Asia Pac. J. Math. 2026 **13**:139

APPLYING TRI-SIMULATION FUNCTIONS IN G-METRIC SPACES TO NONLINEAR INTEGRAL EQUATIONS FOR GENERALIZED FIXED POINT RESULTS

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Received Jul. 5, 2026

Abstract. This study introduces tri-simulation functions to provide new fixed point results for generalized $\aleph Z_G$-contractive mappings in complete G-metric spaces (GMS) . The approach in question combines triangular $\aleph$-admissibility with a generalized contractive framework, without requiring mapping continuity. Suitable admissibility and sequential requirements yield a fixed-point existence theorem, and an extra natural assumption yields a uniqueness result. These findings broaden a number of simulation functions and acceptable contraction fixed point in (GMS). We evaluate the solvability of a non-linear Hammerstein integral equation. We prove the existence and its uniqueness of integral equation of standard solution by developing an appropriate operator on a complete G-metric function space and validating the theorem's assumptions. The application shows tri-simulation's efficacy in nonlinear functional analysis and integral equations.

2020 Mathematics Subject Classification. 47H10.

Key words and phrases: fixed point; G-metric space; tri-simulation function; generalized $\aleph\mathcal{Z}_G$-contraction; triangular $\aleph$-admissible mapping; Hammerstein integral equation.

1. Introduction and Preliminaries

Due to its many uses in math, engineering, economics, optimization theory, differential equations, and integral equations, fixed point theory has grown to become one of the busiest areas of nonlinear analysis. We can trace the roots of this theory back to the famous Banach contraction principle [1]. This principle says that fixed points must exist and be unique in complete metric spaces. As time has gone on, many experts have built on Banach's theorem and made it more general and useful in various abstract settings.

The generalization of metric spaces is one of the important ways this progress is going in. Mustafa and Sims [2] came up with the idea of a G-metric space in 2006 as a natural development of classical metric spaces. They found out what the basic topological features of these spaces are and proved a

DOI: 10.28924/APJM/13-139

number of fixed point theorems. Many people were interested in their work, and it led to new ways of studying fixed point problems in generalized metric systems. Consequently, numerous authors conducted research on fixed point results in G-metric spaces under various contractive conditions. In particular, various researchers, including Mustafa and Sims [2], Abbas and Rhoades [3], and others, have devised fixed point techniques that incorporate admissibility conditions, weak contractions, cyclic contractions, and generalized contractive mappings. Numerous researchers have recently conducted additional research on generalized metric structures. In particular, Hashoosh and Dasher [12] utilized the concept of a tri-simulation function to introduce novel findings in fixed-point theory within S-metric spaces. Their work contributed to the expansion of classical fixed point techniques beyond the confines of standard metric spaces.

Hashoosh and Ali [7] established a new method for investigating fixed points in $\mathcal{G}$-spaces, demonstrating the usefulness of generalized contractive methods in nonlinear analysis. These findings have spurred research on generalized distance structures and current contraction principles.

Khojasteh, Shukla, and Radenović [4] pioneered the theory by introducing simulation functions and creating a coherent framework for investigating contractive conditions. Their method allowed them to regard numerous classical contractions as special examples of a general contractive scheme. Later, Karapınar [5] improved this methodology and produced fixed-point results using metric space simulation functions.

Researchers became inspired to develop a number of generalized kinds of simulation-based contractions as a result of the success of simulation functions. Tri-simulation functions were offered as powerful tools for creating larger classes of contractive mappings and achieving stronger fixed-point findings [8]. This approach represented one of the significant advances made in the field. By expanding a large number of fixed point concepts that are already well-known, the tri-simulation approach has proven to be a successful method for giving new applications to nonlinear issues.

According to Samet, Vetro, and Vetro [6], the introduction of admissibility requirements has significantly contributed to the development of current fixed point theory. In addition to considerably expanding the range of mappings that are being considered, such conditions make it possible to derive fixed point conclusions without applying stringent continuity assumptions. There is a close connection between the nonlinear analysis techniques used in fixed-point theory and a number of other research topics, such as variational inequalities, optimization theory, and equilibrium problems, for example. It is possible to obtain recent contributions in these areas in the following sources: [9–11,14].

The goal of this work is to demonstrate novel fixed-point findings for extended $\aleph\, Z_G$-contractive mapping in complete GMS via tri-simulation functions, taking inspiration from the aforementioned developments. By combining admissibility procedures with generalized contractive conditions, the strategy that has been proposed eliminates the continuity assumption that has been enforced in a number of additional publications that are related. A further application of the results obtained is to analyze the existence of solutions for a class of nonlinear Hammerstein integral equations and to determine whether these solutions are unique.

