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A NOTE ON MINIMAL POLYNOMIAL OPERATORS FOR SINGULAR MATRIX SYSTEMS

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Abstract. While the concept and use of minimal polynomials is classical in the context of linear algebra, their direct application to derive a matrix operator for singular matrices are not widely utilized. In this paper, we develop a residual matrix derived directly from the coefficients of the minimal polynomial of a singular matrix. We introduce the necessary properties of the proposed residual matrix in order to solve singular systems. In particular, we establish the necessary and sufficient conditions in order to solve singular matrix equations and initial value problems involving singular coefficient matrices. This approach forms an algebraic alternative to traditional methods that rely on pseudoinverses or spectral decompositions with potential advantages in both simplicity and computational efficiency.

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1. Introduction

Matrix theory plays a fundamental role in various branches of mathematics, engineering, and computational sciences [1–3]. The characteristic polynomial and the Cayley-Hamilton theorem play central roles in understanding and manipulating square matrices. The characteristic polynomial of a square matrix provides fundamental information about its eigenvalues, whereas the Cayley-Hamilton theorem establishes that every square matrix satisfies its own characteristic equation [4,5]. For a singular matrix, The minimal polynomial is essential to understand the algebraic structure and properties [6,7]. The Cayley-Hamilton theorem insures that a square matrix A satisfies its own characteristic polynomial $P_A(\lambda)$. Yet, the characteristic polynomial may not be the polynomial of lowest degree with this property. The minimal polynomial of A is defined with the least degree property which makes it particularly useful in analyzing and manipulating singular matrices. Although, the concept and applications of

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minimal polynomial are well-established in the literature, more can be said in this area especially when it comes to constructing operators, such as residual matrices, that act on the matrix in a minimal and structured way.

Linear systems involving singular coefficient matrices pose significant theoretical and computational difficulties, primarily because the standard approach of matrix inversion is no longer applicable. Traditional strategies for solving systems as $Ax = b$ includes the utilizing generalized inverses such as the Moore-Penrose pseudoinverse or the Drazin inverse [8–10]. Such methods offer generalized solutions by extending the notion of invertibility. While effective in many applications, these techniques often rely on matrix decompositions that are not always straightforward and leads to computational complexities. Recent years have witnessed renewed interest in the study of singular and descriptor systems, particularly through algebraic and computational approaches. Krylov-based and projection methods have been developed for singular linear and skew-symmetric systems, focusing on iterative and spectral properties rather than explicit operator construction [11]. Similarly, new frameworks for continuous-time descriptor systems emphasize structural solvability through generalized Sylvester formulations [12]. In parallel, modern treatments of singular matrix polynomials and eigenvalue problems aim to characterize minimal bases and null spaces for polynomial matrices [13,14]. Despite these advances, the direct use of the minimal polynomial itself to construct an algebraic operator that governs the solvability of singular systems remains unexplored.

In this paper, we introduce an alternative technique based on using the minimal polynomial of A to directly construct an residual matrix. Unlike previous approaches, the proposed method leverages known algebraic properties (the minimal polynomial) to construct solutions without requiring pseudoinverses or spectral decompositions. By focusing on algebraic characteristics rather than numerical inverses, this approach not only preserves important structural properties but also offers practical advantages in settings such as symbolic computation and the analysis of constrained or singular systems. We apply the proposed residual matrix as basis to produce solutions for singular matrix equations and initial value problems involving singular coefficient matrices. To the best of our knowledge, this represents the first instance where the minimal polynomial is used not merely for spectral analysis, but as a tool to define a practical residual operator that yields unique solutions in structured singular systems.

The remainder of the paper is organized as follows. Section 2 presents the main theoretical results, including key definitions, proofs, and an illustrative example that establishes the properties of the proposed residual matrix. Section 3 unifies the algebraic and geometric developments of the residual framework and demonstrates its applications to singular systems and matrix equations. Section 4 presents specific applications to singular differential and matrix equations that highlight the practical advantages of the proposed approach. Finally, Section 5 concludes the paper with a summary of findings, theoretical implications, and directions for future research.

