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THE WEIGHTED LOGARITHMIC LOMAX-G FAMILY OF DISTRIBUTIONS: PROPERTIES WITH APPLICATIONS TO MECHANICAL AND PETROLEUM DATASETS

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Abstract. This paper introduces a new function that can be utilized in the $T-X$ transformer technique, thereby adding one more member to the list of such functions in Alzaatreh et al. [1]. Using this new function, we construct and study the weighted logarithmic Lomax-G family of distributions, also known as the WLL-G family, which extends the baseline model G by introducing two additional parameters. The study includes practical derivations of the mathematical and statistical properties of this family. Moreover, we estimate the family parameters using both maximum likelihood and Bayesian methods and conduct a simulation study to compare the results. The potential of this family is demonstrated through a comparative analysis with a wide range of well-known models and on various real-world data sets.

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1. Introduction

The ongoing advancement in all fields of science and knowledge means new data sets emerge every day that need to be handled. It is well known that statistical distributions provide an invaluable tool for modeling such data sets; therefore, the need to build more distributions is attracting increased

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attention these days. This tendency toward generating new statistical models is motivated primarily by the sheer number of characteristics that modern data sets exhibit, such as high skewness, multi-modality, heavy tails, or complex dependence structures. It is worth mentioning that most classical distributions fail to address these requirements properly. Therefore, many researchers are continuously building new, more flexible statistical models or modifying existing distributions to address these requirements. Many of these models have found invaluable applications in multiple disciplines such as finance, environmental science, engineering, reliability analysis, and biomedical research. The work in this direction ensures that researchers and data scientists can build more reliable models, which in turn give more accurate predictions. Consequently, the field of distribution theory remains a vibrant and essential area of research, directly supporting the data-driven decision-making that underpins modern scientific discovery and industrial innovation. As a result, the field of statistical modeling and distribution theory occupies an essential role in driving decision-making that is embedded in every aspect of scientific discoveries and industrial innovation.

One of the easiest methods is to modify an existing model by adding one or more parameters. This technique proved effective in providing greater flexibility for handling larger and newer data sets. Many researchers have presented several methods in this regard during the last 30 years. Marshall and Olkin [2] presented the MO generator to expand existing families of distributions. They argued that their method enjoys a stability property in the sense that applying the technique twice yields nothing new. Eugene et al. [3] introduced the beta-normal distribution by exploiting the logit of the beta random variable. This distribution proved to be a good choice to model symmetric, skewed, and bimodal distributions, and they demonstrated its flexibility by applying it to two empirical data sets. Alzaatreh et al. [1] studied a new method called the $T-X$ transformer to construct new families by exploiting two random variables X and T. They demonstrated that many of the statistical models are indeed special cases of the $T-X$ families. Al-Shomrani et al. [4] defined the Topp-Leone family of distributions. They used the Topp-Leone exponential distribution to model a data set concerning the ordered failure of components. Cordeiro and de Castro [5] generalized existing models by exploiting Kumaraswamy's density function to give the KW-G family. They gave useful expressions for the moments and the moments of the order statistics. Salama et al. [6] introduced the length-biased weighted exponentiated inverted exponential distribution. Abbasi et al. [7] used the truncation method to study the right-truncated Xgamma-G family of distributions. They used the right-truncated Xgamma-exponential distribution to efficiently fit three data sets and showed that this model's performance exceeded that of many other well-known models. Hassan et al. [8] obtained the weighted power Lomax distribution and its length-biased version. Bantan et al. [9] discussed Bayesian analysis in partially accelerated life tests for the weighted Lomax distribution. Gemeay et al. [10] defined and studied the three-parameter power Mira distribution, which has found applications in insurance loss data sets. El Gazar et al. [11]

provided an estimation of the inverse power Ailamujia and the truncated inverse power Ailamujia distributions based on censoring scheme. Orji et al. [12] developed a new odd reparameterized exponential transformed-X family of distributions and applied it to certain public health data sets. Hammad et al. [13] investigated new modified Liu estimators to resolve multicollinearity in the beta model. El-Alosey et al. [14] provided a new version of the geometric distribution and compared its performance with multiple discrete distributions. More recently, Aldallal [15, 16] introduced and studied certain statistical models for Laplace and Rayleigh current record distributed data, which have found applications in the industrial sector. Many other ways to define new distributions have been studied; see, for example, [17–20].

The $T-X$ transformer method in [1] depends on a so-called linking function $W(y)$ that satisfies certain conditions. The authors of [1] emphasized studying new $T-X$ families by constructing new linking functions $W(y)$. Motivated by this, we propose a new linking function called the weighted logarithmic function, which is defined by the formula

$$W(y)=\frac{y}{1-y}\log\frac{1}{1-y}. \tag{1}$$

Evidently, the function $W(y)$ is defined on the interval $(-\infty,1)$. Moreover, it is not hard to see that $W(y)\sim y^2$ as $y\to 0$, and $W(y)$ diverges to infinity as $y\to 1^-$. By analyzing the derivative $W'(y)$, we conclude that $W(y)$ is decreasing on $(-\infty,0)$ and increasing on $(0,1)$, with $y=0$ being the global minimum point of $W(y)$. Figure 1 illustrates the shape of the function $W(y)$.

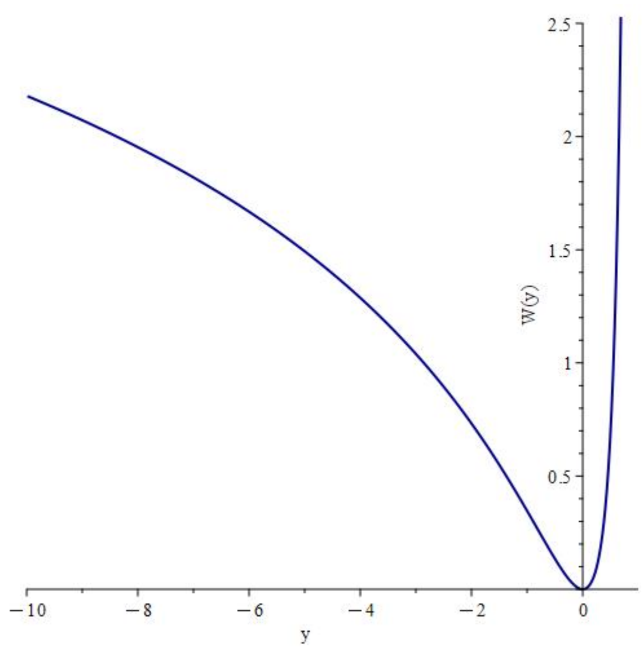

FIGURE 1. Graphical representation for the function $W(y)$.

Using this function, we construct a family of continuous distributions called the weighted logarithmic Lomax family, or the WLL family for short, as follows: Let $r(t) = (\theta/a)[1 + (t/a)]^{-(\theta+1)}$ be the density function of the Lomax distribution where $t \geq 0$ with the parameters vector $(\theta, a) \in \mathbb{R}_+^2$. Let $G(x, \xi)$ be an arbitrary cumulative function of a random variable X with parameter vector ξ. With this setup, it is not hard to check that $W[G(x, \xi)]$ fulfills the conditions of [1, Equation 2.1], so we obtain the cumulative function $F(x; \theta, a, \xi)$ of the WLL-G family

$$\begin{aligned} F(x;\theta,a,\xi) &= \int_0^{W(G(x,\xi))} r(t)dt = \int_0^{\frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)}} \frac{\theta}{a}\left(1+\frac{t}{a}\right)^{-(\theta+1)} dt \\ &= 1 - \left[1 + \frac{1}{a}\frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)}\right]^{-\theta}. \end{aligned} \tag{2}$$

By differentiating this last equation, we get the WLL-G density function $f(x; \theta, a, \xi)$

$$\begin{aligned} f(x;\theta,a,\xi) &= \frac{\theta}{a}\left[1 + \frac{1}{a}\frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)}\right]^{-(\theta+1)} \\ &\quad \times \left[G(x,\xi) + \log \frac{1}{1-G(x,\xi)}\right] \frac{g(x,\xi)}{[1-G(x,\xi)]^2}, \end{aligned} \tag{3}$$

where $g(x, \xi)$ is the density function associated with $G(x, \xi)$. The extra parameter θ is a shape parameter, while the parameter a is a scale parameter. These two parameters give the baseline model a considerable amount of flexibility and control over certain statistical characteristics, such as skewness and kurtosis, as illustrated in Figure 2. Hereafter, a random variable X with pdf given by Equation (3) will be denoted by $X \sim$ WLL-$G(\theta, a, \xi)$. The survival function $S(x; \theta, a, \xi)$ and the hazard function $H(x; \theta, a, \xi)$ are given by

$$S(x;\theta,a,\xi) = \left[1 + \frac{1}{a}\frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)}\right]^{-\theta}, \tag{4}$$

and

$$\begin{aligned} H(x;\theta,a,\xi) &= \frac{\theta}{a}\left[1 + \frac{1}{a}\frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)}\right]^{-1} \\ &\quad \times \left[G(x,\xi) + \log \frac{1}{1-G(x,\xi)}\right] \frac{g(x,\xi)}{[1-G(x,\xi)]^2}. \end{aligned} \tag{5}$$

respectively. The main contributions of this study can be summarized as follows:

(1) We introduce a new linking function $W(y)$ to generate novel statistical distributions within the $T - X$ framework, which has not been previously applied in the literature, hence expanding the range of functions in the $T - X$ method.

(2) We provide a novel closed-form formula for the quantile function via the special r-Lambert function. This new method can be readily implemented in other statistical distributions, which is pivotal for further inferential computations.

(3) We conduct a thorough theoretical investigation of the model by deriving many of its properties. In particular, we obtain formulas for the Rényi and Shannon entropies via certain special

functions, thus providing yet another tool in the information-theoretic aspects of statistical models.

(4) We demonstrate the flexibility and practical utility of the proposed method in generating new statistical distributions. In particular, we show that this new method can provide models that exhibit superior performance when applied to real data sets in comparison to many other existing models.

The organization of the paper is as follows: Section 2 introduces three different distributions emerging from the construction above. In Section 3, we derive some statistical and mathematical properties of the WLL-G family, such as the quantile function, the series expansion of the density function, moments and their generating function, and incomplete moments. Section 4 includes a derivation of entropies and the order statistics of the WLL-G family. The estimation of the family's parameters is provided in Section 5. Section 6 presents a simulation study using a Monte Carlo algorithm that compares two estimation methods: the MLEs and the Bayesian methods. The potential and flexibility of the WLL-G family are demonstrated in Section 7 by conducting a comparative study with a wide range of well-known models and on various real-world data sets. The paper is concluded in Section 8.

2. Special models

In this section, we present three distributions from the WLL-G family and provide plots of their density and hazard functions. The exponential distribution's simple form and its wide applicability in engineering problems, reliability analysis, and survival studies make it an ideal choice to demonstrate the structure and methodology of the proposed approach. Therefore, the weighted logarithmic Lomax exponential distribution, or the WLLE distribution for short, will be used in the empirical aspect of our study in Section 7.

2.1. **The weighted logarithmic Lomax exponential distribution (WLLE).** By setting $G(x,\xi) = 1 - \exp(-bx)$ and $g(x,\xi) = b\exp(-bx)$ in Equation (3), where $x > 0$ and $\xi = b > 0$, we obtain the density function of the weighted logarithmic Lomax exponential distribution

$$f(x;\theta,a,b) = \frac{\theta b}{a}(e^{bx} + bxe^{bx} - 1)\left[1 + \frac{b}{a}x(e^{bx} - 1)\right]^{-(\theta+1)}. \tag{6}$$

A random variable X with pdf given as Equation (6) will be denoted by $X \sim \mathrm{WLLE}(\theta,a,b)$ where its cumulative function, survival function, and the hazard rate function are given by

$$F(x;\theta,a,b) = 1 - \left[1 + \frac{b}{a}x(e^{bx} - 1)\right]^{-\theta}, \tag{7}$$

$$S(x;\theta,a,b) = \left[1 + \frac{b}{a}x(e^{bx} - 1)\right]^{-\theta}, \tag{8}$$

and

$$H(x;\theta,a,b)=\frac{\theta b}{a}(e^{bx}+bxe^{bx}-1)\left[1+\frac{b}{a}x(e^{bx}-1)\right]^{-1},\tag{9}$$

respectively. Figure 2 demonstrates plots of the density function and the hazard rate function discussed above for selected values of parameters. The top-right and the bottom-left figures show how the added parameters θ and a affect the shape properties (skewness and kurtosis) of the density function.

2.2. **The weighted logarithmic Lomax Weibull distribution (WLLW).** The density function of the weighted logarithmic Lomax Weibull distribution is obtained by setting $G(x,\xi)=1-\exp(-bx^{\lambda})$ and $g(x,\xi)=b\lambda x^{\lambda-1}\exp(-bx^{\lambda})$ in Equation (3) where $x>0$ and $\xi=(b,\lambda)\in\mathbb{R}_+^2$. Consequently, this density function has the form

$$f(x;\theta,a,b,\lambda)=\frac{\theta\lambda b}{a}x^{\lambda-1}(e^{bx^{\lambda}}+bx^{\lambda}e^{bx^{\lambda}}-1)\left[1+\frac{b}{a}x^{\lambda}(e^{bx^{\lambda}}-1)\right]^{-(\theta+1)}.\tag{10}$$

Similarly, the cumulative function, the survival function, and the hazard rate function are given by

$$F(x;\theta,a,b,\lambda)=1-\left[1+\frac{b}{a}x^{\lambda}(e^{bx^{\lambda}}-1)\right]^{-\theta},\tag{11}$$

$$S(x;\theta,a,b,\lambda)=\left[1+\frac{b}{a}x^{\lambda}(e^{bx^{\lambda}}-1)\right]^{-\theta},\tag{12}$$

and

$$H(x;\theta,a,b,\lambda)=\frac{\theta\lambda b}{a}x^{\lambda-1}(e^{bx^{\lambda}}+bx^{\lambda}e^{bx^{\lambda}}-1)\left[1+\frac{b}{a}x^{\lambda}(e^{bx^{\lambda}}-1)\right]^{-1},\tag{13}$$

respectively. A random variable X with a four-parameter density function given by Equation (10) will be denoted by $X\sim\text{WLLW}(\theta,a,b,\lambda)$. Plots of this random variable's density function and hazard rate function for selected values of parameters are shown in Figure 3.