There are two parts to this study that make it significant. With the use of tri-simulation functions and triangular $\aleph$-admissibility criteria, it offers a fresh theoretical development of fixed point theory in GMS. The practical aspect is that it proves the previously found conclusions hold for nonlinear integral equations, which further links nonlinear analysis to abstract fixed-point theory. We go over the fundamental concepts and conclusions of the study in this section.

Definition 1.1. [2] Let $X \neq \phi$, and let $G : X \times X \to [0, \infty)$ satisfies the following conditions:

i. $G(\mu, \ell, z) = 0$ if and only if $\zeta = \ell = z$;
ii. $G(\zeta, \zeta, \ell) > 0$ whenever $\zeta \neq \ell$;
iii. $G(\zeta, \zeta, \ell) \leq G(\zeta, \ell, z)$ for all $\zeta, \ell, z \in \zeta$ with $\ell \neq z$;
vi. $G(\zeta, \ell, z) = G(\zeta, z, \ell) = G(\ell, z, \zeta) = \cdots$
v. $G(\zeta, \ell, z) \leq G(\zeta, a, a) + G(a, \ell, z)$ for all $\zeta, \ell, z, a \in X$.

Then, (X, G) is called a G-metric space GMS.

Definition 1.2. [2] Consider that (X, G) is a GMS and $\{\zeta_n\}$ be a sequence in X.

i. $\{\zeta_n\}$ converges to $\zeta \in X$ if, $\lim_{n\to\infty} G(\zeta, \zeta_n, \zeta_n) = 0$
ii. $\{\zeta_n\}$ is called a G-Cauchy sequence if, $\lim_{n,m\to\infty} G(\zeta_n, \zeta_m, \zeta_m) = 0$
iii. The space (X, G) is called complete if every G-Cauchy sequence converges in X.

Definition 1.3. [8] Consider that $M : X \to X$ and $\aleph : X \times X \times X \to [0, \infty)$. The mapping M is called $\aleph$-admissible if $\aleph(\zeta, \ell, \ell) \geq 1$ implies $\aleph(M\zeta, M\ell, M\ell) \geq 1$ for all $\zeta, \ell \in X$.

Definition 1.4. [4] A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is named a simulation if

i. $\zeta(0, 0) = 0$, for all $t, s > 0$;
ii. $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$
iii. $t_n \to r > 0, \qquad s_n \to r > 0.$

Definition 1.5. [8] Considering $T : \mathbb{R}^3_+ \to \mathbb{R}$ is a mapping then T named to be a tri-simulation in the event that you satisfy one of the prerequisites specified below:

$T_1)T(z, \ell, \zeta) < \zeta - \ell z \quad \forall \zeta, \ell > 0, z \geq 0;$

T_2) Let $\{\zeta_n\}$, $\{\ell_n\}$ and $\{z_n\}$ are sequences in $(0, \infty)$ where $\ell_n < \zeta_n$ for $n \in N$ such that $\lim_{n\to\infty} z_n \geq 1$ and $\lim_{n\to\infty} \ell_n = \lim_{n\to\infty} \zeta_n > 0 \to \lim_{n\to\infty} \sup T(\zeta_n, \ell_n, z_n) < 0$

2. Results

Establishing novel fixed point findings for generalized standardized $\aleph Z_G$-contractive mappings in complete GMS is the aim of this section. This is accomplished by utilizing the framework of tri-simulation functions and triangular $\aleph$-admissibility. In furtherance of serving as the theoretical framework for the application that is detailed in the following section, the theorems that were established offer sufficient requirements for the existence of fixed points and ensure that they are entirely unique.

Definition 2.1. Consider that $M \neq \phi$ defined on GMS, $\aleph : \mathrm{X} \times \mathrm{X} \times \mathrm{X} \to [0, \infty)$, $\forall \zeta, \mathrm{v}, \varpi \in \mathrm{X}$ and let $M : X \to X$, then a $\aleph$-admissible triangular mapping is defined as one that meets the aforementioned criteria, If

1- A mapping M is $\aleph$-admissible;

2- A mapping $\aleph(\zeta, \zeta, v) \geq 1$ and $\aleph(v, v, \varpi) \geq 1$ after that $\aleph(\zeta, \zeta, \varpi) \geq 1$.