2. Main Results

In what follows, $A \in \mathbb{R}^{n\times n}$ denotes a singular matrix. We begin by recalling the characteristic and minimal polynomials and proceed to define the residual matrix derived from them.

Definition 2.1 *The characteristic polynomial of A is*

$$P_A(\lambda) = \det(\lambda I - A). \tag{1}$$

Definition 2.2 *The minimal polynomial $m_A(\lambda)$ is the unique monic polynomial of least degree such that*

$$m_A(A) = 0. \tag{2}$$

If A is singular, then the constant term of $m_A(\lambda)$ vanishes and the polynomial can be written as

$$m_A(\lambda) = \lambda^k + b_1\lambda^{k-1} + b_2\lambda^{k-2} + \cdots + b_{k-1}\lambda, \qquad b_{k-1} \neq 0,$$

with $1 \le k \le n$.

By the Cayley–Hamilton theorem, A satisfies its characteristic polynomial $P_A(A) = 0$. Minimality implies

$$A^k + b_1A^{k-1} + b_2A^{k-2} + \cdots + b_{k-1}A = 0, \tag{3}$$

and the preceding nonvanishing condition ensures that

$$A^{k-1} + b_1A^{k-2} + \cdots + b_{k-1}I \neq 0. \tag{4}$$

building on Equation 4, we define the residual matrix as following:

Definition 2.3 *Let $A \in \mathbb{R}^{n\times n}$ be singular with minimal polynomial $m_A(\lambda)$ as in Equation 3. The residual matrix associated with A is defined by*

$$E_A = A^{k-1} + b_1A^{k-2} + b_2A^{k-3} + \cdots + b_{k-1}I. \tag{5}$$

Then $E_AA = AE_A = 0$.

Following Definition 2.3, we get the following results:

Theorem 2.1 *Let A be singular and E_A defined by Equation 5. If $b_{k-1} \neq 0$, then*

$$\ker(E_A) = \mathrm{Range}(A). \tag{6}$$

Proof Suppose $y \in \ker(E_A)$, i.e. $E_A y = 0$. Then

$$A^{k-1}y + b_1 A^{k-2}y + \cdots + b_{k-1}y = 0,$$

which can be rearranged as

$$y = -\frac{1}{b_{k-1}}(A^{k-1}y + b_1 A^{k-2}y + \cdots + b_{k-2}Ay).$$

Each term on the right-hand side lies in $\mathrm{Range}(A)$; hence $y \in \mathrm{Range}(A)$.

Conversely, if $y = Ax$ for some x, then $E_A y = E_A A x = A E_A x = 0$, implying $y \in \ker(E_A)$. □

Corollary 2.1 *Under the conditions of Theorem 2.1,*

$$\dim(\ker(A)) + \dim(\ker(E_A)) = n. \tag{7}$$

Remark 2.1 *For any singular matrix A, the linear system $Ax = y$ is consistent if and only if $y \in \ker(E_A)$. This provides an immediate algebraic test for compatibility without computing any inverse or decomposition.*

Example 2.1 *Consider*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, A is singular with rank 2. A direct computation gives the minimal polynomial

$$m_A(\lambda) = \lambda(\lambda - 1)^2 = \lambda^3 - 2\lambda^2 + \lambda,$$

so that $k = 3$, $b_1 = -2$, and $b_2 = 1$.

According to Equation 5,

$$E_A = A^2 - 2A + I.$$

By explicit multiplication,

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that $E_A A = 0$ and $A E_A = 0$. Moreover, $\ker(E_A) = \mathrm{span}\{(1,0,0)^\top, (0,1,0)^\top\} = \mathrm{Range}(A)$, confirming Theorem 2.1.

This example demonstrates how the residual matrix acts as a projector onto the complement of the range of A and provides immediate insight into the structure of the singular system $Ax = y$.

3. Algebraic and Geometric Properties of the Residual Matrix

Building upon the theoretical foundation introduced in Section 2, this section integrates the algebraic construction and geometric interpretation of the residual matrix into a single framework. It first establishes the general solution of singular linear systems in terms of the residual operator and then explores its relation to projectors, the matrix index, and representative applications to differential and matrix equations.