2.3. **The weighted logarithmic Lomax gamma distribution (WLLG).** The probability density function of the weighted logarithmic Lomax gamma distribution is found by putting $G(x,\xi)=\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)}$ and $g(x,\xi)=\frac{b^{\lambda}}{\Gamma(\lambda)}x^{\lambda-1}\exp(-bx)$ in Equation (3) where $x>0$ and $\xi=(b,\lambda)\in\mathbb{R}_+^2$, so it takes the form

$$\begin{aligned}f(x;\theta,a,b,\lambda)=&\frac{\theta b^{\lambda}\Gamma(\lambda)}{a}\frac{x^{\lambda-1}e^{-bx}}{[\Gamma(\lambda)-\gamma(\lambda,bx)]^2}\left[1+\frac{1}{a}\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right]^{-(\theta+1)}\\&\times\left[\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)}+\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right].\end{aligned}\tag{14}$$

A random variable X with a four-parameter density function given by Equation (14) will be denoted by $X\sim\text{WLLG}(\theta,a,b,\lambda)$. Its cdf function, survival function, and hazard rate function are easily found to be

$$F(x;\theta,a,b,\lambda)=1-\left[1+\frac{1}{a}\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right]^{-\theta},\tag{15}$$

$$S(x;\theta,a,b,\lambda)=\left[1+\frac{1}{a}\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right]^{-\theta},\tag{16}$$

and

$$H(x;\theta,a,b,\lambda)=\frac{\theta b^{\lambda}\Gamma(\lambda)}{a}\frac{x^{\lambda-1}e^{-bx}}{[\Gamma(\lambda)-\gamma(\lambda,bx)]^{2}}\left[1+\frac{1}{a}\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right]^{-1}$$
$$\times\left[\frac{\gamma(\lambda,bx)}{\Gamma(\lambda)}+\log\frac{\Gamma(\lambda)}{\Gamma(\lambda)-\gamma(\lambda,bx)}\right], \tag{17}$$

respectively. Multiple plots of the density function and the hazard rate function are given in Figure 4 for selected values of parameters.

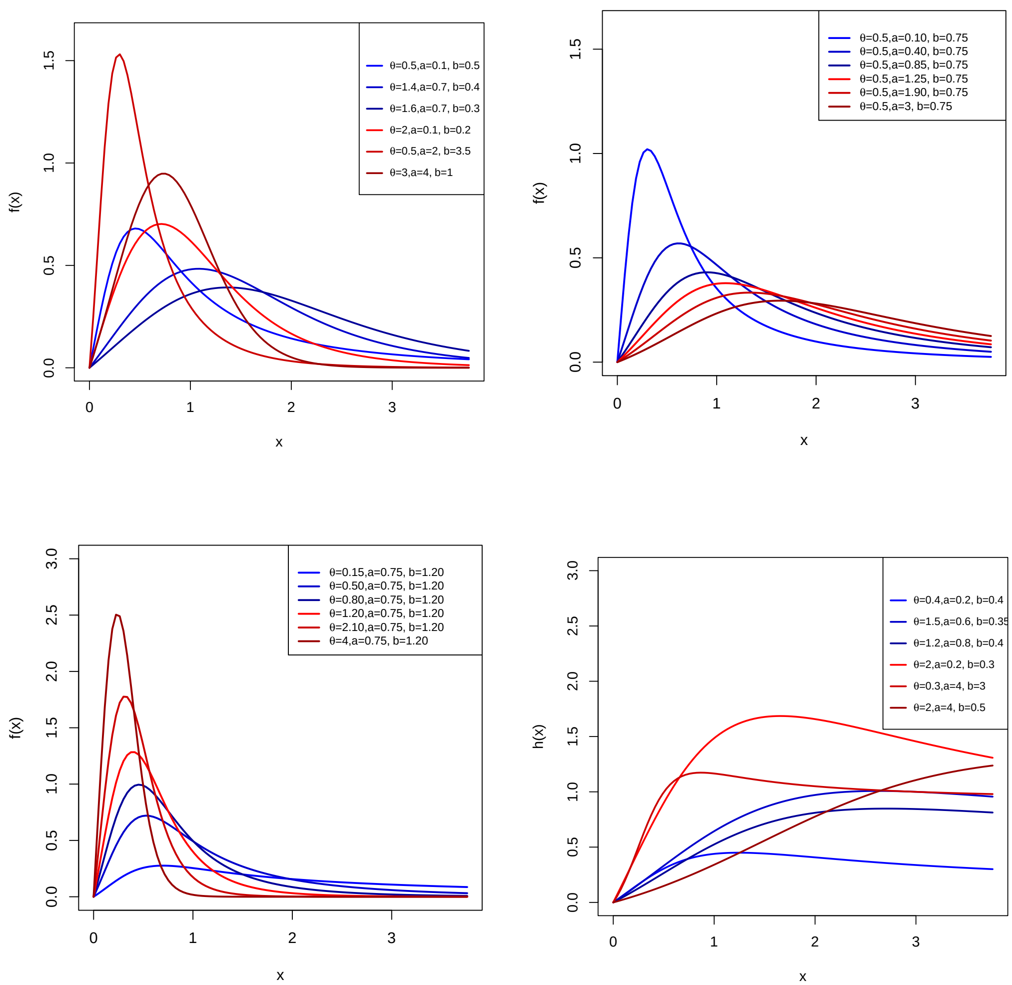

FIGURE 2. The density function and the hazard function of the WLLE distribution for selected values of parameters.

3. Statistical and mathematical properties of the WLL-G family

3.1. **Quantile function.** In this part of the paper, we derive a useful expression for the quantile function $Q_{\text{WLL-}G}(u)$ of the WLL-G family. This derivation relies on the r-Lambert function of Mező

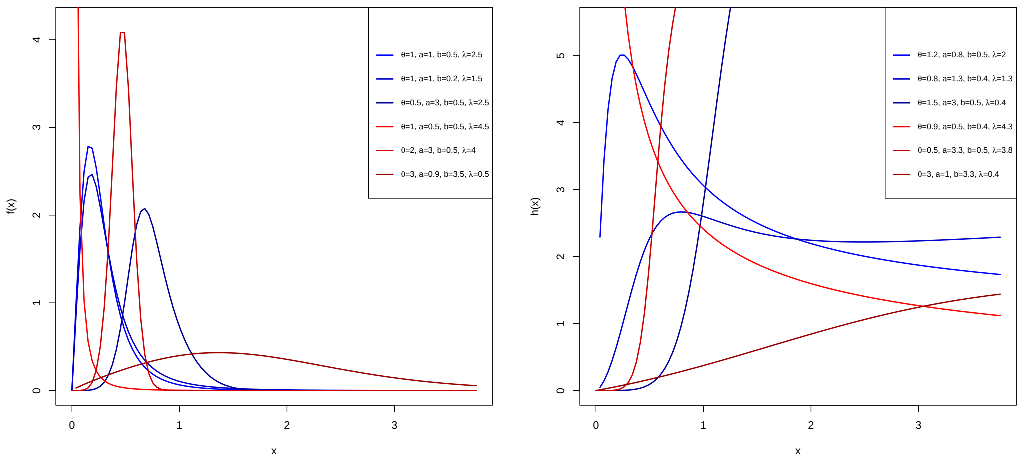

FIGURE 3. The density function and the hazard function of the WLLW distribution for selected values of parameters.

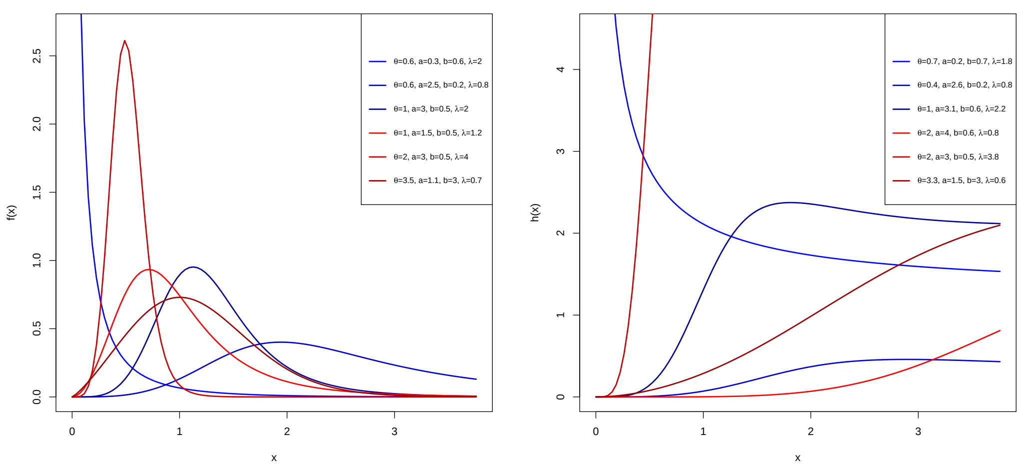

FIGURE 4. The density function and the hazard function of the WLLG distribution for selected values of parameters.

and Baricz [21]. Let U be a $[0,1]$ uniform random variable. If we set $F(x;\theta,a,\xi)=u$ where $F(x;\theta,a,\xi)$ is given by Equation (2), then some algebra gives

$$\exp\left[\phi(u;\theta,a)\frac{1-G(x,\xi)}{G(x,\xi)}\right][1-G(x,\xi)]=1, \tag{18}$$

where $\phi(u;\theta,a)=a[(1/1-u)^{\frac{1}{\theta}}-1]$. By making the change of variables $y=\frac{1-G(x,\xi)}{G(x,\xi)}$, Equation (18) becomes

$$\exp[\phi(u;\theta,a)y]\frac{y}{y+1}=1. \tag{19}$$

Invoking [21, Theorem 3] yields

$$y = \frac{W_{-1}[\phi(u;\theta,a)]}{\phi(u;\theta,a)}. \tag{20}$$

By substituting $\frac{1-G(x,\xi)}{G(x,\xi)}$ for y in Equation (20) and solving for $G(x,\xi)$, we get

$$G(x,\xi) = \frac{\phi(u;\theta,a)}{\phi(u;\theta,a) + W_{-1}[\phi(u;\theta,a)]}. \tag{21}$$

Finally, the quantile function of the WLL-G family takes the form

$$Q_{\text{WLL-}G}(u) = Q_G\left\{\frac{\phi(u;\theta,a)}{\phi(u;\theta,a) + W_{-1}[\phi(u;\theta,a)]}\right\}, \tag{22}$$

where $Q_G(u)$ is the quantile function of the cdf function $G(x,\xi)$. It is worth mentioning that this approach can be applied in many other statistical models especially the ones that include the natural logarithmic function or an exponential function.

3.2. **Useful expansions.** In this subsection, we derive expansions of the cdf and pdf given by Equations (2) and (3), respectively. Given a cdf function $G(x)$ and a constant $e > 0$, one can define the exponentiated-G model whose cdf function is $T_e(x) = G(x)^e$. Thus, the pdf of this new distribution is $t_e(x) = eG(x)^{e-1}g(x)$, where $g(x)$ is the pdf associated with $G(x)$. We denote by $X \sim \text{Exp}^c(G)$ a random variable having an exponentiated-G density function with exponent c. Many works discussed the exponentiated-G model for various $G(x)$; see, for example, [22–24].

Starting with Equation (3), if we apply the generalized binomial expansion, we get

$$\begin{aligned} f(x;\theta,a,\xi) = &\sum_{i=0}^{\infty}\binom{\theta+i}{i}\frac{(-1)^i\theta}{a^{i+1}}\frac{g(x,\xi)}{[1-G(x,\xi)]^{i+2}} \\ &\times\left\{G(x,\xi)^{i+1}\left[\log\frac{1}{1-G(x,\xi)}\right]^i + G(x,\xi)^i\left[\log\frac{1}{1-G(x,\xi)}\right]^{i+1}\right\}. \end{aligned} \tag{23}$$

Here, we make use of a result due to Gradshteyn and Ryzhik [25], which gives a formula for a series raised to a positive integer

$$\left(\sum_{j=0}^{\infty}\delta_j u^j\right)^n = \sum_{j=0}^{\infty}\Delta_{n,j}u^j, \tag{24}$$

where $\Delta_{n,0} = \delta_0^n$ and the coefficients $\Delta_{n,j}$ are determined recursively from the relation (for $j = 1, 2, \ldots$)

$$\Delta_{n,j} = \frac{1}{j\delta_0}\sum_{p=1}^{j}[p(n+1)-j]\delta_p\Delta_{n,j-p}. \tag{25}$$

Applying the series $\log\frac{1}{1-u} = \sum_{j=0}^{\infty}\frac{u^{j+1}}{j+1}$ and using Equations (24) and (25) yields

$$f(x;\theta,a,\xi) = \sum_{i,j=0}^{\infty}\binom{\theta+i}{i}\frac{(-1)^i\theta}{a^{i+1}}(c_{i,j}+d_{i+1,j})\frac{g(x,\xi)}{[1-G(x,\xi)]^{i+2}}G(x,\xi)^{2i+j+1}, \tag{26}$$

where $c_{i,0} = d_{i+1,0} = 1$,

$$c_{i,j} = \frac{1}{j}\sum_{p=1}^{j}[p(i+1) - j]\frac{c_{i,j-p}}{p+1}, \tag{27}$$

and

$$d_{i+1,j} = \frac{1}{j}\sum_{p=1}^{j}[p(i+2) - j]\frac{d_{i+1,j-p}}{p+1}. \tag{28}$$

Writing $G(x,\xi)^{2i+j+1} = \{1 - [1 - G(x,\xi)]\}^{2i+j+1}$ and applying the generalized binomial expansion twice gives

$$f(x;\theta,a,\xi) = \sum_{l=0}^{\infty} \varrho_l t_{l+1}(x,\xi), \tag{29}$$

where

$$\varrho_l = \sum_{i,j,k=0}^{\infty} \binom{\theta+i}{i}\binom{2i+j+1}{k}\binom{k-i-2}{l}\frac{(-1)^{i+k+l}\theta}{a^{i+1}(l+1)}(c_{i,j} + d_{i+1,j}), \tag{30}$$

and $t_{l+1}(x,\xi) = (l+1)G(x,\xi)^l g(x,\xi)$ is the exponentiated-G distribution with exponent $(l+1)$. Integrating Equation (29) yields the expansion of Equation (2) as

$$F(x;\theta,a,\xi) = \sum_{l=0}^{\infty} \varrho_l T_{l+1}(x,\xi), \tag{31}$$

where $T_{l+1}(x,\xi) = G(x,\xi)^{(l+1)}$ is the distribution function of the exponentiated-G distribution with exponent $(l+1)$. Equations (31) and (29) show that the cdf and the pdf functions of the WLL-G family are multilinear mixtures of the cdf and the pdf functions of the exponentiated-G distributions with varying exponents, respectively. Equations (31) and (29) are useful for obtaining further properties of the WLL-G family, and we will make use of them later.

3.3. **Moments and their generating function.** The nth order moment μ'_n of a random variable $X \sim$ WLL-$G(\theta,a,\xi)$ can be computed using Equation (29)

$$\mu'_n = E(X^n) = \sum_{l=0}^{\infty} \varrho_l E(Y_{l+1}^n), \tag{32}$$

where $Y_{l+1} \sim \text{Exp}^{l+1}(G)$ is a random variable that follows the exponentiated-G distributions with exponent $(l+1)$. Nadarajah and Kotz [23] computed closed-form expressions for various exponentiated-G distributions. Alternatively, Equation (32) can also be written as

$$\mu'_n = E(X^n) = \sum_{l=0}^{\infty} \varrho_l \vartheta_{n,l}, \tag{33}$$

where $\vartheta_{n,l} = (l+1)\int_0^1 u^l Q_G(u)^n du$. The moment generating function $M(t) = E[\exp(tX)]$ of the WLL-G family can be obtained using Equation (29) as

$$M(t) = \sum_{l=0}^{\infty} \varrho_l M_{l+1}(t), \tag{34}$$

where $M_{l+1}(t)$ is the moment generating function of $Y_{l+1} \sim \text{Exp}^{l+1}(G)$. Another form for this function is

$$M(t) = \sum_{l=0}^{\infty} \varrho_l \varpi(t, l) \tag{35}$$

where $\varpi(t, l) = (l + 1) \int_0^1 u^l \exp[tQ_G(u)]du$.