Definition 2.2. Consider that (X, G) is a GMS and $M : X \to X$ be a self-mapping. If M is a generalized $\aleph Z_G$-contraction to a tri-simulation function T, then

$$T\Big(\aleph(\zeta, \nu, \nu), G(M\zeta, M\nu, M\nu), \Theta(\zeta, \nu)\Big) \geq 0$$

$\forall\ \zeta, \nu \in X$, in which

$$\Theta(\zeta, \nu) =: \max\Big\{G(\zeta, \nu, \nu), G(\zeta, M\zeta, M\zeta), G(\nu, M\nu, M\nu)\Big\}$$

Lemma 2.1. *Consider that $a_n = G(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$ If $a_n \to r$, then, $\Theta(\zeta_n, \zeta_{n+1}) \to r$.*

Proof. Given that $\Theta(\zeta_n, \zeta_{n+1}) =: \max\{a_n, a_n, a_{n+1}\}$, and

$$a_n \to r, \qquad a_{n+1} \to r,$$

it is consequent that

$$\Theta(\zeta_n, \zeta_{n+1}) \to r$$

□

Lemma 2.2. *Consider (X, G) is a GMS, and let $\{\zeta_n\}$ be defined by the equation $\zeta_{n+1} = M\zeta_n$, where $n \geq 0$. Suppose that*

$$G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) < \Theta(\zeta_n, \zeta_{n+1}),$$

such that

$$\Theta(\zeta_n, \zeta_{n+1}) = \max\Big\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}), G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})\Big\}$$

Moreover,

$$G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) < G(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$$

Proof. Consider that

$$G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})<\Theta(\zeta_n,\zeta_{n+1}),\tag{2.1}$$

in which

$$\Theta(\zeta_n,\zeta_{n+1})=\max\Big\{G(\zeta_n,\zeta_{n+1},\zeta_{n+1}),G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})\Big\}$$

Take, on the other hand, the scenario under which

$$\Theta(\zeta_n,\zeta_{n+1})=G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})\tag{2.2}$$

When we plug in (2.1) into (2.2), we get the following:

$$G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})<G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2}),$$

since this is not possible. Which is why,

$$\Theta(\zeta_n,\zeta_{n+1})\neq G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})$$

As a result, the maximum level that is possible must exist at

$$G(\zeta_n,\zeta_{n+1},\zeta_{n+1}).$$

Then,

$$\Theta(\zeta_n,\zeta_{n+1})=G(\zeta_n,\zeta_{n+1},\zeta_{n+1})\tag{2.3}$$

Putting together equations (2.1) and (2.3), we obtain

$$G(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2})<G(\zeta_n,\zeta_{n+1},\zeta_{n+1})$$

□

Theorem 2.1. *If (X,G) is a complete GMSand let $M:X\to X$ be a self-mapping. If $\exists\,\aleph:X\times X\times X\to[0,\infty)$ and a tri-simulation mapping $T:\mathbb{R}^3_+\to\mathbb{R}$ in which*

$$T\Big(\aleph(\zeta,\ell,\ell),G(M\zeta,M\ell,M\ell),\Theta(\zeta,\ell)\Big)\geq 0,\tag{2.4}$$

for each $\zeta,\ell\in X$, such that

$$\Theta(\zeta,\ell)=\max\Big\{G(\zeta,\ell,\ell),G(\zeta,M\zeta,M\zeta),G(\ell,M\ell,M\ell)\Big\}$$

Assume further that

(1) *M is triangular $\aleph$-admissible;*

(2) *$\exists\quad\zeta_0\in X$ s.t.*

$$\aleph(\zeta_0,M\zeta_0,M\zeta_0)\geq 1;$$

(3) *$\{\aleph(\zeta_n,\zeta_{n+1},\zeta_{n+1})\}$ is bounded. Further, whenever $\{\zeta_n\}\in X$ satisfies $\aleph(\zeta_n,\zeta_{n+1},\zeta_{n+1})\geq 1$, $n\in\mathbb{N}$, and $\zeta_n\to u$, after than there is a subsequence $\{\zeta_{n(k)}\}$ such that $\aleph(\zeta_{n(k)},u,u)\geq 1$ for all $k\in\mathbb{N}$.*

That is, there is a fixed point for M.