3.1. General Solution for Singular Linear Systems.

While Remark 2.1 establishes the algebraic test for consistency, i.e.that the system $Ax = y$ is solvable if and only if $y \in \ker(E_A)$, a complete framework requires a construction of the solution.

If $Ax = y$ is consistent, the general solution is given by the sum of a particular solution, x_p, and the homogeneous solution, x_h:

$$x = x_p + x_h$$

The homogeneous component $x_h \in \ker(A)$ is found by solving $Ax_h = 0$. Since $\ker(A) = \text{Range}(E_A)$, as shown in subsection 3.2, the homogeneous solution can be parameterized as:

$$x_h = E_A z, \quad \text{for any vector } z \in \mathbb{R}^n.$$

The particular solution x_p must satisfy $Ax_p = y$. Because the system is consistent, we can define x_p by using the Drazin inverse A^D, which is known to be a polynomial in A and is necessary for an algebraic solution that complements E_A. Since E_A annihilates the null space of A and $\ker(A)$ is spanned by the columns of E_A, the solution to $Ax = y$ can be expressed as:

$$x = A^D y + E_A z,$$

where A^D is the Drazin inverse of A and z is an arbitrary vector. This formula integrates the proposed residual matrix framework, based on the minimal polynomial, with established generalized inverse theory by clearly separating the dynamic $(A^D y)$ and static $(E_A z)$ components of the solution.

3.2. Relationship to Projectors and the Matrix Index.

The properties of the residual matrix E_A establish a direct, algebraically constructed link to the geometry of the singular matrix A.

3.2.1. *The Projector E_A.*

The matrix E_A, defined in 2.1, is not a projection matrix in the classical sense $(E_A^2 = E_A)$ unless $k = 1$, but it is a projection operator up to a scalar multiple. Specifically, the minimal polynomial $m_A(\lambda)$ can be factored as $m_A(\lambda) = \lambda^k \tilde{m}_A(\lambda)$, where $\tilde{m}_A(0) = b_{k-1} \neq 0$.

The property $E_A A = AE_A = 0$ implies that $\text{Range}(E_A) \subset \ker(A)$ and $\text{Range}(A) \subset \ker(E_A)$. Together with Theorem 2.1, this establishes the following fundamental duality:

$$\ker(E_A) = \text{Range}(A) \quad \text{and} \quad \text{Range}(E_A) = \ker(A^k).$$

Furthermore, the matrix E_A can be used to construct the projector onto $\ker(A)$. If we define the polynomial $p(\lambda) = \frac{1}{\lambda} m_A(\lambda)$, then $p(A) = E_A$. For the simplest singular case where the index $k = 1$, the Drazin inverse A^D is $A^D = \frac{1}{\tilde{m}_A(0)} \tilde{m}_A(A) = \frac{1}{b_0} E_A$, and the projector onto the null space of A is $P_{\text{null}} = I - AA^D$. The residual matrix E_A thus provides the non-invertible part of the Drazin inverse, simplifying the consistency check.

3.2.2. *Connection to the Matrix Index.* The index of a singular matrix A is the smallest non-negative integer k such that $\text{Range}(A^k) = \text{Range}(A^{k+1})$. Crucially, the integer k appearing in the minimal polynomial:

$$m_A(\lambda) = \lambda^k + b_1 \lambda^{k-1} + \cdots + b_{k-1} \lambda,$$

is exactly the index of the eigenvalue $\lambda = 0$ (the algebraic index of the matrix A). This is the same index used in the analysis of Differential Algebraic Equations (DAEs) to characterize hidden constraints and differentiability requirements. The residual matrix E_A therefore provides a tool whose order $(k-1)$ is directly related to the complexity and solvability index of the underlying singular system.

4. Applications

The residual matrix introduced in Section 2 can be used as an algebraic tool for analyzing the solvability of singular systems that arise frequently in applied mathematics, control theory, and DAEs. In this section we present two representative applications: singular initial value problems and matrix equations. Both illustrate how the minimal polynomial framework provides transparent criteria for existence and consistency.