A numerical representation of the WLLE distribution first four moments, variance, coefficients of variation, skewness, and kurtosis are presented in Tables 1 and 2. The numerical results presented in Tables 1 and 2 reveal systematic patterns in the distributional behavior under varying parameter configurations. For both fixed $a = 0.5$ and $a = 2.5$, increasing the parameter θ while holding b constant produces monotonic decreases in all raw moments $(\mu'_1, \mu'_2, \mu'_3, \mu'_4)$, variance σ^2, and the coefficient of variation (CV), indicating greater concentration of probability mass near the origin with reduced relative dispersion. The skewness and kurtosis measures also decline substantially with increasing θ, transitioning from highly right-skewed, leptokurtic distributions toward more symmetric, mesokurtic forms. For instance, when $b = 0.4$ in Table 1, skewness falls from 5.99702 at $\theta = 0.15$ to 1.13299 at $\theta = 3.5$, while kurtosis decreases from 11.8605 to 5.04734 over the same range. Comparing the two tables, the parameter a has a pronounced influence on the distribution's scale and shape. At equivalent b and θ combinations, Table 2 $(a = 2.5)$ yields smaller moment magnitudes and lower CV values than Table 1 $(a = 0.5)$ consistently, demonstrating that larger a reduces both absolute and relative variability.

3.4. **Incomplete moments.** Incomplete moments play a pivotal role in statistical sciences. They appear in numerous theoretical derivations, such as the calculation of mean deviations and the construction of Bonferroni and Lorenz curves. These calculations have wide applications across various disciplines, including medicine, reliability, economics, and insurance. Let X be a random variable with a pdf function $f(x)$. The nth order incomplete moment $m(n, y)$ of X can be computed using the formula

$$m(n, y) = \int_{-\infty}^{y} x^n f(x)dx. \tag{36}$$

Now, if $X \sim \text{WLL-}G(\theta, a, \xi)$, then with the aid of Equation (29), the nth order incomplete moment of the WLL-G family becomes

$$m(n, y) = \sum_{l=0}^{\infty} \varrho_l m_{l+1}(n, y), \tag{37}$$

where $m_{l+1}(n, y)$ is the nth order incomplete moment of $Y_{l+1} \sim \text{Exp}^{l+1}(G)$. As we have seen with regard to the moments and their generating function, the incomplete moments can be computed alternatively as

$$m(n, y) = \sum_{l=0}^{\infty} \varrho_l \varphi(n, y, l), \tag{38}$$

where $\varphi(n, y, l) = (l + 1) \int_0^{G(y,\xi)} u^l Q_G(u)^n du$.

TABLE 1. Numerical results of first four moments, variance, coefficients of variation, skewness, and kurtosis for $a = 0.5$.

Parameters		Measures							
b	θ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	skewness	kurtosis
0.4	0.15	12.4557	356.795	16479.9	1.04343×10^6	201.649	1.14006	5.99702	11.8605
0.4	0.4	4.24966	37.2211	555.407	11958.2	19.1615	1.03006	7.8075	15.1751
0.4	0.8	2.23155	8.44633	51.2809	462.051	3.46649	0.83433	6.9062	15.1689
0.4	1.3	1.54101	3.55093	11.7594	54.7455	1.17621	0.703778	4.35577	11.5193
0.4	1.7	1.28683	2.35525	5.84687	19.3636	0.699325	0.649859	3.01968	9.08372
0.4	2.2	1.09566	1.64853	3.22047	8.00053	0.448067	0.610938	2.07833	7.1581
0.4	2.8	0.950672	1.21216	1.94781	3.84208	0.308385	0.584138	1.49075	5.86609
0.4	3.5	0.838326	0.927764	1.26855	2.07891	0.224974	0.565787	1.13299	5.04734
0.7	0.15	7.11757	116.505	3074.96	111253.	65.8447	1.14006	5.99702	11.8605
0.7	0.4	2.42837	12.1538	103.633	1275.01	6.25683	1.03006	7.8075	15.1751
0.7	0.8	1.27517	2.75798	9.56845	49.2649	1.13192	0.83433	6.9062	15.1689
0.7	1.3	0.880579	1.15949	2.19418	5.83708	0.384068	0.703778	4.35577	11.5193
0.7	1.7	0.73533	0.769061	1.09096	2.0646	0.228351	0.649859	3.01968	9.08372
0.7	2.2	0.62609	0.538296	0.600904	0.853034	0.146308	0.610938	2.07833	7.1581
0.7	2.8	0.543241	0.395808	0.36344	0.409651	0.100697	0.584138	1.49075	5.86609
0.7	3.5	0.479044	0.302943	0.236698	0.221658	0.0734608	0.565787	1.13299	5.04734
1.2	0.15	4.15192	39.6439	610.366	12881.9	22.4055	1.14006	5.99702	11.8605
1.2	0.4	1.41655	4.13568	20.5706	147.632	2.12906	1.03006	7.8075	15.1751
1.2	0.8	0.743851	0.938481	1.89929	5.70433	0.385166	0.83433	6.9062	15.1689
1.2	1.3	0.513671	0.394548	0.435535	0.67587	0.13069	0.703778	4.35577	11.5193
1.2	1.7	0.428942	0.261694	0.216551	0.239057	0.0777028	0.649859	3.01968	9.08372
1.2	2.2	0.365219	0.18317	0.119277	0.0987719	0.0497853	0.610938	2.07833	7.1581
1.2	2.8	0.316891	0.134685	0.0721411	0.0474331	0.034265	0.584138	1.49075	5.86609
1.2	3.5	0.279442	0.103085	0.0469835	0.0256655	0.0249971	0.565787	1.13299	5.04734
1.8	0.15	2.76794	17.6195	180.849	2544.57	9.958	1.14006	5.99702	11.8605
1.8	0.4	0.944368	1.83808	6.095	29.1618	0.946249	1.03006	7.8075	15.1751
1.8	0.8	0.495901	0.417103	0.562753	1.12678	0.171185	0.83433	6.9062	15.1689
1.8	1.3	0.342447	0.175354	0.129047	0.133505	0.0580843	0.703778	4.35577	11.5193
1.8	1.7	0.285962	0.116309	0.0641631	0.0472212	0.0345346	0.649859	3.01968	9.08372
1.8	2.2	0.243479	0.081409	0.0353412	0.0195105	0.0221268	0.610938	2.07833	7.1581
1.8	2.8	0.211261	0.0598599	0.0213752	0.00936949	0.0152289	0.584138	1.49075	5.86609
1.8	3.5	0.186295	0.0458155	0.013921	0.00506973	0.0111098	0.565787	1.13299	5.04734
2.4	0.15	2.07596	9.91097	76.2958	805.118	5.60137	1.14006	5.99702	11.8605
2.4	0.4	0.708276	1.03392	2.57133	9.22698	0.532265	1.03006	7.8075	15.1751
2.4	0.8	0.371926	0.23462	0.237412	0.356521	0.0962915	0.83433	6.9062	15.1689
2.4	1.3	0.256835	0.0986369	0.0544418	0.0422419	0.0326724	0.703778	4.35577	11.5193
2.4	1.7	0.214471	0.0654236	0.0270688	0.0149411	0.0194257	0.649859	3.01968	9.08372
2.4	2.2	0.182609	0.0457925	0.0149096	0.00617325	0.0124463	0.610938	2.07833	7.1581
2.4	2.8	0.158445	0.0336712	0.00901764	0.00296457	0.00856625	0.584138	1.49075	5.86609

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Table 1 – *Continued from previous page*

Parameters		Measures							
b	θ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	skewness	kurtosis
2.4	3.5	0.139721	0.0257712	0.00587293	0.0016041	0.00624927	0.565787	1.13299	5.04734
3.0	0.15	1.66077	6.34302	39.0634	329.776	3.58488	1.14006	5.99702	11.8605
3.0	0.4	0.566621	0.661709	1.31652	3.77937	0.34065	1.03006	7.8075	15.1751
3.0	0.8	0.297541	0.150157	0.121555	0.146031	0.0616265	0.83433	6.9062	15.1689
3.0	1.3	0.205468	0.0631276	0.0278742	0.0173023	0.0209104	0.703778	4.35577	11.5193
3.0	1.7	0.171577	0.0418711	0.0138592	0.00611987	0.0124324	0.649859	3.01968	9.08372
3.0	2.2	0.146088	0.0293072	0.0076337	0.00252856	0.00796564	0.610938	2.07833	7.1581
3.0	2.8	0.126756	0.0215496	0.00461703	0.00121429	0.0054824	0.584138	1.49075	5.86609
3.0	3.5	0.111777	0.0164936	0.00300694	0.000657037	0.00399953	0.565787	1.13299	5.04734
3.5	0.15	1.42351	4.66018	24.5997	178.005	2.63379	1.14006	5.99702	11.8605
3.5	0.4	0.485675	0.486153	0.829062	2.04001	0.250273	1.03006	7.8075	15.1751
3.5	0.8	0.255035	0.110319	0.0765476	0.0788238	0.0452766	0.83433	6.9062	15.1689
3.5	1.3	0.176116	0.0463795	0.0175534	0.00933933	0.0153627	0.703778	4.35577	11.5193
3.5	1.7	0.147066	0.0307624	0.00872768	0.00330335	0.00913404	0.649859	3.01968	9.08372
3.5	2.2	0.125218	0.0215318	0.00480723	0.00136485	0.00585231	0.610938	2.07833	7.1581
3.5	2.8	0.108648	0.0158323	0.00290752	0.000655442	0.00402789	0.584138	1.49075	5.86609
3.5	3.5	0.0958087	0.0121177	0.00189358	0.000354652	0.00293843	0.565787	1.13299	5.04734
4.5	0.15	1.10718	2.81912	11.5743	65.141	1.59328	1.14006	5.99702	11.8605
4.5	0.4	0.377747	0.294093	0.39008	0.746542	0.1514	1.03006	7.8075	15.1751
4.5	0.8	0.19836	0.0667364	0.0360162	0.0288456	0.0273896	0.83433	6.9062	15.1689
4.5	1.3	0.136979	0.0280567	0.00825903	0.00341773	0.00929349	0.703778	4.35577	11.5193
4.5	1.7	0.114385	0.0186094	0.00410644	0.00120886	0.00552553	0.649859	3.01968	9.08372
4.5	2.2	0.0973917	0.0130254	0.00226184	0.000499469	0.00354029	0.610938	2.07833	7.1581
4.5	2.8	0.0845042	0.00957759	0.00136801	0.000239859	0.00243662	0.584138	1.49075	5.86609
4.5	3.5	0.0745179	0.00733048	0.000890946	0.000129785	0.00177757	0.565787	1.13299	5.04734

Table 2. Numerical results of first four moments, variance, coefficient of variation, skewness, and kurtosis for $a = 2.5$.

Parameters		Measures							
a	θ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	skewness	kurtosis
0.4	0.15	15.2347	452.128	20970.	1.32828×10^6	220.033	0.973667	5.10948	10.7078
0.4	0.4	6.31164	65.8238	1038.05	22667.8	25.987	0.807673	4.94369	11.0066
0.4	0.8	3.80056	20.7872	156.243	1560.93	6.34291	0.662669	3.30168	8.98051
0.4	1.3	2.81678	10.6607	51.6018	312.397	2.72648	0.586202	1.90456	6.67863
0.4	1.7	2.42274	7.68868	30.2786	144.468	1.819	0.556683	1.33728	5.57918
0.4	2.2	2.11136	5.7381	18.9387	73.9066	1.28026	0.535903	0.957357	4.77156
0.4	2.8	1.86546	4.428	12.5806	41.5501	0.948076	0.521959	0.719685	4.22895
0.4	3.5	1.66866	3.51654	8.78505	25.223	0.732112	0.512768	0.572175	3.87341
0.7	0.15	8.70553	147.634	3912.76	141624.	71.8475	0.973667	5.10948	10.7078
0.7	0.4	3.60665	21.4935	193.688	2416.9	8.48555	0.807673	4.94369	11.0066

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Table 2 — continued from previous page

Parameters		Measures							
a	θ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	skewness	kurtosis
0.7	0.8	2.17175	6.78765	29.1532	166.43	2.07115	0.662669	3.30168	8.98051
0.7	1.3	1.60959	3.48106	9.62832	33.3085	0.890278	0.586202	1.90456	6.67863
0.7	1.7	1.38443	2.51059	5.64966	15.4035	0.593958	0.556683	1.33728	5.57918
0.7	2.2	1.20649	1.87367	3.53376	7.88009	0.418043	0.535903	0.957357	4.77156
0.7	2.8	1.06597	1.44588	2.34739	4.43016	0.309576	0.521959	0.719685	4.22895
0.7	3.5	0.953521	1.14826	1.63919	2.68933	0.239057	0.512768	0.572175	3.87341
1.2	0.15	5.07823	50.2365	776.666	16398.5	24.4481	0.973667	5.10948	10.7078
1.2	0.4	2.10388	7.31376	38.4462	279.85	2.88744	0.807673	4.94369	11.0066
1.2	0.8	1.26685	2.30969	5.78678	19.2708	0.704768	0.662669	3.30168	8.98051
1.2	1.3	0.938928	1.18453	1.91118	3.85676	0.302942	0.586202	1.90456	6.67863
1.2	1.7	0.807581	0.854298	1.12143	1.78356	0.202111	0.556683	1.33728	5.57918
1.2	2.2	0.703787	0.637567	0.701435	0.912428	0.142251	0.535903	0.957357	4.77156
1.2	2.8	0.621819	0.492	0.465947	0.512964	0.105342	0.521959	0.719685	4.22895
1.2	3.5	0.556221	0.390727	0.325372	0.311395	0.0813458	0.512768	0.572175	3.87341
1.8	0.15	3.38548	22.3273	230.123	3239.22	10.8658	0.973667	5.10948	10.7078
1.8	0.4	1.40259	3.25056	11.3915	55.279	1.28331	0.807673	4.94369	11.0066
1.8	0.8	0.844569	1.02653	1.7146	3.80657	0.31323	0.662669	3.30168	8.98051
1.8	1.3	0.625952	0.526457	0.566275	0.761829	0.134641	0.586202	1.90456	6.67863
1.8	1.7	0.538388	0.379688	0.332276	0.352308	0.0898269	0.556683	1.33728	5.57918
1.8	2.2	0.469191	0.283363	0.207832	0.180233	0.0632226	0.535903	0.957357	4.77156
1.8	2.8	0.414546	0.218667	0.138058	0.101326	0.0468186	0.521959	0.719685	4.22895
1.8	3.5	0.370814	0.173656	0.0964066	0.06151	0.0361537	0.512768	0.572175	3.87341
2.4	0.15	2.53911	12.5591	97.0832	1024.91	6.11203	0.973667	5.10948	10.7078
2.4	0.4	1.05194	1.82844	4.80578	17.4906	0.721861	0.807673	4.94369	11.0066
2.4	0.8	0.633427	0.577421	0.723347	1.20442	0.176192	0.662669	3.30168	8.98051
2.4	1.3	0.469464	0.296132	0.238897	0.241047	0.0757354	0.586202	1.90456	6.67863
2.4	1.7	0.403791	0.213575	0.140179	0.111472	0.0505276	0.556683	1.33728	5.57918
2.4	2.2	0.351893	0.159392	0.0876793	0.0570267	0.0355627	0.535903	0.957357	4.77156
2.4	2.8	0.310909	0.123	0.0582433	0.0320602	0.0263355	0.521959	0.719685	4.22895
2.4	3.5	0.27811	0.0976818	0.0406716	0.0194622	0.0203365	0.512768	0.572175	3.87341
3.0	0.15	2.03129	8.03784	49.7066	419.802	3.9117	0.973667	5.10948	10.7078
3.0	0.4	0.841553	1.1702	2.46056	7.16416	0.461991	0.807673	4.94369	11.0066
3.0	0.8	0.506741	0.36955	0.370354	0.493331	0.112763	0.662669	3.30168	8.98051
3.0	1.3	0.375571	0.189524	0.122315	0.098733	0.0484707	0.586202	1.90456	6.67863
3.0	1.7	0.323033	0.136688	0.0717716	0.0456591	0.0323377	0.556683	1.33728	5.57918
3.0	2.2	0.281515	0.102011	0.0448918	0.0233581	0.0227601	0.535903	0.957357	4.77156
3.0	2.8	0.248727	0.07872	0.0298206	0.0131319	0.0168547	0.521959	0.719685	4.22895
3.0	3.5	0.222488	0.0625163	0.0208238	0.0079717	0.0130153	0.512768	0.572175	3.87341
3.5	0.15	1.74111	5.90535	31.3021	226.599	2.8739	0.973667	5.10948	10.7078
3.5	0.4	0.721331	0.85974	1.54951	3.86703	0.339422	0.807673	4.94369	11.0066
3.5	0.8	0.43435	0.271506	0.233226	0.266288	0.0828462	0.662669	3.30168	8.98051