Proof. Let $\zeta_n = M\zeta_{n-1}, \qquad n \geq 1$ Since $\aleph(\zeta_0, M\zeta_0, M\zeta_0) \geq 1$, and according to the inductive explanation, since M is triangular $\aleph$-admissible,

$$\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \geq 1, \qquad n \geq 0. \tag{2.5}$$

By equation (2.4) with the variables $\zeta = \zeta_n$ and $\ell = \zeta_{n+1}$, one can able to understand the following:

$$T\Big(\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}), G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}), \Theta(\zeta_n, \zeta_{n+1})\Big) \geq 0$$

As T is a tri-simulation function, (T_1) implies

$$\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1})\, G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) < \Theta(\zeta_n, \zeta_{n+1})$$

Since

$$\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \geq 1,$$

it is consequent that

$$G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) < \Theta(\zeta_n, \zeta_{n+1})$$

$$\Theta(\zeta_n, \zeta_{n+1}) = G(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$$

So,

$$G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) < G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \tag{2.6}$$

Therefore, the above sequence is decreasing and bounded below by 0. Consequently, there is $r \geq 0$ where

$$\lim_{n\to\infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = r$$

Suppose that $r > 0$. Passing to the limit in (2.6) Since

$$\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \geq 1 \quad (n \geq 0),$$

one can get

$$\liminf_{n\to\infty} \aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \geq 1$$

Hence, by passing to a suitable subsequence if necessary, there is a sequence $z_n = \aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$ such that

$$\liminf_{n\to\infty} z_n \geq 1$$

Set

$$\zeta_n = \Theta(\zeta_n, \zeta_{n+1}), \qquad \ell_n = G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}), \qquad z_n = \aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$$

By (2.6), we have

$$\ell_n < \zeta_n$$

for every n, and by Lemma 2.1,

$$\zeta_n \to r, \qquad \ell_n \to r$$

Hence all assumptions of (T_2) are satisfied. So, using property (T_2) of the tri-simulation function, we obtain

$$0 \leq \limsup_{n\to\infty} T\Big(\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}), G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}), \Theta(\zeta_n, \zeta_{n+1})\Big) < 0,$$

which is impossible. Thus

$$\lim_{n\to\infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0 \tag{2.7}$$

Then, ζ_n is a G-Cauchy. Alternatively, let pretend ζ_n is not a G-Cauchy sequence. If $\varepsilon > 0$, then there are two subsequences ζ_{n_k} and ζ_{m_k} of ζ_n with $m_k > n_k$ in which

$$G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) \geq \varepsilon, \qquad k \in \mathbb{N} \tag{2.8}$$

Choose m_k as the smallest number bigger than n_k that satisfies (2.8) for every $k \in \mathbb{N}$. After that

$$G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1}) < \varepsilon \tag{2.9}$$

By the rectangular inequality and since (2.9) is the case, it follows that

$$\varepsilon \leq G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) < \varepsilon + G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{m_k})$$

Since

$$\lim_{n\to\infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0,$$

So

$$\lim_{k\to\infty} G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{m_k}) = 0$$

Then,

$$\lim_{k\to\infty} G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) = \varepsilon. \tag{2.10}$$

Here, our claim that

$$\aleph(\zeta_n, \zeta_m, \zeta_m) \geq 1, \qquad \forall, m > n. \tag{2.11}$$

Indeed, taking into consideration the construction of ζ_n,

$$\aleph(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \geq 1, \qquad n \in \mathbb{N}$$

Suppose that

$$\aleph(\zeta_n, \zeta_m, \zeta_m) \geq 1$$

Since

$$\aleph(\zeta_m, \zeta_{m+1}, \zeta_{m+1}) \geq 1,$$

and M is triangular $\aleph$-admissible, we have

$$\aleph(\zeta_n, \zeta_{m+1}, \zeta_{m+1}) \geq 1$$

So, from induction, (2.11) holds, and by the contractive condition (2.4) with $\zeta = \zeta_{n_k}$ and $\ell = \zeta_{m_k}$, we get

$$T\Big(\aleph(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}), G(\zeta_{n_k+1}, \zeta_{m_k+1}, \zeta_{m_k+1}), \Theta(\zeta_{n_k}, \zeta_{m_k})\Big) \geq 0 \tag{2.12}$$

The rectangular inequality, on the other hand, states that

$$G(\zeta_{n_k+1}, \zeta_{m_k+1}, \zeta_{m_k+1}) \leq G(\zeta_{n_k+1}, \zeta_{n_k}, \zeta_{n_k}) + G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) + G(\zeta_{m_k}, \zeta_{m_k+1}, \zeta_{m_k+1}),$$

and

$$G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) \leq G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{n_k+1}) + G(\zeta_{n_k+1}, \zeta_{m_k+1}, \zeta_{m_k+1}) + G(\zeta_{m_k+1}, \zeta_{m_k}, \zeta_{m_k})$$

