq g^{-1}(g(M_\alpha)) = M_\alpha.$$

The fact that $\mathcal{N}_1$ refines $\mathcal{M}$ is thus established. Therefore, the space $(\mathrm{T}, \phi, \mathfrak{J})$ satisfies β_θ-$\mathfrak{J}$-paracompactness. □

7. Conclusions

In this paper, we have introduced and systematically developed the notions of β_θ-$\mathfrak{J}$-compactness and β_θ-$\mathfrak{J}$-paracompactness within the framework of ITP spaces. By employing β_θ-$\mathfrak{J}$-open sets—defined in terms of the θ-closure operator and the ideal-induced $*$-topology—we have established a unified setting that generalizes both classical covering properties and their existing ideal-based extensions. Several characterizations of these new concepts have been obtained, along with preservation results under appropriate classes of mappings. In addition, we have explored the relationships between β_θ-$\mathfrak{J}$-paracompactness and separation axioms, particularly β_θ-$\mathfrak{J}$-regularity and β_θ-$\mathfrak{J}$-normality. Collectively, the results presented here enrich the general theory of ideal topology and furnish effective tools for investigating generalized continuity, separation axioms, and associated covering properties within ideal-theoretic frameworks.

The relationships among the generalized compactness and paracompactness properties introduced in this work can be summarized by the following implications:

$$\begin{array}{ccc} \beta_\theta\text{-}\mathfrak{J}\text{-compact} & \Longrightarrow & \text{compact} \\ \beta_\theta\text{-}\mathfrak{J}\text{-compact} & \Longrightarrow & \text{compatibly } \beta_\theta\text{-}\mathfrak{J}\text{-compact} \\ \text{compatibly } \beta_\theta\text{-}\mathfrak{J}\text{-compact} & \Longrightarrow & \beta_\theta\text{-}\mathfrak{J}\text{-paracompact} \\ \text{paracompact} & \Longrightarrow & \beta_\theta\text{-}\mathfrak{J}\text{-paracompact} \end{array}$$

Moreover, under suitable separation and refinement conditions, the following implications hold:

$$\beta_\theta\text{-}\mathfrak{J}\text{-paracompact} + \text{Hausdorff} + \beta_\theta\text{-}\mathfrak{J}\text{-open sets are finitely } \cap\text{-closed} \Longrightarrow \beta_\theta\text{-}\mathfrak{J}\text{-regular}$$

$$\beta_\theta\text{-}\mathfrak{I}\text{-paracompact} + \text{Hausdorff} + \mathfrak{I} \text{ closed under arbitrary unions}$$
$$+\beta_\theta\text{-}\mathfrak{I}\text{-open sets are finitely } \cap\text{-closed} \Longrightarrow \beta_\theta\text{-}\mathfrak{I}\text{-normal.}$$

It should be noted, however, that the proofs of the foregoing implications depend crucially on the finite intersection-closure of the β_θ-$\mathfrak{I}$-open sets. As demonstrated in Example 2.3, this property is not generally satisfied in arbitrary ITP spaces. Consequently, the results presented above are established not for the full class of ITP spaces, but rather for the subclass in which the β_θ-$\mathfrak{I}$-open sets are closed under finite intersections. This constitutes a limitation of the present framework and indicates a direction for future refinement.

The proposed framework also suggests several potential applications. In generalized continuity theory, β_θ-$\mathfrak{I}$-open sets may be used to define new classes of continuous mappings, which could be relevant in areas such as digital topology and computational geometry, particularly when the ideal $\mathfrak{I}$ models approximation or uncertainty. Regarding separation axioms, the established connections among β_θ-$\mathfrak{I}$-paracompactness, regularity, and normality provide a useful foundation for investigating extension problems and handling singularities in ITP spaces. In covering-based models of information systems, β_θ-$\mathfrak{I}$-locally finite refinements may offer useful tools for representing data structures in which the ideal $\mathfrak{I}$ captures negligible or insignificant information. In addition, these concepts may be extended to structures arising in topological algebra, such as ideal topological groups or rings, allowing further investigation of compactness and paracompactness within algebraic systems endowed with topology.

The present work naturally opens up several avenues for future exploration. These include proposing the concept of β_θ-$\mathfrak{I}$-connectedness and examining the associated topologies on function spaces, as well as broadening the current framework to encompass fuzzy or soft topological spaces endowed with ideal structures. Another worthwhile direction lies in establishing metrization theorems for β_θ-$\mathfrak{I}$-paracompact spaces. Furthermore, a study of β_θ-$\mathfrak{I}$-partitions of unity could yield significant applications in analysis. Lastly, the construction of pertinent counterexamples would serve to elucidate the interrelations and hierarchies among various generalized covering notions, thereby reinforcing the bridges between ideal topology and other branches of mathematics.

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The authors contributed to this work in the following ways:

- Chawalit Boonpok: Conceptualization; formal analysis; investigation; methodology; validation; original draft; review and support editing
- Areeyuth Sama-Ae: Project administration; conceptualization; original draft; formal analysis; validation; methodology; supervision; review and editing.

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