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Table 2 — continued from previous page

Parameters		Measures							
a	θ	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CV	skewness	kurtosis
3.5	1.3	0.321918	0.139242	0.0770266	0.0532936	0.0356111	0.586202	1.90456	6.67863
3.5	1.7	0.276885	0.100424	0.0451972	0.0246456	0.0237583	0.556683	1.33728	5.57918
3.5	2.2	0.241298	0.0749466	0.0282701	0.0126081	0.0167217	0.535903	0.957357	4.77156
3.5	2.8	0.213195	0.0578351	0.0187791	0.00708826	0.012383	0.521959	0.719685	4.22895
3.5	3.5	0.190704	0.0459304	0.0131136	0.00430293	0.00956228	0.512768	0.572175	3.87341
4.5	0.15	1.35419	3.57237	14.7279	82.9239	1.73853	0.973667	5.10948	10.7078
4.5	0.4	0.561035	0.52009	0.729055	1.41514	0.205329	0.807673	4.94369	11.0066
4.5	0.8	0.337828	0.164244	0.109734	0.0974481	0.0501168	0.662669	3.30168	8.98051
4.5	1.3	0.250381	0.084233	0.0362416	0.0195028	0.0215425	0.586202	1.90456	6.67863
4.5	1.7	0.215355	0.0607501	0.0212656	0.00901908	0.0143723	0.556683	1.33728	5.57918
4.5	2.2	0.187676	0.0453381	0.0133013	0.00461395	0.0101156	0.535903	0.957357	4.77156
4.5	2.8	0.165818	0.0349867	0.00883573	0.00259395	0.00749097	0.521959	0.719685	4.22895
4.5	3.5	0.148325	0.027785	0.00617002	0.00157466	0.00578459	0.512768	0.572175	3.87341

4. Entropies and order statistics

In this subsection, we derive expressions for two popular entropy measures, namely the Rényi entropy [26] and the Shannon entropy [27]. Moreover, we derive useful expressions for the densities of the order statistics of the WLL-G family.

4.1. Rényi entropy.

The Rényi entropy $I_R(b)$ of a random variable X is defined to be

$$I_R(b) = \frac{1}{1-b} \log \left[\int_{-\infty}^{\infty} f(x)^b \, dx \right], \tag{39}$$

where $b > 0$, $b \neq 1$, and $f(x)$ is the pdf function of X. For a real number γ, a result of Flajolet and Odlyzko [28, Theorem 3A] states that (for $u \in (0,1)$)

$$\left(\log \frac{1}{1-u} \right)^{\gamma} = u^{\gamma} + \gamma \sum_{j=0}^{\infty} s_j(j+\gamma) u^{j+\gamma+1}, \tag{40}$$

where $s_j(y)$ are polynomials related to the Stirling polynomials $S_j(y)$ by $s_{j-1}(y) = \frac{S_j(y)}{j!(w+1)}$ [29]. Starting with Equation (3), we have

$$\begin{aligned} f(x;\theta,a,\xi)^b = \left(\frac{\theta}{a}\right)^b & \left[1 + \frac{1}{a} \frac{G(x,\xi)}{1-G(x,\xi)} \log \frac{1}{1-G(x,\xi)} \right]^{-b(\theta+1)} \\ & \times \left[G(x,\xi) + \log \frac{1}{1-G(x,\xi)} \right]^b \frac{g(x,\xi)^b}{[1-G(x,\xi)]^{2b}}. \end{aligned} \tag{41}$$

Applying the generalized binomial expansion to the first square brackets yields

$$\begin{aligned} f(x;\theta,a,\xi)^b = \left(\frac{\theta}{a}\right)^b \sum_{i=0}^{\infty} \binom{b(\theta+1)+i-1}{i} \frac{(-1)^i}{a^i} \frac{G(x,\xi)^i g(x,\xi)^b}{[1-G(x,\xi)]^{i+2b}} \left[\log \frac{1}{1-G(x,\xi)}\right]^{i+b} \\ \times \left[1 + \frac{G(x,\xi)}{\log \frac{1}{1-G(x,\xi)}}\right]^b. \end{aligned} \tag{42}$$

Using Equation (40) and the generalized binomial expansion, $f(x;\theta,a,\xi)^b$ takes the form

$$f(x;\theta,a,\xi)^b = \left(\frac{\theta}{a}\right)^b \left[\sum_{i,k=0}^{\infty} \alpha_{i,k} G(x,\xi)^{2i+k+b} g(x,\xi)^b + \sum_{i,k,l=0}^{\infty} \beta_{i,k,l} G(x,\xi)^{2i+k+l+b+1} g(x,\xi)^b\right], \tag{43}$$

where

$$\alpha_{i,k} = \sum_{j=0}^{\infty} \binom{b(\theta+1)+i-1}{i} \binom{b}{j} \binom{i+2b}{k} \frac{(-1)^{i+k}}{a^i}, \tag{44}$$

and

$$\beta_{i,k,l} = \sum_{j=0}^{\infty} \binom{b(\theta+1)+i-1}{i} \binom{b}{j} \binom{i+2b}{k} \frac{(-1)^{i+k} s_l(i-j+l+b)}{a^i} (i-j+b). \tag{45}$$

Therefore, the Rényi entropy of the WLL-G family is given by

$$I_R(b) = \frac{b}{1-b} \log\left(\frac{\theta}{a}\right) + \frac{1}{1-b} \log\left(\sum_{i,k=0}^{\infty} \alpha_{i,k} I_{i,k}(\xi,b) + \sum_{i,k,l=0}^{\infty} \beta_{i,k,l} \bar{I}_{i,k,l}(\xi,b)\right), \tag{46}$$

where

$$I_{i,k}(\xi,b) = \int_{-\infty}^{\infty} G(x,\xi)^{2i+k+b} g(x,\xi)^b dx, \tag{47}$$

and

$$\bar{I}_{i,k,l}(\xi,b) = \int_{-\infty}^{\infty} G(x,\xi)^{2i+k+l+b+1} g(x,\xi)^b dx. \tag{48}$$

4.2. **Shannon entropy.** The Shannon entropy of a random variable X that has a pdf function $f(x)$ is by definition equal to $E[-\log f(X)]$. It is the special limit $b \uparrow 1$ of the Rényi entropy $I_R(b)$. For the WLL-G family, a simple computation gives

$$\begin{aligned} E[-\log f(X;\theta,a,\xi)] = \log\left(\frac{a}{\theta}\right) - E[\log g(X,\xi)] + 2E\{\log[1-G(X,\xi)]\} - E\left[\log\log\frac{1}{1-G(X,\xi)}\right] \\ + (\theta+1)E\left\{\log\left[1 + \frac{1}{a}\frac{G(X,\xi)}{1-G(X,\xi)} \log\frac{1}{1-G(X,\xi)}\right]\right\} \\ - E\left(\log\left\{1 - \frac{G(X,\xi)}{\log[1-G(X,\xi)]}\right\}\right). \end{aligned} \tag{49}$$

Following [30], let $A(a_1,a_2,a_3)$ be the function defined by

$$A(a_1,a_2,a_3) = \int_0^1 \frac{y^{a_1}}{(1-y)^{a_2}} \log[(1-y)]^{a_3} dy. \tag{50}$$

Furthermore, Cordeiro et al. [31, Equation 31] gave a useful expression for the double logarithm function $(u \in (0,1))$

$$\log\log\frac{1}{1-u} = u + \sum_{i,j=0}^{\infty}\frac{(-1)^i}{i+1}e_{i+1,j}u^{i+j+1}, \tag{51}$$

where $e_{i+1,0} = 1/2^{(i+1)}$ and (for $j = 1, 2, \ldots$)

$$e_{i+1,j} = \frac{2}{j}\sum_{p=1}^{j}\frac{p(i+2)-j}{p+2}e_{i+1,j-p}. \tag{52}$$

Proposition 4.1. *Let X be a random variable such that $X \sim$ WLL-$G(\theta, a, \xi)$. Then, the Shannon entropy has the form*

$$\begin{aligned}
E[-\log f(X;\theta,a,\xi)] \quad = \quad & \log\left(\frac{a}{\theta}\right) - E[\log g(X,\xi)] + \sum_{i=0}^{\infty}\alpha_i[A(i+1,i+2,i+1) - A(i,i+2,i+2)] \\
& + \sum_{i,j=0}^{\infty}\beta_{i,j}\Bigg\{\Bigg[A(i+j+2,i+2,i-j-1) + A(i+1,i-j+1,i) \\
& + \frac{\theta+1}{a^{j+1}}A(i+j+1,i+j+3,i+j+2)\Bigg] - \Bigg[A(i+j+1,i+2,i-j) \\
& + A(i,i-j+1,i+1) + \frac{\theta+1}{a^{j+1}}A(i+j+2,i+j+3,i+j+1)\Bigg]\Bigg\} \\
& + \sum_{i,j,k=0}^{\infty}\gamma_{i,j,k}[A(i+j+k+1,i+2,i+1) - A(i+j+k+2,i+2,i)].
\end{aligned} \tag{53}$$

where $\alpha_i = \binom{\theta+i}{i}\frac{2\theta}{a^{i+1}}$, $\beta_{i,j} = \frac{\alpha_i}{2(j+1)}$, and $\gamma_{i,j,k} = (-1)^j e_{j+1,k}\beta_{i,j}$.

The term $E[\log g(X,\xi)]$ can be computed numerically for most baseline models. One formula for it, using Equation (3), is

$$E[\log g(X,\xi)] = \sum_{l=0}^{\infty}(l+1)\varrho_l I_l, \tag{54}$$

where $I_l = \int_0^1 u^l \log g[Q_G(u)]du$.

4.3. **Order statistics.** Order statistics is an essential notion in statistical theory and has far-reaching applications in real life. Let $X_1, \ldots, X_n$ be a random sample drawn from the WLL-G family. The density function of the ith order statistic $X_{i:n}$ is

$$\begin{aligned}
f_{i:n}(x;\theta,a,\xi) &= R\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f(x;\theta,a,\xi)F(x;\theta,a,\xi)^{j+i-1} \\
&= R\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\sum_{l=0}^{\infty}(l+1)\varrho_l G(x,\xi)^l g(x,\xi)\right]\left[\sum_{m=0}^{\infty}\varrho_m G(x,\xi)^{m+1}\right]^{j+i-1},
\end{aligned} \tag{55}$$

where $R = n!/[(i-1)!(n-i)!]$. Using Equations (24) and (25), we have

$$\left[\sum_{m=0}^{\infty} \varrho_m G(x,\xi)^{m+1}\right]^{j+i-1} = \sum_{m=0}^{\infty} \rho_{j+i-1,m} G(x,\xi)^{j+i+m-1}, \tag{56}$$

where $\rho_{j+i-1,0} = \varrho_0^{j+i-1}$ and for $m = 1, 2, \ldots$, we have

$$\rho_{j+i-1,m} = \frac{1}{m\varrho_0} \sum_{p=1}^{m} [p(i+j) - m] \varrho_p \rho_{j+i-1,m-p}. \tag{57}$$

It follows that

$$f_{i:n}(x;\theta,a,\xi) = \sum_{j=0}^{n-i} \sum_{l,m=0}^{\infty} \beta_{j,l,m} t_{i+j+l+m}(x,\xi), \tag{58}$$

where

$$\beta_{j,l,m} = (-1)^j \binom{n-i}{j} R \frac{(1+l)\varrho_l \rho_{j+i-1,m}}{i+j+l+m}. \tag{59}$$

Equation (58) implies that the density functions of the WLL-G family are multilinear combinations of exponentiated-G density functions. As a result, many properties of these order statistics can be understood from the corresponding properties of the exponentiated-G distributions.