By the formula (2.10) in conjunction with passing to the limit as k approaches infinity

$$\lim_{n\to\infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0,$$

we deduce that

$$\lim_{k\to\infty} G(\zeta_{n_k+1}, \zeta_{m_k+1}, \zeta_{m_k+1}) = \varepsilon \tag{2.13}$$

Further,

$$\Theta(\zeta_{n_k}, \zeta_{m_k}) = \max\{G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}), G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{n_k+1}), G(\zeta_{m_k}, \zeta_{m_k+1}, \zeta_{m_k+1})\}$$

Taking into consideration the fact that (2.10), we have the limit goes to 0, therefore

$$\lim_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}) = \varepsilon \tag{2.14}$$

In conclusion, by utilizing the terms (2.11–2.14) and the property (T_2) of the tri-simulation function T, we are able to derive

$$0 \leq \limsup_{k\to\infty} T\Big(\aleph(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}), G(\zeta_{n_k+1}, \zeta_{m_k+1}, \zeta_{m_k+1}), \Theta(\zeta_{n_k}, \zeta_{m_k})\Big) < 0,$$

which is not possible. This contradiction demonstrates that the sequence ζ_n is a G-Cauchy sequence. If (X, G) is complete and ζ_n is a G-Cauchy sequence, $\exists\, u \in X$ and

$$\lim_{n\to\infty} \zeta_n = u \tag{2.15}$$

According to assumption (3), a subsequence $\zeta_{n(k)}$ of ζ_n exists and

$$\aleph(\zeta_{n(k)}, u, u) \geq 1 \tag{2.16}$$

Using the contractive condition (2.4) for $\zeta = \zeta_{n(k)}$ and $\ell = u$, we have

$$T\Big(\aleph(\zeta_{n(k)}, u, u), G(M\zeta_{n(k)}, Mu, Mu), \Theta(\zeta_{n(k)}, u)\Big) \geq 0 \tag{2.17}$$

The property (T_1) implies that $\aleph(\zeta_{n(k)}, u, u) \geq 1$.

$$G(M\zeta_{n(k)}, Mu, Mu) < \Theta(\zeta_{n(k)}, u) \tag{2.18}$$

By (2.15) and given that

$$M\zeta_{n(k)} = \zeta_{n(k)+1},$$

we have

$$\lim_{k\to\infty} \zeta_{n(k)} = u \quad \text{and} \quad \lim_{k\to\infty} M\zeta_{n(k)} = \lim_{k\to\infty} \zeta_{n(k)+1} = u$$

Therefore,

$$\lim_{k\to\infty} \Theta(\zeta_{n(k)}, u) = 0 \tag{2.19}$$

Adding (2.18) with (2.19), we get

$$\lim_{k\to\infty} G(M\zeta_{n(k)}, Mu, Mu) = 0 \tag{2.20}$$

With rectangle inequality, we have

$$G(u, Mu, Mu) \leq G(u, M\zeta_{n(k)}, M\zeta_{n(k)}) + G(M\zeta_{n(k)}, Mu, Mu)$$

Passing to the limit as $k \to \infty$ and using (2.15) and (2.20), we get

$$G(u, Mu, Mu) = 0.$$

Hence,

$$Mu = u$$

Which is a desired. □

Remark 1. *Theorem 2 can be reduced to the analogous simulation-function fixed point theorem of Khojasteh et al. if the tri-simulation function is selected as $T(a, b, c) = c - b$ and if $\aleph(\zeta, \ell, \ell) = 1$ for all $\zeta, \ell \in X$. This can be shown by comparison with the results that have already been obtained. Therefore, the current theorem expands the conventional framework in an appropriate manner by integrating the following:*

(1) *triangular $\aleph$-admissibility,*
(2) *tri-simulation functions,*
(3) *generalized (Z_G)-contractive mappings in GMS,*
(4) *fixed−point existence without continuity assumptions.*

Theorem 2.2. *Assume all key theorem hypotheses. Furthermore, suppose*

$$\aleph(\zeta, \nu, \nu) > 1$$

for all distinct fixed points $\zeta, \nu \in Fix(M)$.