4.1. **Initial value problems with singular coefficients.** Let $A, B \in \mathbb{R}^{n \times n}$ be singular matrices and consider the first order system

$$B\, x'(t) = A\, x(t), \qquad x(0) = x_0. \tag{8}$$

Classical approaches, such as the use of the Drazin inverse, often require a decomposition of B and A into their regular and singular parts. Here, the residual matrices associated with A and B provide an alternative route.

From Remark 2.1, any admissible solution of Equation 8 must satisfy $x(t) \in \ker(E_B A)$ for all t. Using this property we obtain the following theorem.

Theorem 4.1 *For every $u \in \mathbb{R}^n$, the function*

$$x(t) = \exp(E_B A t)\, E_A u \tag{9}$$

is a solution of $B\, x'(t) = A\, x(t)$.

Proof Differentiating Equation 9 gives $x'(t) = E_B A \exp(E_B At)\, E_A u$. Since $B(E_B A) = AE_B A = 0$ by definition of the residuals, it follows directly that

$$B\, x'(t) - A\, x(t) = B(E_B A e^{E_B At} E_A u) - Ae^{E_B At} E_A u = 0. \tag{10}$$

Hence, $x(t)$ satisfies Equation 8. $\square$

The expression presented by Equation 9 highlights that the evolution of $x(t)$ is confined to the subspace $\ker(E_B A)$, which contains all consistent initial conditions for the singular system. Unlike the pseudoinverse-based formulation, this approach remains fully algebraic and free of decompositions.

Corollary 4.1 *If an external forcing term $f(t) \in C(I, \mathbb{R}^n)$ is present,*

$$B\, x'(t) = A\, x(t) + f(t), \tag{11}$$

then the system is equivalent to the pair

$$E_A B\, x'(t) = E_A f(t), \tag{12}$$

and

$$E_B(Ax(t) + f(t)) = 0. \tag{13}$$

Equation 12 describes the dynamic evolution on the consistent subspace, while Equation 13 ensures that the solution remains compatible with the singular structure.

Example 4.1 *Consider*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $E_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $E_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Equation 8 reduces to*

$$\begin{cases} 0 = x_1'(t), \\ x_2'(t) = 0, \end{cases}$$

whose solutions are constant vectors $x(t) = (c_1, c_2)^\top$. Using Equation 9 gives $E_B A = 0$ and hence $\exp(E_B At) = I$, so $x(t) = E_A u = (0, u_2)^\top$, consistent with the algebraic constraints. The residual matrices thus isolate the dynamically admissible components without explicit elimination of variables.

4.2. **Singular matrix equations.** A second important class of problems involves matrix equations of the form

$$AXB = C, \qquad A, B, C \in \mathbb{R}^{n\times n}, \tag{14}$$

where A and B may be singular. Such equations occur in control design, model reduction, and the solution of coupled systems. Determining whether a solution X exists typically requires generalized inverses or Kronecker canonical forms; however, the residual matrices yield simple algebraic consistency conditions.

Lemma 4.1 *Equation 14 is consistent if and only if*

$$E_A C = 0 \quad \textit{and} \quad CE_B = 0.$$

Proof Multiplying Equation 14 on the left by E_A and using $E_A A = 0$ gives $E_A C = 0$. Similarly, multiplying on the right by E_B and using $BE_B = 0$ yields $CE_B = 0$. These conditions are also sufficient, as any X satisfying Equation 14 necessarily enforces them. □

Lemma 4.2 *For the Sylvester equation*

$$AX + XB = C,$$

with A, B, C singular, the equation is solvable if the following condition hold:

$$E_A C E_B = 0. \tag{15}$$

Proof Suppose there exists X satisfying $AX + XB = C$. Premultiplying by E_A and postmultiplying by E_B yields

$$E_A(AX + XB)E_B = E_A C E_B.$$

Because the residual matrices satisfy $E_A A = 0$ and $BE_B = 0$, both terms on the left vanish:

$$E_A(AX + XB)E_B = (E_A A)XE_B + E_A X(BE_B) = 0.$$

Hence $E_A C E_B = 0$. □

Example 4.2 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $E_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $E_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Condition gives

$$E_A C E_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

so a solution exists. Direct computation yields $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$*, confirming consistency.*

This small example demonstrates the simplicity of the residual matrix test compared with constructing pseudoinverses.