5. Parameters estimation

5.1. **Likelihood estimation.**

One of the most commonly used techniques for parameter estimation is the maximum likelihood method. This method provides confidence intervals for the distribution's parameters. Moreover, it constructs simple and reasonably efficient approximations of a model's parameters. Now, we derive the maximum-likelihood estimators for the WLL-G family. Let $x_1, \ldots, x_n$ be the observations recorded for the random variable $X \sim$ WLL-$G(\theta, a, \xi)$ where $\xi = (\xi_1, \ldots, \xi_k)^\top$ is the parameters vector of the parent distribution. The log-likelihood function for $\nu = (\theta, a, \xi)^\top$ is

$$\begin{aligned} \ell_n(\nu) = n\log\theta - n\log a - (\theta+1)\sum_{i=1}^{n} \log\left\{1 - \frac{1}{a}\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\log[1-G(x_i,\xi)]\right\} \\ + \sum_{i=1}^{n} \log\left\{G(x_i,\xi) - \log[1-G(x_i,\xi)]\right\} + \sum_{i=1}^{n} \log g(x_i,\xi) - 2\sum_{i=1}^{n} \log[1-G(x_i,\xi)]. \end{aligned} \tag{60}$$

The components of the score function $U_n(\nu) = (\partial\ell_n(\nu)/\partial\theta, \partial\ell_n(\nu)/\partial a, \partial\ell_n(\nu)/\partial\xi_1, \ldots, \partial\ell_n(\nu)/\partial\xi_k)^\top$ are as follows

$$\frac{\partial\ell_n(\nu)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log\left\{1 - \frac{1}{a}\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\log[1-G(x_i,\xi)]\right\}, \tag{61}$$

$$\frac{\partial\ell_n(\nu)}{\partial a} = -\frac{n}{a} + \frac{\theta+1}{a} \sum_{i=1}^{n} \frac{G(x_i,\xi)\log[1-G(x_i,\xi)]}{G(x_i,\xi)\log[1-G(x_i,\xi)] - a[1-G(x_i,\xi)]}, \tag{62}$$

and (for $j = 1, 2, \ldots$)

$$\frac{\partial \ell_n(\nu)}{\partial \xi_j} = \sum_{i=1}^{n} \frac{g_{\xi_j}(x_i, \xi)}{g(x_i, \xi)} + 2\sum_{i=1}^{n} \frac{G_{\xi_j}(x_i, \xi)}{1 - G(x_i, \xi)} + \sum_{i=1}^{n} \frac{[2 - G(x_i, \xi)]G_{\xi_j}(x_i, \xi)}{[1 - G(x_i, \xi)]\{G(x_i, \xi) - \log[1 - G(x_i, \xi)]\}}$$
$$+ (\theta + 1)\sum_{i=1}^{n} \frac{\{G(x_i, \xi) - \log[1 - G(x_i, \xi)]\}G_{\xi_j}(x_i, \xi)}{[1 - G(x_i, \xi)]\{G(x_i, \xi)\log[1 - G(x_i, \xi)] - a[1 - G(x_i, \xi)]\}},$$

where $G_{\xi_j}(x_i, \xi)$ and $g_{\xi_j}(x_i, \xi)$ denote the derivative with respect to ξ_j of $G(x_i, \xi)$ and $g(x_i, \xi)$, respectively. Setting these equations to zero and solving the resulting system numerically yields the maximum-likelihood estimators for the WLL-G family. It is worth noting that the regularity conditions guaranteeing the existence and uniqueness of the maximum likelihood estimators for the WLL-G family ultimately depend on the cdf $G(x, \xi)$ of the baseline model and its corresponding pdf $g(x, \xi)$, as given in Equation (60). The observed information matrix $I(\nu)$ can be obtained by differentiating $U_n(\nu)$ and numerically evaluating the result. Under the standard regularity conditions, the distribution of the maximum likelihood estimates $\hat{\nu}$ is approximated by the multivariate normal distribution $\mathcal{N}(0, I(\hat{\nu})^{-1})$ as $n \to \infty$. These approximations are then used to construct suitable confidence intervals.

5.2. **Bayesian estimation.** In this subsection, we focus on the Bayesian estimation procedure for determining the parameters of the WLL-G family. Both point and interval estimates of the parameters are derived using the likelihood function.

5.2.1. *Model and prior specification.* In several simulation scenarios, the Bayesian estimators under the independent Gamma priors used in this study yielded smaller mean squared errors and shorter posterior credible intervals than the corresponding asymptotic MLEs-based intervals. This behavior can be largely attributed to two factors. First, the Gamma priors act as a form of regularization (shrinkage) that reduces estimator variance and stabilizes numerical optimization when sample sizes are small or moderate. Second, posterior-based credible intervals can exhibit better finite-sample trade-offs between length and coverage than asymptotic confidence intervals based on the Fisher information. Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote the observed sample and let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_p)^\top$ be the vector of unknown model parameters (recall that $\boldsymbol{\nu} = (\theta, a, \xi)^\top$, where $\xi = (\xi_1, \ldots, \xi_k)^\top$ is the parameter of the parent model, and $\nu_j > 0$ when applicable). The likelihood function of the WLL-G family is

$$L(\boldsymbol{\nu} \mid \mathbf{x}) = \left(\frac{\theta}{a}\right)^n \prod_{i=1}^{n} \frac{G(x_i, \xi) - \log[1 - G(x_i, \xi)]}{\left\{1 - \frac{1}{a}\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\log[1 - G(x_i, \xi)]\right\}^{\theta+1}} \frac{g(x_i, \xi)}{[1 - G(x_i, \xi)]^2}. \tag{63}$$

We adopt independent Gamma priors for positive parameters

$$\nu_j \sim \text{Gamma}(\alpha_j, \beta_j), \qquad j = 1, \ldots, p, \tag{64}$$

Consequently, the prior distribution of α_j and β_j, is formulated as follows

$$\pi(\nu_j) \propto \nu_j^{\alpha_j - 1} e^{-\nu_j \beta_j}, \qquad j = 1, \ldots, p. \tag{65}$$

The joint prior is $\pi(\boldsymbol{\nu}) = \prod_{j=1}^{p} \pi(\nu_j)$, and the posterior (up to normalization) is

$$\pi(\boldsymbol{\nu} \mid \mathbf{x}) \propto L(\boldsymbol{\nu} \mid \mathbf{x})\, \pi(\boldsymbol{\nu}). \tag{66}$$

Therefore, $\pi(\boldsymbol{\nu} \mid \mathbf{x})$ takes the form

$$\begin{aligned}\pi(\boldsymbol{\nu} \mid \mathbf{x}) &= \theta^{\alpha_1 - 1} e^{-\theta\beta_1} a^{\alpha_2 - 1} e^{-a\beta_2} \prod_{e=1}^{k} \xi_e^{\alpha_e - 1} e^{-\xi_e \beta_e} \\ &\times \left(\frac{\theta}{a}\right)^n \prod_{i=1}^{n} \frac{G(x_i,\xi) - \log[1 - G(x_i,\xi)]}{\left\{1 - \frac{1}{a}\frac{G(x_i,\xi)}{1-G(x_i,\xi)} \log[1 - G(x_i,\xi)]\right\}^{\theta+1}} \frac{g(x_i,\xi)}{[1 - G(x_i,\xi)]^2}.\end{aligned} \tag{67}$$

From the posterior sample, we report the posterior mean, posterior median, and the posterior mode (MAP) for each parameter. Interval summaries consist of $(1-\alpha)$ equal-tailed credible intervals and highest-posterior-density (HPD) intervals computed from the posterior draws. For any scalar function of parameters $\phi = \phi(\boldsymbol{\nu})$, the posterior distribution of ϕ is obtained by applying ϕ to each posterior draw and summarising the resulting sample. All Bayesian sampling in this work was performed using the Metropolis–Hastings (MH) algorithm. We implemented a random-walk MH on an unconstrained scale (log-transform for strictly positive parameters). Key implementation details are in Algorithm 1.

Algorithm 1 Random-walk Metropolis–Hastings (schematic)

(1) Inputs: data $\mathbf{x}$, prior hyperparameters $\{(\alpha_j, \beta_j)\}_{j=1}^{p}$, proposal covariance Σ_{prop}, total iterations S, burn-in B, number of chains m.

(2) For each chain $c = 1, \ldots, m$:

  (a) Initialize $\boldsymbol{\nu}^{(0)}$ (e.g. at MLEs or randomized near it); set $\eta^{(0)} = \log \boldsymbol{\nu}^{(0)}$.

  (b) For $t = 1, \ldots, S$:

    (i) Propose $\eta^{(\text{prop})} \sim \mathcal{N}(\eta^{(t-1)}, \Sigma_{\text{prop}})$ and set $\boldsymbol{\nu}^{(\text{prop})} = \exp(\eta^{(\text{prop})})$.

    (ii) Compute log acceptance ratio

$$\rho = \log \pi(\boldsymbol{\nu}^{(\text{prop})} \mid \mathbf{x}) - \log \pi(\boldsymbol{\nu}^{(t-1)} \mid \mathbf{x}) \tag{68}$$

    (Jacobian cancels due to symmetric transform if identical transforms used).

    (iii) With probability $\min\{1, \exp(\rho)\}$ set $\eta^{(t)} = \eta^{(\text{prop})}$ (accept), otherwise set $\eta^{(t)} = \eta^{(t-1)}$ (reject).

  (c) Transform retained draws: $\boldsymbol{\theta}^{(t)} = \exp(\eta^{(t)})$.

  (d) Discard first B draws, retain $S - B$ draws.

(3) Combine retained draws from all chains and compute posterior summaries and Monte Carlo standard errors.

In our experiments we used $S = 12{,}000$ iterations per chain, burn-in $B = 2{,}000$, and $m \geq 3$ chains unless otherwise stated.

6. Simulation study

In this part, a Monte-Carlo simulation procedure is used to compare two estimation techniques for estimating the parameters of the WLLE distribution over time using the R programming language: the MLEs method and the Bayesian estimation method under the square error loss function. In our approach, we have opted for independent gamma distributions as our priors, recognizing the significance of the prior distribution choice in parameter estimation. A WLLE distribution data is utilized to create 10,000 random samples for Monte-Carlo experiments, where x represents the WLLE distribution lifetime for various actual parameter values and sample sizes n of $50, 100$, and 200. The best estimation techniques reduce the bias and mean square error (MSE) of the estimators. In addition to the asymptotic confidence intervals (LACI) and Bayesian credible intervals (LCCI), two nonparametric bootstrap approaches were employed to assess interval estimation accuracy, namely the bootstrap–percentile (BP–CI) and the bootstrap–t (BT–CI) methods. The detailed computational steps of these two procedures are summarized in Algorithms 2 and 3, respectively, and their corresponding interval lengths (LBCIP and LBCIT) are reported in Tables 3, 4, 5, and 6.

Algorithm 2 Bootstrap–Percentile Confidence Interval (BP–CI)

(1) Inputs: sample $\mathbf{x} = (x_1, \ldots, x_n)$, estimator $\hat{\boldsymbol{\nu}} = t(\mathbf{x})$, number of bootstrap replicates B, and confidence level $1-\alpha$.

(2) For each $b = 1, \ldots, B$:

  (a) Draw a bootstrap sample $\mathbf{x}^{*(b)}$ of size n with replacement from $\mathbf{x}$.

  (b) Compute the bootstrap estimate $\hat{\boldsymbol{\nu}}^{*(b)} = t(\mathbf{x}^{*(b)})$.

(3) Sort $\{\hat{\boldsymbol{\nu}}^{*(b)}\}_{b=1}^{B}$ in ascending order to obtain $\hat{\boldsymbol{\nu}}^{*(1)} \le \cdots \le \hat{\boldsymbol{\nu}}^{*(B)}$.

(4) Determine indices $L = \lceil (\alpha/2)B \rceil$ and $U = \lceil (1-\alpha/2)B \rceil$.

(5) The $(1-\alpha)$ bootstrap–percentile confidence interval is

$$\mathrm{CI}_{\mathrm{BP}} = \left[\hat{\boldsymbol{\nu}}^{*(L)},\, \hat{\boldsymbol{\nu}}^{*(U)}\right]. \tag{69}$$

Algorithm 3 Bootstrap–t (Studentized) Confidence Interval (BT–CI)

(1) Inputs: sample $\mathbf{x}$, estimator $\hat{\boldsymbol{\nu}} = t(\mathbf{x})$, estimated standard error $\widehat{\mathrm{se}}(\hat{\boldsymbol{\nu}})$, number of bootstrap replicates B, confidence level $1-\alpha$.

(2) For each $b = 1, \ldots, B$:

  (a) Draw a bootstrap sample $\mathbf{x}^{*(b)}$ of size n with replacement from $\mathbf{x}$.

  (b) Compute $\hat{\boldsymbol{\nu}}^{*(b)} = t(\mathbf{x}^{*(b)})$ and its standard error $\widehat{\mathrm{se}}^{*(b)}$.

  (c) Calculate the studentized statistic

$$T^{*(b)} = \frac{\hat{\boldsymbol{\nu}}^{*(b)} - \hat{\boldsymbol{\nu}}}{\widehat{\mathrm{se}}^{*(b)}}. \tag{70}$$

(3) Sort $\{T^{*(b)}\}_{b=1}^{B}$ to obtain ordered values $T^{*(1)} \le \cdots \le T^{*(B)}$.

(4) Determine indices $L = \lceil (\alpha/2)B \rceil$ and $U = \lceil (1-\alpha/2)B \rceil$.

(5) The $(1-\alpha)$ bootstrap–t confidence interval is

$$\mathrm{CI}_{\mathrm{BT}} = \left[\,\hat{\boldsymbol{\nu}} - T^{*(U)}\,\widehat{\mathrm{se}}(\hat{\boldsymbol{\nu}}),\ \hat{\boldsymbol{\nu}} - T^{*(L)}\,\widehat{\mathrm{se}}(\hat{\boldsymbol{\nu}})\,\right]. \tag{71}$$

The simulation results of the point-estimating methods suggested in this paper are summarized in Tables 3, 4, 5, and 6. To perform the necessary comparison between various point-estimating methods, we consider the bias and the MSE values. These tables reveal the following observations:

(1) Biases and MSEs go smaller as the sample size grows.

(2) We find that the Bayesian estimation outperforms the classical estimation techniques for estimating θ, b, and a with respect to MSE.

(3) We note that the lengths of credible confidence intervals (LCCI) are smaller than the lengths of asymptotic confidence intervals (LACI) for θ, b, and a.

(4) By results in bootstrapping, we note that the lengths of confidence intervals of bootstrap for the Bayesian estimation method are smaller than the lengths of confidence intervals of bootstrap for the MLEs method for θ, b, and a.

(5) Sometimes, the length of the bootstrap-t confidence interval (LBCIT) is smaller than the length of the bootstrap-p confidence interval (LBCIP) for θ, b, and a.

Table 3. The MLEs and Bayesian estimation results for the parameters of the WLLE distribution with $a = 1.5$.