Proof. Main theorem is that $Fix(M)$ is nonempty. Instead, assume two separate fixed−points $u, v \in Fix(M)$ with $u \neq v$. As u and v are fixed points of M, we obtain $Mu = u$ and $Mv = v$. Moreover, by assumption, $\aleph(u, v, v) \geq 1$. Using the generalized $\aleph$ Z_G-contractive condition (3.1) for $\zeta = u$ and $\ell = v$, we get

$$T\Big(\aleph(u, v, v), G(Mu, Mv, Mv), \Theta(u, v)\Big) \geq 0, \tag{2.21}$$

in which

$$\Theta(u, v) =: \max\Big\{G(u, v, v), G(u, Mu, Mu), G(v, Mv, Mv)\Big\}$$

Due to the fact that u and v are fixed positions, it makes sense that

$$G(u, Mu, Mu) = G(u, u, u) = 0,$$

and

$$G(v, Mv, Mv) = G(v, v, v) = 0$$

Therefore,

$$\Theta(u, v) = G(u, v, v) \tag{2.22}$$

Also,

$$G(Mu, Mv, Mv) = G(u, v, v) \tag{2.23}$$

By entering (2.22) and (2.23) into the equation (2.21), we are able to obtain

$$T\Big(\aleph(u, v, v), G(u, v, v), G(u, v, v)\Big) \geq 0 \tag{2.24}$$

Since$\aleph(u, v, v) \geq 1$ and $G(u, v, v) > 0$, property $(T1)$ yields

$$T(\aleph(u, v, v), G(u, v, v), G(u, v, v)) < G(u, v, v) - \aleph(u, v, v)G(u, v, v) \leq 0$$

It contradicts 2.24. Since our assumption is incorrect

$$u = v$$

As a consequence, we can conclude that $Fix(M)$ has precisely one element.As a result, M possesses a singular fixed point. □

Example 2.1. *Let $X = [0, \infty)$ provided by the GMS as $G(\zeta, \ell, z) = |\zeta - \ell| + |\ell - z| + |z - \zeta|, \forall\, \zeta, \ell, z \in X$. Let us define the mapping $M : X \to X$ by*

$$M(\zeta) = \frac{\zeta}{4} + 1$$

Define too

$$\aleph : X \times X \times X \to [0, \infty)$$

by

$$\aleph(\zeta,\ell,z)=\begin{cases}2, & \zeta,\ell,z\geq 1,\\ 0, & \textit{otherwise}.\end{cases}$$

Additionally, consider a tri-simulation. $T(a,b,c)=c-ab$ *We demonstrate that Theorem 3.1 fully meets its assumptions. For all* $\zeta,\ell\in X$ *with* $\aleph(\zeta,\ell,\ell)\geq 1$, *we obtain* $\zeta\geq 1, \qquad \ell\geq 1$ *Since*

$$M(\zeta)=\frac{\zeta}{4}+1\geq 1,$$

and

$$M(\ell)=\frac{\ell}{4}+1\geq 1,$$

it follows that

$$\aleph(M\zeta,M\ell,M\ell)=2\geq 1$$

Hence M *is* $\aleph$*-admissible. Moreover, if* $\aleph(\zeta,\ell,\ell)\geq 1$, *and* $\aleph(\ell,z,z)\geq 1$ *so,* $\zeta,\ell,z\geq 1$. *This implies that* $\aleph(\zeta,z,z)=2\geq 1$. *We conclude that,* M *is triangular* $\aleph$*-admissible. Next, let* $\zeta_0=1$ *Then,*

$$M(\zeta_0)=\frac{5}{4},$$

So,

$$\aleph(\zeta_0,M\zeta_0,M\zeta_0)=2\geq 1$$

At this point, for inexplicable $\zeta,\ell\in X$ *with* $\aleph(\zeta,\ell,\ell)\geq 1$, *we have*

$$G(M\zeta,M\ell,M\ell)=2|M\zeta-M\ell|$$

Since

$$M\zeta-M\ell=\frac{\zeta-\ell}{4},$$

it follows that

$$G(M\zeta,M\ell,M\ell)=\frac{1}{2}|\zeta-\ell|.$$

In contrast,

$$G(\zeta,\ell,\ell)=2|\zeta-\ell|$$

Hence

$$G(M\zeta,M\ell,M\ell)=\frac{1}{4}G(\zeta,\ell,\ell) \tag{2.25}$$

Since

$$\Theta(\zeta,\ell)=\max\Big\{G(\zeta,\ell,\ell),G(\zeta,M\zeta,M\zeta),G(\ell,M\ell,M\ell)\Big\},$$