5. Conclusions

The study has introduced a minimal polynomial-based framework for analyzing and solving singular matrix systems. By deriving a residual matrix directly from the coefficients of the minimal polynomial, we established a purely algebraic operator that replaces the need for pseudoinverses or spectral decompositions. The residual matrix E_A provides immediate criteria for consistency, characterizes the interaction between the kernel and range of singular matrices, and connects naturally to the matrix index and projector structures. The unified treatment presented in this paper demonstrates that minimal-polynomial residuals not only clarify the underlying algebraic geometry of singular systems but also serve as practical tools for constructing and analyzing solutions to differential and matrix equations. This approach complements existing generalized inverse techniques by offering an explicit and computationally transparent alternative. Future work will focus on extending the residual framework to nonlinear and parameter-dependent singular systems, as well as exploring its symbolic implementation in computational algebra environments.

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References

[1] A.A. Hadi, Matrix Application in Engineering Problems, in: DT. Tran, G. Jeon, T.D.L. Nguyen, J. Lu, T.D. Xuan, (eds) Intelligent Systems and Networks, ICISN 2021, Lecture Notes in Networks and Systems, Vol. 243, Springer, Singapore, (2021). https://doi.org/10.1007/978-981-16-2094-2_69.

[2] Z. Bai, J. Pan, Matrix Analysis and Computations, SIAM, Philadelphia, 2021. https://doi.org/10.1137/1.9781611976632.

[3] J. Sjöberg, Descriptor Systems and Control Theory, Linköping University Electronic Press, 2005.

[4] P. Kunkel, V. Mehrmann, Differential-Algebraic Equations, EMS Press, 2006. https://doi.org/10.4171/ETB/28.

[5] T. Bruell, Explicit Solutions of Regular Linear Discrete-Time Descriptor Systems with Constant Coefficients, Electron. J. Linear Algebr. 18 (2009), 317–338. https://doi.org/10.13001/1081-3810.1316.

[6] K. Conrad, the Minimal Polynomial and Some Applications, Technical report, Department of Mathematics, University of Connecticut, 2014.

[7] M. Neunhöffer, C.E. Praeger, Computing Minimal Polynomials of Matrices, LMS J. Comput. Math. 11 (2008), 252–279. https://doi.org/10.1112/s1461157000000590.

[8] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Springer, Singapore, 2018. https://doi.org/10.1007/978-981-13-0146-9.

[9] L. Lu, Constructing Solutions of the Yang–Baxter–Like Matrix Equation for Singular Matrices, J. Comput. Appl. Math. 436 (2024), 115403. https://doi.org/10.1016/j.cam.2023.115403.

[10] P.K. Baruah, D.K.K. Vamsi, IVPs for Singular Interface Problems, Adv. Dyn. Syst. Appl. 3 (2008), 209–227.

[11] K. Du, J. Fan, X. Sun, F. Wang, Y. Zhang, On Krylov Subspace Methods for Skew-Symmetric and Shifted Skew-Symmetric Linear Systems, Adv. Comput. Math. 50 (2024), 78. https://doi.org/10.1007/s10444-024-10178-9.

[12] K.N. Pirbazari, M. Mansouri, A New Approach to Solve the Continuous-Time Linear Descriptor Systems Based on Generalized Upper Triangular Form and Positivity Criteria, Numer. Algebr. Control. Optim. 15 (2025), 1116–1131. https://doi.org/10.3934/naco.2025006.

[13] V.B. Khazanov, Construction of a Minimal Basis of the Right Null Space of a Singular Multiparameter Polynomial Matrix, J. Math. Sci. 249 (2020), 290–300. https://doi.org/10.1007/s10958-020-04943-6.

[14] M.E. Hochstenbach, C. Mehl, B. Plestenjak, Numerical Methods for Eigenvalues of Singular Polynomial Eigenvalue Problems, Linear Algebr. Appl. 719 (2025), 1–33. https://doi.org/10.1016/j.laa.2025.04.002.