$a = 1.5$				MLEs		Bayes		CI		MLEs		Bayes	
θ	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
2	0.7	50	θ	0.0204	0.3168	0.1430	0.2532	2.2059	1.8919	0.0693	0.0705	0.0605	0.0602
			a	0.1129	0.3268	0.2228	0.2350	2.1979	1.6884	0.0700	0.0685	0.0546	0.0538
			b	0.0409	0.0280	0.0602	0.0233	0.6367	0.5507	0.0201	0.0202	0.0170	0.0171
		100	θ	-0.0058	0.2621	0.0624	0.0708	2.0077	1.0146	0.0627	0.0626	0.0320	0.0320
			a	0.1023	0.3143	0.1320	0.0739	2.0532	0.9319	0.0698	0.0682	0.0280	0.0282
			b	0.0404	0.0239	0.0377	0.0089	0.6248	0.3402	0.0202	0.0202	0.0112	0.0111
		200	θ	-0.0003	0.2573	0.0448	0.0368	1.9049	0.7315	0.0626	0.0616	0.0226	0.0227
			a	0.0878	0.3058	0.0786	0.0298	2.0320	0.6029	0.0674	0.0675	0.0188	0.0190
			b	0.0386	0.0207	0.0291	0.0059	0.6031	0.2787	0.0198	0.0198	0.0091	0.0090
	1.5	50	θ	0.3061	1.1059	0.1689	0.2001	3.9458	1.6244	0.1228	0.1243	0.0518	0.0507
			a	0.2085	0.6108	0.2110	0.1946	2.9541	1.5195	0.0914	0.0930	0.0471	0.0471
			b	0.0070	0.0954	0.0725	0.0466	1.2111	0.7977	0.0400	0.0402	0.0237	0.0239
		100	θ	0.3001	0.9163	0.0944	0.0688	3.5650	0.9600	0.1146	0.1146	0.0302	0.0306
			a	0.2189	0.6013	0.1126	0.0638	2.9175	0.8868	0.0904	0.0917	0.0269	0.0270
			b	0.0032	0.0690	0.0412	0.0185	1.0304	0.5091	0.0347	0.0346	0.0155	0.0155
		200	θ	0.2676	0.7207	0.0639	0.0339	3.1597	0.6774	0.0986	0.0989	0.0214	0.0213
			a	0.2135	0.5886	0.0827	0.0306	2.8901	0.6047	0.0877	0.0877	0.0183	0.0183
			b	0.0117	0.0658	0.0287	0.0112	1.0053	0.3997	0.0321	0.0330	0.0120	0.0121
	0.7	50	θ	0.1171	0.1328	0.1139	0.0998	1.3536	1.1557	0.0425	0.0432	0.0364	0.0367

continued on next page

Table 3 – continued

$a = 1.5$				MLEs		Bayes		CI		MLEs		Bayes	
θ	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
			a	0.0714	0.0964	0.1907	0.2148	1.1851	1.6566	0.0368	0.0370	0.0508	0.0505
			b	0.0101	0.0455	0.0732	0.0638	0.8354	0.9478	0.0252	0.0253	0.0307	0.0306
			θ	0.1150	0.1276	0.0893	0.0731	1.3265	1.0008	0.0434	0.0430	0.0326	0.0339
		100	a	0.0829	0.1098	0.1214	0.0679	1.2584	0.9042	0.0399	0.0402	0.0297	0.0296
			b	0.0209	0.0560	0.0663	0.0443	0.9242	0.7834	0.0283	0.0284	0.0246	0.0246
			θ	0.1296	0.1713	0.0819	0.0535	1.5415	0.8483	0.0487	0.0498	0.0267	0.0268
		200	a	0.0709	0.1342	0.0766	0.0316	1.4096	0.6295	0.0451	0.0490	0.0202	0.0198
			b	0.0081	0.0547	0.0426	0.0242	0.9166	0.5868	0.0290	0.0290	0.0185	0.0187
			θ	0.1407	0.2247	0.1094	0.0713	1.7755	0.9556	0.0575	0.0579	0.0301	0.0301
		50	a	0.1996	0.6909	0.1918	0.1966	3.1646	1.5678	0.1041	0.1036	0.0494	0.0487
			b	0.0293	0.2462	0.0623	0.0958	1.9426	1.1893	0.0593	0.0596	0.0376	0.0372
			θ	0.1482	0.2018	0.0812	0.0410	1.6633	0.7278	0.0545	0.0547	0.0235	0.0243
	1.5	100	a	0.2123	0.9757	0.1393	0.0780	3.7834	0.9497	0.1130	0.1167	0.0308	0.0308
			b	0.0018	0.2543	0.0311	0.0391	1.9776	0.7657	0.0642	0.0643	0.0229	0.0232
			θ	0.1448	0.2488	0.0686	0.0282	1.8720	0.6008	0.0614	0.0634	0.0187	0.0194
		200	a	0.1807	0.6613	0.0834	0.0347	3.1096	0.6533	0.0953	0.0958	0.0217	0.0217
			b	0.0206	0.2293	0.0198	0.0235	1.8761	0.5966	0.0605	0.0601	0.0196	0.0192

Table 4. The MLEs and Bayesian estimation results for the parameters of the WLLE distribution with $a = 0.4$.

$a = 0.4$				MLEs		Bayes		CI		MLEs		Bayes	
θ	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
0.6			θ	0.0306	0.1922	0.0861	0.0650	1.7152	0.9415	0.0508	0.0662	0.0310	0.0312
		50	a	0.2446	0.2603	0.1323	0.1011	1.7561	1.1340	0.0537	0.0537	0.0369	0.0371
			b	0.1663	0.1904	0.0919	0.1013	1.5820	1.1950	0.0515	0.0511	0.0387	0.0391
			θ	0.0380	0.1685	0.0747	0.0494	1.6029	0.8206	0.0485	0.0574	0.0251	0.0250
	0.7	100	a	0.2226	0.2496	0.1168	0.0795	1.7542	1.0063	0.0527	0.0528	0.0314	0.0309
			b	0.1455	0.1739	0.0725	0.0551	1.5330	0.8757	0.0497	0.0509	0.0264	0.0264
			θ	0.0253	0.1076	0.0620	0.0344	1.2829	0.6858	0.0388	0.0410	0.0220	0.0223
		200	a	0.2124	0.2356	0.0997	0.0585	1.7081	0.8647	0.0518	0.0518	0.0271	0.0272
			b	0.1354	0.1620	0.0541	0.0348	1.4650	0.6997	0.0485	0.0503	0.0223	0.0220
			θ	0.0306	0.1922	0.0861	0.0650	1.7152	0.9415	0.0508	0.0662	0.0310	0.0312
		50	a	0.2446	0.2603	0.1323	0.1011	1.7561	1.1340	0.0537	0.0537	0.0369	0.0371
			b	0.1663	0.1904	0.0919	0.1013	1.5820	1.1950	0.0515	0.0511	0.0387	0.0391
			θ	0.0380	0.1685	0.0747	0.0494	1.6029	0.8206	0.0502	0.0574	0.0251	0.0250
	1.5	100	a	0.2226	0.2496	0.1168	0.0795	1.7542	1.0063	0.0518	0.0528	0.0314	0.0309

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Table 4 – continued

$a = 0.4$				MLEs		Bayes		CI		MLEs		Bayes	
θ	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
			b	0.1455	0.1739	0.0725	0.0551	1.5330	0.8757	0.0497	0.0505	0.0264	0.0264
		200	θ	0.0253	0.1076	0.0620	0.0344	1.2829	0.6858	0.0388	0.0410	0.0220	0.0223
			a	0.2124	0.2456	0.0997	0.0585	1.7481	0.8647	0.0508	0.0515	0.0271	0.0272
			b	0.1415	0.1620	0.0541	0.0348	1.5065	0.6997	0.0453	0.0502	0.0223	0.0220
3	0.7	50	θ	-0.0196	0.1396	0.0837	0.1358	1.4633	1.4076	0.0454	0.0475	0.0454	0.0460
			a	0.0247	0.1536	0.0202	0.1383	1.5338	1.4800	0.0471	0.0470	0.0459	0.0456
			b	0.0114	0.0106	0.0403	0.0101	0.4019	0.3602	0.0134	0.0134	0.0114	0.0113
		100	θ	-0.0192	0.1342	0.0369	0.0372	1.4158	0.7428	0.0425	0.0453	0.0232	0.0229
			a	0.0246	0.1424	0.0201	0.0479	1.4903	0.7496	0.0416	0.0468	0.0240	0.0238
			b	0.0188	0.0103	0.0262	0.0050	0.3918	0.2567	0.0123	0.0123	0.0079	0.0078
		200	θ	-0.0090	0.1214	0.0307	0.0179	1.3815	0.5112	0.0416	0.0426	0.0157	0.0157
			a	0.0333	0.1393	0.0166	0.0189	1.4719	0.4730	0.0414	0.0446	0.0157	0.0156
			b	0.0100	0.0068	0.0133	0.0032	0.3200	0.2173	0.0102	0.0101	0.0068	0.0068
	1.5	50	θ	0.1300	0.6261	0.1029	0.1237	3.0611	1.3188	0.0956	0.0994	0.0428	0.0420
			a	0.1618	0.8031	0.1905	0.1454	3.4570	1.2955	0.1128	0.1060	0.0402	0.0403
			b	0.0182	0.0429	0.0696	0.0268	0.8091	0.5809	0.0246	0.0263	0.0182	0.0181
		100	θ	0.0751	0.6008	0.0406	0.0336	3.0257	0.7011	0.0949	0.0910	0.0210	0.0212
			a	0.0915	0.7173	0.0966	0.0409	3.3022	0.6972	0.1064	0.0911	0.0216	0.0216
			b	0.0032	0.0415	0.0381	0.0130	0.7834	0.4214	0.0236	0.0240	0.0132	0.0132
		200	θ	0.1084	0.4652	0.0402	0.0182	3.0138	0.5038	0.0927	0.0906	0.0162	0.0163
			a	0.1499	0.6806	0.0703	0.0207	3.2471	0.4926	0.1050	0.0901	0.0154	0.0158
			b	0.0136	0.0402	0.0258	0.0098	0.7251	0.3749	0.0234	0.0235	0.0122	0.0123

Table 5. The MLEs and Bayesian estimation results for the parameters of the WLLE distribution with $\theta = 3$.

$\theta = 3$				MLE		Bayes		CI		MLE		Bayes	
a	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
1.8	0.5	50	θ	0.3935	0.4634	0.0820	0.1382	1.9414	1.4222	0.2284	0.2318	0.0435	0.0433
			a	0.6590	0.5726	0.2051	0.2009	2.8962	1.5632	0.3669	0.3664	0.0494	0.0493
			b	0.0205	0.0097	0.0302	0.0058	0.1815	0.2743	0.0291	0.0299	0.0087	0.0088
		100	θ	0.3777	0.4557	0.0281	0.0410	1.8351	0.7866	0.2261	0.2245	0.0258	0.0261
			a	0.6274	0.4971	0.1196	0.0580	2.7871	0.8200	0.3162	0.3174	0.0264	0.0263
			b	0.0143	0.0096	0.0196	0.0028	0.1535	0.1915	0.0243	0.0267	0.0062	0.0061
		200	θ	0.3756	0.4016	0.0278	0.0203	1.4618	0.5479	0.1825	0.1837	0.0175	0.0175
			a	0.5988	0.2523	0.0812	0.0274	1.9475	0.5652	0.2582	0.2590	0.0178	0.0177
			b	0.0012	0.0092	0.0129	0.0019	0.1503	0.1652	0.0184	0.0182	0.0054	0.0054

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Table 5 – continued

$\theta = 3$				MLE		Bayes		CI		MLE		Bayes	
a	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
	2.2	50	θ	0.5661	0.9467	0.1145	0.1250	2.7464	1.3121	0.2540	0.2574	0.0404	0.0405
			a	1.0627	1.0391	0.1864	0.1430	3.5040	1.2906	0.3112	0.3184	0.0413	0.0413
			b	0.2382	0.3661	0.0737	0.0400	1.0744	0.7294	0.1030	0.1029	0.0235	0.0234
		100	θ	0.5628	0.4636	0.0606	0.0346	1.7992	0.6899	0.2339	0.2306	0.0226	0.0220
			a	1.0521	0.6222	0.1043	0.0486	2.0783	0.7619	0.2687	0.2679	0.0243	0.0239
			b	0.2249	0.2025	0.0383	0.0161	0.7317	0.4752	0.0909	0.0942	0.0147	0.0145
		200	θ	0.5422	0.3802	0.0366	0.0198	1.3873	0.5322	0.2139	0.2117	0.0175	0.0177
			a	1.0209	0.6057	0.0645	0.0225	1.7728	0.5304	0.2574	0.2574	0.0176	0.0176
			b	0.2028	0.2017	0.0289	0.0119	0.6724	0.4118	0.0911	0.0911	0.0128	0.0127
3.5	0.5	50	θ	0.1049	0.3781	0.0815	0.1415	1.7923	1.4405	0.1737	0.1706	0.0474	0.0472
			a	0.1871	0.3925	0.1199	0.0624	2.3037	0.8591	0.2106	0.2057	0.0281	0.0281
			b	0.0179	0.0046	0.0226	0.0032	0.1231	0.2030	0.0115	0.0118	0.0061	0.0062
		100	θ	0.0368	0.1484	0.0305	0.0365	1.0676	0.7395	0.1392	0.1418	0.0239	0.0238
			a	0.1803	0.4117	0.0655	0.0181	1.8791	0.4612	0.2041	0.1925	0.0146	0.0146
			b	0.0167	0.0025	0.0133	0.0017	0.0825	0.1548	0.0103	0.0104	0.0047	0.0047
		200	θ	0.0351	0.0912	0.0274	0.0192	0.7096	0.5321	0.1158	0.1180	0.0167	0.0167
			a	0.1693	0.1886	0.0378	0.0081	1.0690	0.3207	0.1759	0.1763	0.0106	0.0107
			b	0.0155	0.0022	0.0105	0.0015	0.0718	0.1453	0.0101	0.0101	0.0046	0.0046
	2.2	50	θ	0.5476	0.6740	0.1052	0.1154	2.2906	1.2669	0.2218	0.2247	0.0409	0.0413
			a	0.9225	0.9162	0.1118	0.0626	3.3276	0.8782	0.3159	0.3131	0.0285	0.0286
			b	0.1133	0.1691	0.0520	0.0292	0.7456	0.6387	0.0699	0.0701	0.0200	0.0198
		100	θ	0.4068	0.3218	0.0596	0.0358	1.5217	0.7047	0.1945	0.1952	0.0226	0.0221
			a	0.8754	0.4165	0.0525	0.0160	2.5285	0.4519	0.3019	0.3026	0.0142	0.0141
			b	0.1045	0.1383	0.0255	0.0140	0.5901	0.4534	0.0631	0.0673	0.0144	0.0143
		200	θ	0.3807	0.2944	0.0373	0.0190	1.2664	0.5208	0.1951	0.1820	0.0168	0.0170
			a	0.8239	0.3139	0.0389	0.0083	2.1629	0.3226	0.2933	0.2330	0.0101	0.0104
			b	0.0915	0.1143	0.0173	0.0109	0.5002	0.4040	0.0608	0.0657	0.0124	0.0124

Table 6. The MLEs and Bayesian estimation results for the parameters of the WLLE distribution with $\theta = 0.5$.