we get

$$\Theta(\zeta,\ell)\geq G(\zeta,\ell,\ell)$$

The result of combining this with 2.25 is as follows:

$$G(M\zeta, M\ell, M\ell) \leq \frac{1}{4}\,\Theta(\zeta,\ell)$$

Then,

$$T\Big(\aleph(\zeta,\ell,\ell), G(M\zeta, M\ell, M\ell), \Theta(\zeta,\ell)\Big)$$

$$= \Theta(\zeta,\ell) - \aleph(\zeta,\ell,\ell)G(M\zeta, M\ell, M\ell)$$

Since

$$\aleph(\zeta,\ell,\ell) = 2,$$

and

$$G(M\zeta, M\ell, M\ell) \leq \frac{1}{4}\Theta(\zeta,\ell),$$

One can get

$$T\Big(\aleph(\zeta,\ell,\ell), G(M\zeta, M\ell, M\ell), \Theta(\zeta,\ell)\Big) \geq \Theta(\zeta,\ell) - \frac{1}{2}\Theta(\zeta,\ell) = \frac{1}{2}\Theta(\zeta,\ell) \geq 0$$

As a result, M is a generalized $\aleph Z_G$-contractive mapping variable.Assumptions 2.1 in Theorem 2.1 are all valid. M possesses a singular fixed point.

This example shows that the fixed point theorem applies to complete GMS.

3. Application to Nonlinear Integral Equations

We demonstrate an application of the main theorem in this part, which will allow us to determine whether or not a solution exists for a nonlinear Hammerstein integral problem.

Take into consideration the integral equation.

$$\zeta(t) = \varphi(t) + \int_0^1 \mho(t,s,\zeta(s))\,ds, \qquad t \in [0,1], \tag{3.1}$$

in which

$$\varphi : [0,1] \to \mathbb{R}$$

and

$$\mho : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$$

are continuous functions. Consider the possibility that the GMS belongs to the set $X = C([0,1],\mathbb{R})$.

$$G(\zeta,\ell,z) = \|\zeta-\ell\|_\infty + \|\ell - z\|_\infty + \|z-\zeta\|_\infty,$$

for all $\zeta,\ell,z \in X$. As is generally known, (X,G) is a complete G-metric space. Suppose these conditions hold:

(1) There is a continuous function

$$p : [0,1] \times [0,1] \to [0,\infty)$$

in which

$$|\mho(t,s,u) - \mho(t,s,v)| \leq p(t,s)|u-v|$$

for each $t, s \in [0,1]$ and $u, v \in \mathbb{R}$.

(2)

$$\sup_{t\in[0,1]} \int_0^1 p(t,s)\,ds = q < \frac{1}{2}$$

(3)

$$\aleph(\zeta, \ell, \ell) = 1$$

for all $\zeta, \ell \in X$.

Define the operator $M : X \to X$ by

$$(M\zeta)(t) = \varphi(t) + \int_0^1 \mho(t,s,\zeta(s))\,ds \tag{3.2}$$

It is important to note that the fixed points of M correspond with the solutions of the integral equation known as (3.1).

Despite the fact that the application is provided with the specific tri-simulation function $T(a,b,c) = c - ab$, the existence conclusion that was achieved is still valid for any tri-simulation function that satisfies $(T1)$–$(T2)$. Because of this, the application serves as a particular instance of the general theory that was defined in Section 2.

Theorem 3.1. *Given that* (1), (2), *and* (3) *are true, the nonlinear integral equation* (3.1) *has a singular solution in the set* $C([0,1], \mathbb{R})$.

Proof. Assume that $\zeta, \ell \in X$. So,

$$|(M\zeta)(t) - (M\ell)(t)|$$

$$= \left| \int_0^1 \Big(\mho(t,s,\zeta(s)) - \mho(t,s,\ell(s)) \Big) ds \right|$$