$\theta = 0.5$				MLE		Bayes		CI		MLE		Bayes	
a	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
		50	θ	0.0740	0.1754	0.0769	0.0507	0.8761	0.8297	0.0528	0.0578	0.0248	0.0249
			a	0.0769	0.0957	0.1096	0.0804	1.0087	1.0469	0.0409	0.0407	0.0339	0.0338
			b	0.0327	0.0969	0.0639	0.0595	1.0234	0.9232	0.0338	0.0342	0.0280	0.0278
	0.5	100	θ	0.0694	0.1371	0.0672	0.0452	0.8043	0.7909	0.0430	0.0537	0.0238	0.0240

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Table 6 – continued

$\theta = 0.5$				MLE		Bayes		CI		MLE		Bayes	
a	b	n		Bias	MSE	Bias	MSE	LACI	LCCI	LBCIP	LBCIT	LBCIP	LBCIT
			a	0.0690	0.0848	0.1046	0.0706	1.0011	0.9579	0.0342	0.0339	0.0300	0.0300
			b	0.0310	0.0822	0.0626	0.0431	1.0118	0.7760	0.0336	0.0336	0.0246	0.0248
		200	θ	0.0676	0.0511	0.0665	0.0378	0.7846	0.7168	0.0259	0.0254	0.0230	0.0229
			a	0.0573	0.0786	0.1030	0.0709	0.9506	0.9518	0.0342	0.0339	0.0304	0.0302
			b	0.0291	0.0804	0.0525	0.0336	0.9811	0.6889	0.0314	0.0344	0.0218	0.0217
	2.2	50	θ	0.0472	0.0473	0.0645	0.0249	0.8326	0.5653	0.0280	0.0279	0.0185	0.0185
			a	0.1267	0.1276	0.1331	0.0837	1.3098	1.0072	0.0421	0.0424	0.0309	0.0308
			b	0.1204	0.5029	0.0721	0.0954	2.7410	1.1778	0.0881	0.0844	0.0358	0.0359
		100	θ	0.0416	0.0328	0.0512	0.0168	0.6909	0.4666	0.0208	0.0208	0.0145	0.0145
			a	0.1174	0.1150	0.1019	0.0511	1.2447	0.7911	0.0406	0.0416	0.0254	0.0253
			b	0.0972	0.4094	0.0411	0.0316	2.5773	0.6786	0.0865	0.0837	0.0220	0.0220
		200	θ	0.0344	0.0302	0.0476	0.0148	0.6683	0.4395	0.0201	0.0206	0.0130	0.0129
			a	0.1023	0.1032	0.1004	0.0502	1.1339	0.7928	0.0404	0.0404	0.0257	0.0257
			b	0.0912	0.4102	0.0268	0.0151	2.4685	0.4700	0.0776	0.0775	0.0147	0.0147
3	0.5	50	θ	0.0848	0.1509	0.0746	0.0733	1.1577	1.0204	0.0537	0.0567	0.0323	0.0318
			a	0.1082	0.3915	0.1317	0.0887	2.4169	1.0477	0.0978	0.0756	0.0318	0.0318
			b	0.0242	0.0333	0.0717	0.0405	0.7097	0.7370	0.0237	0.0236	0.0228	0.0228
		100	θ	0.0811	0.1417	0.0729	0.0554	1.1142	0.8692	0.0426	0.0450	0.0274	0.0274
			a	0.1026	0.2569	0.0709	0.0235	2.3917	0.5327	0.0909	0.0903	0.0159	0.0161
			b	0.0114	0.0318	0.0461	0.0240	0.6981	0.5798	0.0220	0.0223	0.0185	0.0185
		200	θ	0.0807	0.0795	0.0605	0.0409	1.0521	0.7571	0.0348	0.0346	0.0233	0.0235
			a	0.0795	0.2335	0.0384	0.0096	2.2499	0.3530	0.0733	0.0741	0.0110	0.0110
			b	0.0106	0.0287	0.0450	0.0197	0.6620	0.5139	0.0211	0.0213	0.0165	0.0168
	2.2	50	θ	0.1424	0.2419	0.0747	0.0343	1.6004	0.6652	0.0600	0.0648	0.0211	0.0208
			a	0.0982	1.1187	0.1166	0.0779	4.1930	0.9946	0.1359	0.1357	0.0323	0.0320
			b	-0.0390	0.3823	0.0292	0.0934	2.4200	1.1930	0.0750	0.0743	0.0387	0.0382
		100	θ	0.1259	0.2255	0.0562	0.0193	1.5441	0.4975	0.0571	0.0582	0.0161	0.0160
			a	0.0835	1.1036	0.0718	0.0268	4.1667	0.5768	0.1340	0.1371	0.0181	0.0181
			b	-0.0367	0.3588	0.0058	0.0400	2.4291	0.7836	0.0746	0.0745	0.0251	0.0250
		200	θ	0.1144	0.2146	0.0430	0.0125	1.2673	0.4047	0.0525	0.0521	0.0132	0.0133
			a	0.0730	0.7064	0.0459	0.0118	3.2840	0.3855	0.1047	0.1019	0.0123	0.0124
			b	-0.0345	0.3598	0.0030	0.0202	2.3459	0.5567	0.0737	0.0736	0.0176	0.0173

7. Applications

Real data sets are used in this section to demonstrate the adaptability of the WLL-G family. We compare the performance of the weighted logarithmic Lomax exponential distribution (WLLE) with a wide range of distributions, such as the exponential distribution (E), the Weibull distribution (W), the power modified kies exponential distribution (PMKE) [32], the modified Kies exponential distribution (MKE) [33], the alpha power exponentiated exponential distribution (APExE) [34], the Frechet Weibull mixture exponential distribution (FWME) [35], the linear exponential distribution (LE) [36], the Marshall Olkin alpha power exponential distribution (MOAPE) [37], the Marshall Olkin exponential distribution (MOE) [2], the Marshall Olkin logisitic exponential distribution (MOLE) [38], the Nadarajah-Haghighi exponential distribution (NHE) [39], the odd exponentiated half logistic exponential distribution (OExHLE) [40], the transmuted exponential distribution (TE) [36], the transmuted generalized exponential distribution (TGE) [41], the generalized gamma distribution (GG) [42], and the exponentiated Weibull distribution (ExW) [43].

To identify the best-fitting model for the real data sets, all models were compared using some well-known information criteria, such as the Akaike information criterion (AIC), the corrected Akaike information (CAIC), the Bayesian information criterion (BIC), and the Hannan-Quinn information criteria (HQIC), along with some goodness-of-fit measures, such as the Anderson-Darling test (A^*), the Cramér-von Mises criterion (W^*), and the Kolmogorov-Smirnov test (KST) with its p-value (p-KST). It is worth noting that, in general, the model that attains the lowest value of these measures is considered superior to the others. In both data sets, the computations were carried out using Mathematica program.

7.1. **The mechanical component failure data set.** This data set consists of twenty-four observations on the times to failure of mechanical components, which were introduced by Murthy et al. [44]. The data set is:

$$30.94, 18.51, 16.62, 51.56, 22.85, 22.38, 19.08, 49.56, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93,$$
$$27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84.$$

The performance of the WLLE distribution compared to the other models for this data set is summarized in Table 7. It is evident from the first row that the WLLE distribution attains the lowest values, and it more closely matches this data set than the other competing models. Figure 7 demonstrates the fitted density function, the cdf function, the survival function, and the p-p plots of the WLLE distribution for the mechanical component failure data set. The estimates' maximality is shown in Figure 5. The existence and uniqueness of the estimates are shown in Figure 6. For a level of significance ($\theta = 0.05$), we conducted the likelihood ratio test (LRT) when $H_0 : a = 1$ and deduced that RLT $= 6.4366$ and its p-value $= 0.01117 < \theta$, so we reject H_0.

7.2. **The petroleum rock samples data set.** This data set consists of forty-eight measurements on petroleum rock samples from a petroleum reservoir. It was introduced and studied by Cordeiro and Brito [45]. The data set values:

$$0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270,$$

$0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250,$

$0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520,$

$0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510,$

$0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650,$

$0.2760160, 0.2538320, 0.2004470$

The performance of the WLLE distribution compared to the other models for this data set is summarized in Table 8. It is evident from the first row that the WLLE distribution attains the lowest values, indicating that it more closely matches this data set than the other competing models. Figure 10 demonstrates the fitted density function, the cdf function, the survival function, and the p-p plots of the WLLE distribution for the petroleum rock samples data set. The maximality of the estimates is shown in Figure 8. The existence and uniqueness of the estimations are shown in Figure 9. For a level of significance ($\theta = 0.05$), we conducted the likelihood ratio test (LRT) when $H_0 : a = 1$ and deduced that RLT $= 18.2918$ and its p-value $= 1.895212 \times 10^{-5} < \theta$, so we reject H_0.

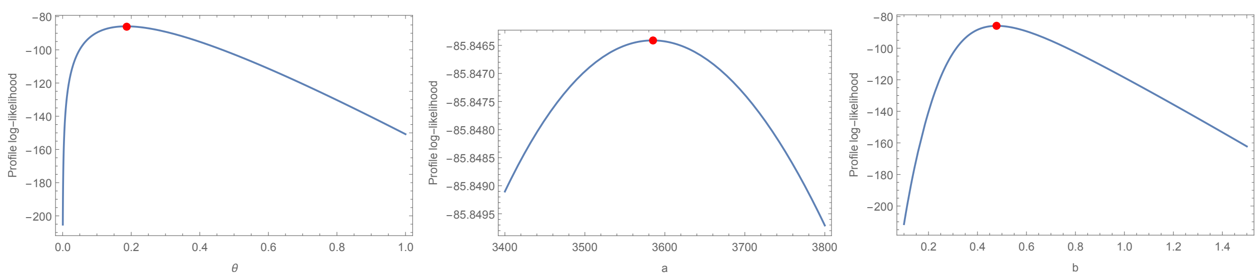

Figure 5. Plots of the profile-likelihood functions of the three estimators for the mechanical component failure data set.

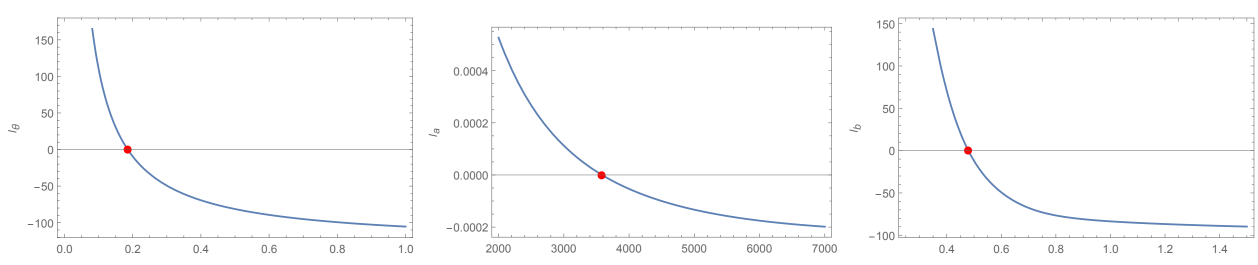

Figure 6. Plots of the existence and uniqueness of the model estimators for the mechanical component failure data set.

TABLE 7. The numerical analysis for the mechanical component failure data set.

Distribution	Estimates	SEs	$-\ell$	AIC	CAIC	BIC	HQIC	A^*	W^*	KST	p-KST
WLLE	$\hat{\theta} = 0.185505$	0.147436	85.8464	177.693	178.893	181.227	178.63	0.150504	0.0191032	0.076196	0.999041
	$\hat{a} = 3584.76$	14444.3									
	$\hat{b} = 0.47921$	0.325385									
E	$\hat{a} = 0.0435305$	0.00888562	99.2231	200.446	200.628	201.624	200.759	3.9769	0.793261	0.359658	0.00402261
W	$\hat{a} = 2.30889$	0.337721	88.891	181.782	182.353	184.138	182.407	0.746669	0.108212	0.143629	0.705176
	$\hat{b} = 26.023$	2.44568									
PMKE	$\hat{\alpha} = 2075.79$	6454.51	88.8923	183.785	184.985	187.319	184.722	0.746978	0.108305	0.143447	0.706659
	$\hat{\lambda} = 0.691339$	0.0056174									
	$\hat{\beta} = 0.000801562$	0.00249269									
MKE	$\hat{\alpha} = 1.6242$	0.252326	90.5153	185.031	185.602	187.387	185.656	1.13581	0.182107	0.17337	0.466326
	$\hat{\lambda} = 0.0254436$	0.00237072									
APExE	$\hat{\alpha} = 0.581324$	0.19262	86.1305	178.261	179.461	181.795	179.199	0.232274	0.0339012	0.116321	0.901508
	$\hat{a} = 0.111696$	0.0190489									
	$\hat{c} = 8.36397$	3.39403									
FWME	$\hat{\alpha} = 1.44161$	0.744237	86.354	180.708	182.813	185.42	181.958	0.180829	0.0234079	0.102053	0.963975
	$\hat{\lambda} = 0.00795722$	1.75356×10^{-7}									
	$\hat{k} = 2.75836$	1.31559									
	$\hat{a} = 2.66724$	1.42401									
LE	$\hat{\beta} = 7.2592 \times 10^{-8}$	0.0314783	89.3342	182.668	183.24	185.024	183.293	0.858101	0.121927	0.151431	0.640887
	$\hat{\theta} = 0.00313193$	0.00174843									
MOAPE	$\hat{\alpha} = 517884$	2.18123×10^{6}	85.9646	177.929	179.129	181.463	178.867	0.18635	0.0222066	0.0977129	0.975965
	$\hat{\lambda} = 0.100161$	0.0478262									
	$\hat{\beta} = 0.224233$	0.40913									
MOE	$\hat{\alpha} = 36.7089$	29.6374	88.9502	181.9	182.472	184.257	182.525	0.588227	0.0627562	0.129532	0.815473
	$\hat{a} = 0.169381$	0.0342867									
MOLE	$\hat{\alpha} = 0.00662727$	0.00115351	89.5068	185.014	186.214	188.548	185.951	0.648135	0.0650405	0.135175	0.772849
	$\hat{\lambda} = 27.4436$	0.0139343									
	$\hat{\beta} = 49.7437$	36.4584									
NHE	$\hat{\alpha} = 9533.42$	138721	94.2489	192.498	193.069	194.854	193.123	2.65633	0.507568	0.299526	0.0269646
	$\hat{\lambda} = 3.11961 \times 10^{-6}$	0.000045396									
OExHLE	$\hat{\alpha} = 4.5031$	1.97577	86.9432	179.886	181.086	183.421	180.824	0.418516	0.0572897	0.146009	0.685648
	$\hat{\lambda} = 0.009436$	0.00650988									
	$\hat{\beta} = 11.335$	9.72874									
TE	$\hat{\beta} = 0.063872$	0.0167808	92.6738	189.348	189.919	191.704	189.973	1.90265	0.325461	0.230461	0.156183
	$\hat{\lambda} = -1.000$	1.11309									
TGE	$\hat{\alpha} = 9.99995$	4.87178	85.9946	177.989	179.189	181.523	178.927	0.216325	0.0268033	0.112979	0.919278
	$\hat{\lambda} = 0.390577$	0.634892									
	$\hat{\beta} = 0.117658$	0.0351094									
GG	$\hat{\alpha} = 4.68377 \times 10^{-6}$	0.000204	86.2315	178.463	179.663	181.997	179.401	0.260788	0.0356138	0.122867	0.861723
	$\hat{\lambda} = 20.2915$	39.7091									
	$\hat{\beta} = 0.279368$	0.540716									
ExW	$\hat{a} = 0.432817$	0.489501	85.6253	177.851	179.051	181.385	178.788	0.152113	0.0191866	0.0837177	0.996011
	$\hat{b} = 0.265299$	1.98979									
	$\hat{c} = 473.966$	3452.81									

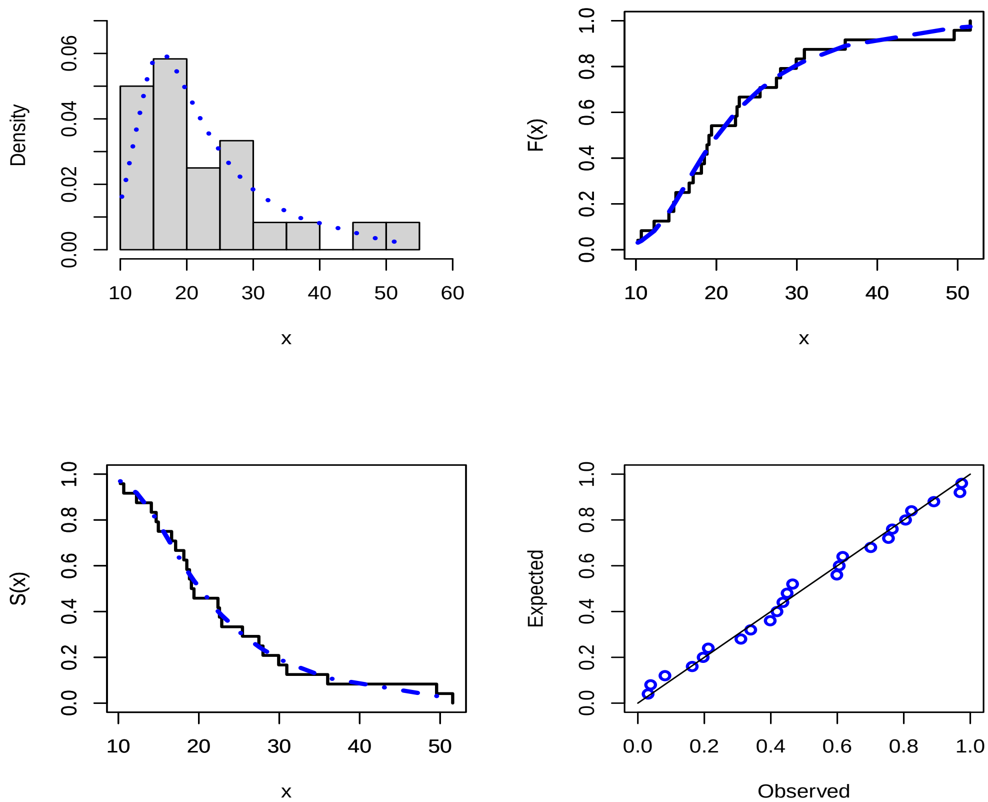

Figure 7. The density function, the cumulative function, the survival function, and the P–P plots of the fitted WLLE distribution for the mechanical component failure data set.

Table 8. The numerical analysis for the petroleum rock samples data set.

Distribution	Estimates	SEs	$-\ell$	AIC	CAIC	BIC	HQIC	A^*	W^*	KST	p-KST
WLLE	$\hat{\theta} = 0.22088$	0.100306	-58.5437	-111.087	-110.542	-105.474	-108.966	0.134087	0.018714	0.063142	0.990913
	$\hat{a} = 12976.4$	35662.7									
	$\hat{b} = 51.9073$	19.6312									
E	$\hat{a} = 4.58483$	0.661763	-25.0922	-48.1844	-48.0974	-46.3132	-47.4772	9.57089	1.95351	0.385829	< 0.000001
W	$\hat{a} = 2.74756$	0.28443	-52.7422	-101.484	-101.218	-97.742	-100.07	1.22945	0.19453	0.1499	0.230955
	$\hat{b} = 0.245226$	0.0136883									
PMKE	$\hat{\alpha} = 2142.21$	6363.99	-52.7397	-99.4794	-98.934	-93.8658	-97.358	1.23024	0.194657	0.149843	0.231332
	$\hat{\lambda} = 0.694049$	0.0026796									
	$\hat{\beta} = 0.0009246$	0.00274721									
MKE	$\hat{\alpha} = 1.93697$	0.210055	-50.0254	-96.0508	-95.7841	-92.3084	-94.6365	1.79581	0.297547	0.167197	0.13658
	$\hat{\lambda} = 2.74208$	0.153152									

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Table 8 – continued from previous page

Distribution	Estimates	SEs	$-\ell$	AIC	CAIC	BIC	HQIC	A^*	W^*	KST	p-KST
APExE	$\hat{\alpha} = 0.269684$	0.521133	-58.3425	-110.685	-110.14	-105.071	-108.564	0.205116	0.0318898	0.0844882	0.883065
	$\hat{a} = 13.601$	4.25451									
	$\hat{c} = 15.7526$	6.93735									
FWME	$\hat{\alpha} = 2.54762$	7966.93	-57.471	-106.942	-106.012	-99.4572	-104.114	0.287488	0.043698	0.0876736	0.854308
	$\hat{\lambda} = 0.166541$	1763.74									
	$\hat{k} = 152.804$	1.94265									
	$\hat{a} = 0.388413$	6075.07									
LE	$\hat{\beta} = 0.100007$	2.96175	-48.4618	-92.9237	-92.657	-89.1813	-91.5094	2.77415	0.466642	0.221837	0.0177536
	$\hat{\theta} = 36.247$	16.2282									
MOAPE	$\hat{\alpha} = 1.63023 \times 10^8$	9.7135×10^8	-58.3712	-110.742	-110.197	-105.129	-108.621	0.192776	0.0291426	0.0767008	0.940256
	$\hat{\lambda} = 12.8762$	4.59289									
	$\hat{\beta} = 0.30854$	0.431933									
MOE	$\hat{\alpha} = 89.9586$	57.1628	-53.3601	-102.72	-102.453	-98.9777	-101.306	0.900118	0.111997	0.122975	0.462291
	$\hat{a} = 21.7117$	2.88208									
MOLE	$\hat{\alpha} = 4.60224$	0.643784	-57.3763	-108.753	-108.207	-103.139	-106.631	0.309182	0.0458472	0.089911	0.832557
	$\hat{\lambda} = 0.641054$	0.862477									
	$\hat{\beta} = 0.00011083$	0.000816									
NHE	$\hat{\alpha} = 20306$	226896	-37.1066	-70.2132	-69.9465	-66.4708	-68.7989	7.51402	1.55615	0.352775	0.0000129541
	$\hat{\lambda} = 0.000162$	0.00181									
OExHLE	$\hat{\alpha} = 10.1147$	3.46322	-57.9614	-109.923	-109.377	-104.309	-107.801	0.302812	0.0488107	0.101253	0.708739
	$\hat{\lambda} = 0.00167245$	0.0125775									
	$\hat{\beta} = 9897.04$	74456.4									
TE	$\hat{\beta} = 2.29242$	0.206029	-25.0922	-46.1844	-45.9177	-42.442	-44.7701	9.5709	1.95352	0.385829	< 0.00001
	$\hat{\lambda} = 1.000$	0.374615									
TGE	$\hat{\alpha} = 16.5758$	6.82908	-58.2336	-110.467	-109.922	-104.854	-108.346	0.23584	0.0374279	0.0914933	0.816499
	$\hat{\lambda} = 0.325214$	0.56318									
	$\hat{\beta} = 14.735$	3.31544									
GG	$\hat{\alpha} = 0.0150166$	0.0275248	-56.9177	-107.835	-107.29	-102.222	-105.714	0.501308	0.0833359	0.123464	0.457178
	$\hat{\lambda} = 8.85294$	3.39489									
	$\hat{\beta} = 0.870093$	0.323977									
ExW	$\hat{a} = 1.10822$	0.298289	-57.8958	177.851	178.396	183.464	179.972	0.310538	0.0515361	0.105163	0.663262
	$\hat{b} = 0.0793379$	0.0413269									
	$\hat{c} = 11.7005$	9.97391									

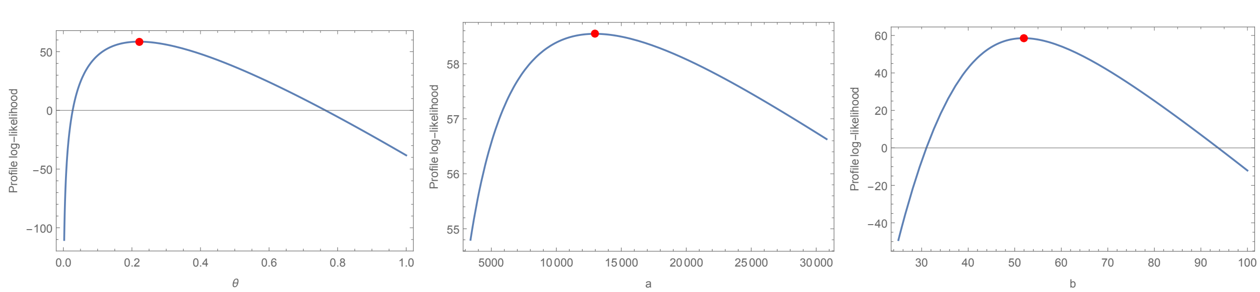

FIGURE 8. Plots of the profile-likelihood functions of the three estimators for the petroleum rock samples data set.

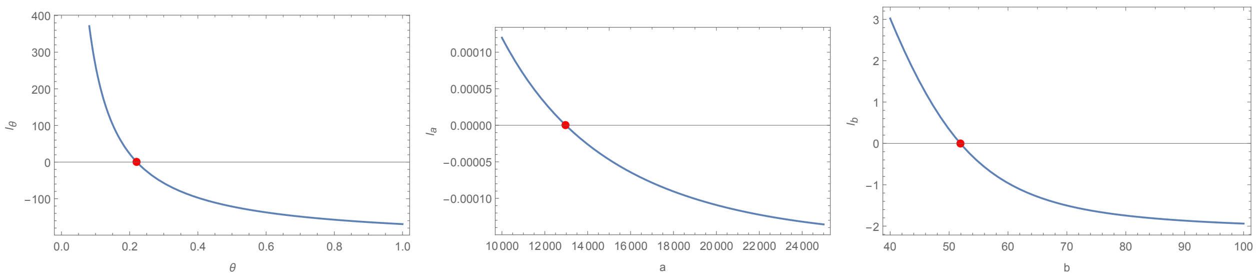

FIGURE 9. Plots of the existence and uniqueness of the model estimators for the petroleum rock samples data set.

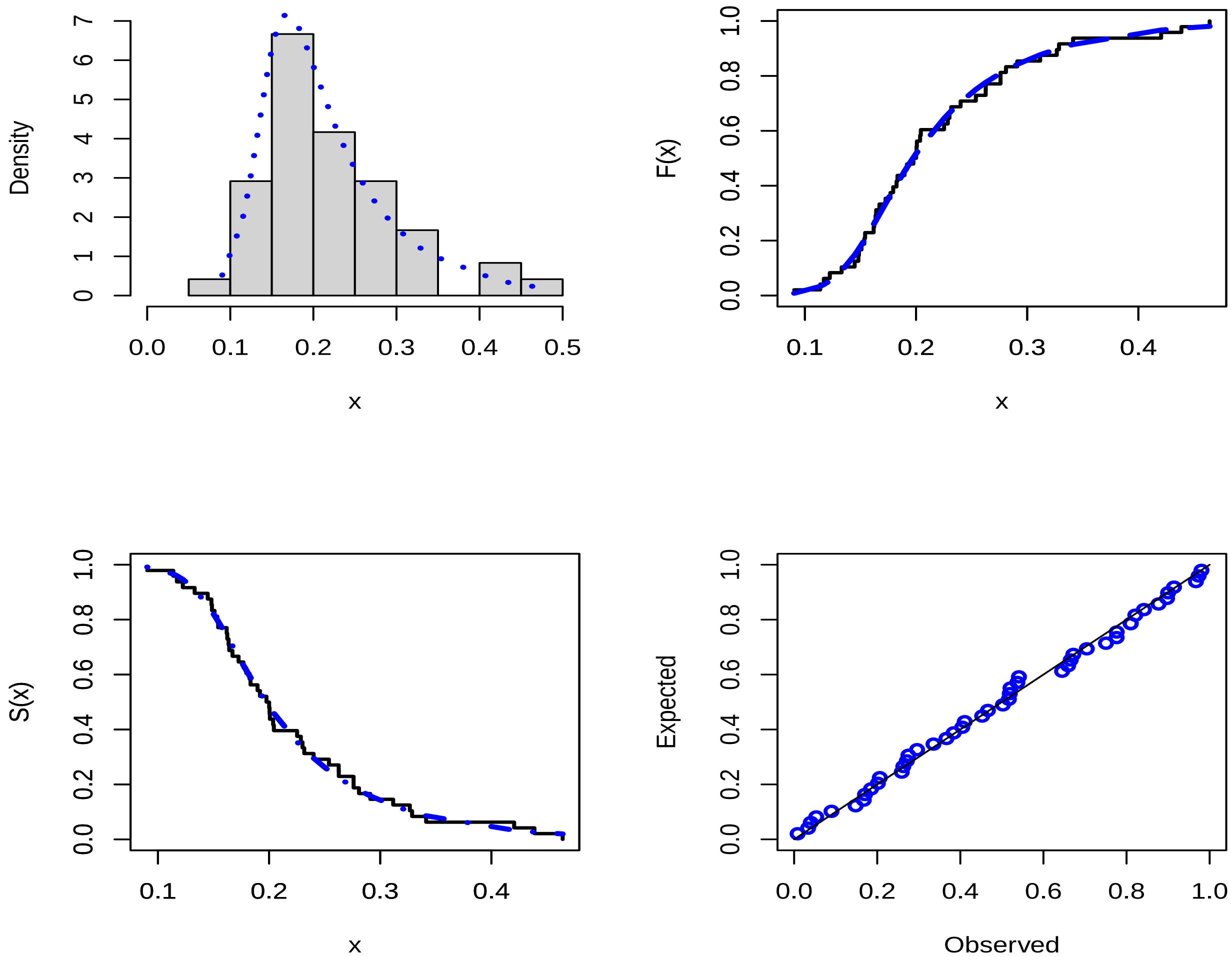

FIGURE 10. The density function, the cumulative function, the survival function, and the P–P plots of the fitted WLLE distribution for the petroleum rock samples data set.

8. Conclusions

In this paper, a new linking function is introduced and utilized to generate a new family of continuous distributions, called the weighted logarithmic Lomax-G family, or the WLL-G family for short, which can be applied in the $T-X$ framework. Three sub-models of this family are given, along with their various statistical functions and plots. Many statistical and mathematical properties of this family were derived, such as the quantile function, series representations of the pdf and cdf, the moments and their generating function, incomplete moments, entropies, order statistics, and parameter estimation. In particular, we provide a novel closed-form formula for the quantile function via the special r-Lambert function, where this new approach can be readily implemented in other statistical distributions. We give a simulation study which uses a Monte Carlo algorithm to estimate the parameters of the weighted logarithmic Lomax exponential distribution using the MLEs and Bayesian methods. The weighted logarithmic Lomax exponential distribution was fitted to real data sets. This distribution outperformed many other known models for these data sets.

Data availability. The corresponding author can provide the data sets created and/or studied during the current work upon reasonable request.

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