Using the first assumption, we are able to obtain

$$|(M\zeta)(t) - (M\ell)(t)| \leq \int_0^1 p(t,s)|\zeta(s) - \ell(s)|ds$$

Hence

$$|(M\zeta)(t) - (M\ell)(t)| \leq \|\zeta - \ell\|_\infty \int_0^1 p(t,s)ds$$

Assuming supremum over $t \in [0,1]$,

$$\|M\zeta - M\ell\|_\infty \leq \|\zeta - \ell\|_\infty \tag{3.3}$$

Since

$$q < \frac{1}{2},$$

one can get

$$G(M\zeta, M\ell, M\ell) = 2\|M\zeta - M\ell\|_\infty$$

$$\leq 2q\|\zeta - \ell\|_\infty$$

But

$$G(\zeta, \ell, \ell) = 2\|\zeta - \ell\|_\infty$$

So,

$$G(M\zeta, M\ell, M\ell) \leq G(\zeta, \ell, \ell) \tag{3.4}$$

Now, define

$$T(a, b, c) = c - ab$$

In other words, T is a function that performs tri-simulation. The formula (3.4)

$$T\Big(1, G(M\zeta, M\ell, M\ell), G(\zeta, \ell, \ell)\Big)$$

$$= G(\zeta, \ell, \ell) - G(M\zeta, M\ell, M\ell)$$

$$\geq (1 - q)G(\zeta, \ell, \ell) \geq 0$$

Thus M is a generalized $\aleph Z_G$-contraction. Since

$$\aleph(\zeta, \ell, \ell) = 1$$

for all $\zeta, \ell \in X$, Map M is triangular and $\aleph$-admissible. We meet all main theorem assumptions. Hence, M has a fixed point $\zeta^* \in X$. Thus,

$$\zeta^*(t) = \varphi(t) + \int_0^1 \mho(t, s, \zeta^*(s))\, ds,$$

for every $t \in [0, 1]$. As a result, the nonlinear integral equation (3.1) has a solution in the form of ζ^*. In order to prove that ζ^* and ℓ^* are distinct, let us suppose that they are two solutions. Following that,

$$G(M\zeta^*, M\ell^*, M\ell^*) = G(\zeta^*, \ell^*, \ell^*)$$

Using the contractive condition results in

$$G(\zeta^*, \ell^*, \ell^*) < G(\zeta^*, \ell^*, \ell^*),$$

which is not possible. Therefore,

$$\zeta^* = \ell^*$$

Because of this, the solution to (3.1) is unique. □

4. Conclusion

The current study uses tri-simulation functions and triangular $\aleph$-admissibility conditions to provide new fixed point solutions for generalized $\aleph Z_G$-contractive mappings in full G-metric spaces.The existence theorem reduces the applicability of classical fixed point approaches by not requiring mapping continuity.

The suggested approach expands on Khojasteh et al.'s simulation-function framework [4] and Samet et al.'s admissibility approach [6]. The tri-simulation methodology developed by Gubran et al. in [**?**] and [8] is extended and a novel contribution is made in the context of entire G-metric spaces. The new theory expands on recent work on generalized contractions in spaces and structures [2], [5]. The following is a condensed version of the most important findings that this study has uncovered:

- i There has been the introduction of a new category of generalized $\aleph Z_G$-contractive mappings that are related with tri-simulation functions in complete G-metric spaces.
- ii The assumption of continuity of the mapping is not necessary to construct a fixed point existence theorem under triangular $\aleph$-admissibility and appropriate sequential conditions.
- iii In accordance with a natural requirement that was imposed on the admissibility function, a uniqueness result has been obtained.
- iv The constructed framework combines tri-simulation functions and admissibility methods into a unified contractive scheme.
- v This paper presents an example to demonstrate the practicality of the theoretical findings.
- vi For the purpose of illustrating the efficacy of the suggested method in nonlinear functional analysis, existence and uniqueness findings were achieved for a nonlinear Hammerstein integral equation.

The obtained results indicate that tri-simulation functions provide a flexible and efficient tool for extending modern fixed point theory beyond classical simulation-function settings and for studying nonlinear problems in generalized metric structures.

5. Future Work

Future research could take several paths:

(1) Applying the findings to partial, rectangular or controlled on GMS.
(2) utilizing tri-simulation functions to study fixed point theorems for mappings satisfying generalized $\aleph ZG$-contractive requirements.
(3) Developing fixed point findings for multi-valued mappings and set-valued operators using tri-simulation functions.

(4) Introduce hybrid contractive conditions using tri-simulation functions and rational, integral, or interpolative contractions.

(5) Studies of linked, tripled, and multidimensional fixed point problems under generalized admissibility.

(6) Applying the theory to fractional differential equations, integro-differential equations, impulsive systems, and boundary value problems.

(7) Studying stability, data dependence, and iterative approximation techniques for generalized $\aleph Z_G$-contractive mappings.

(8) We examine the relationship between tri-simulation functions and current advancements in nonlinear analysis, optimization theory, variational inequalities, and equilibrium problems.

These directions may extend the theory and lead to new nonlinear functional analysis and applied mathematics applications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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