Asia Pac. J. Math. 2026 **13**:141

A UNIFIED THEORY OF RARE CONTINUITY FOR MULTIFUNCTIONS

MONTRI THONGMOON[1], AREEYUTH SAMA-AE[2], CHAWALIT BOONPOK[1,*]

[1]Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand

[*]Corresponding author: chawalit.b@msu.ac.th

Received Jun. 15, 2026

Abstract. This paper is concerned with the concepts of upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are established.

2020 Mathematics Subject Classification. 54C08; 54C60.

Key words and phrases. upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunction; lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunction.

1. Introduction

The concept of rare continuity was first introduced by Popa [23]. Long and Herrington [20], and Jafari [15,16] investigated several properties of rarely continuous functions. Jafari [14] further generalized the notion of rare continuity to rare β-continuity by employing the concept of β-open sets. Caldas [3] introduced a new class of functions, namely rarely $\beta\theta$-continuous functions, by utilizing the notion of β-θ-open sets, and investigated several characterizations of such functions. Jafari [13] introduced and studied the concept of rare α-continuity as a generalization of both rare continuity and weak α-continuity [21]. Thongmoon et al. [28] introduced and investigated the concept of rarely (τ_1, τ_2)-continuous functions. Caldas and Jafari [4] introduced and investigated a new class of functions called rarely g-continuous functions, which is a generalization of both the class of rarely continuous functions and the class of weakly g-continuous functions. Moreover, Caldas et al. [5] introduced and studied the notion of rarely g-continuous multifunctions, which is a generalization of weakly continuous multifunctions [24]. Popa and Noiri [22] introduced and studied s-precontinuous multifunctions, which generalize both s-continuous multifunctions and precontinuous multifunctions. Ekici and Park [12] introduced

DOI: 10.28924/APJM/13-141

and investigated weakly s-precontinuous multifunctions. The notion of weakly s-precontinuous multifunctions is a generalization of s-precontinuous multifunctions introduced by Popa and Noiri [22]. Ekici and Jafari [11] introduced and studied rarely s-precontinuous multifunctions, a generalization of weakly s-precontinuous multifunctions introduced by Ekici and Park [12]. On the other hand, the present authors introduced and investigated the notions of (τ_1, τ_2)-continuous multifunctions [27], almost (τ_1, τ_2)-continuous multifunctions [17], weakly (τ_1, τ_2)-continuous multifunctions [29], almost weakly (τ_1, τ_2)-continuous multifunctions [1], $(\tau_1, \tau_2)\mu$-continuous multifunctions [8], almost $(\tau_1, \tau_2)\mu$-continuous multifunctions [25], weakly $(\tau_1, \tau_2)\mu$-continuous multifunctions [6], almost weakly $(\tau_1, \tau_2)\mu$-continuous multifunctions [18], s-(τ_1, τ_2)-continuous multifunctions [7], weakly s-(τ_1, τ_2)-continuous multifunctions [26] and s-$(\tau_1, \tau_2)p$-continuous multifunctions [30]. Recently, Kong-ied at al. [19] introduced and studied the concepts of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions. In this paper, we introduce the notions of upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [2] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [2] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [2] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [2] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-*Cl*(A) *and* $\tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Cl*$(A)) = \tau_1\tau_2$-*Cl*(A).
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$-*Cl*$(A) \subseteq \tau_1\tau_2$-*Cl*$(B)$.
(3) $\tau_1\tau_2$-*Cl*(A) *is* $\tau_1\tau_2$-*closed.*
(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-*Cl*(A).
(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [31] if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$. A subset R of a bitopological space (X, τ_1, τ_2) is called a *$\tau_1\tau_2$-rare set* if $\tau_1\tau_2$-Int$(R) = \emptyset$.

Lemma 2. [28] *Let (X, τ_1, τ_2) be a bitopological space. Then, $\tau_1\tau_2$-Int$(F \cup R) \subseteq F$ for every $\tau_1\tau_2$-rare set R and every $\tau_1\tau_2$-closed set F.*

Let $\mathcal{P}(X)$ denotes the power set of a nonempty set X. A class $\mu \subseteq \mathcal{P}(X)$ is called a *generalized topology* (briefly, GT) if $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ [10]. A set X with a generalized topology μ on X is called a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called *μ-open sets* and the complements of μ-open sets are called *μ-closed sets*. For $A \subseteq X$, the μ-closure of A [9], $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the μ-interior of A [9], $i_\mu(A)$, is the union of all μ-open sets contained in A. Then, we have $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(A) = X - c_\mu(X - A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce new classes of continuous multifunctions defined from a generalized topological space into a bitopological space, called upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, some characterizations of upper rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower rarely $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are investigated.

Definition 1. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper rarely $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $R_V \cap V = \emptyset$ and a μ-open set U of X containing x such that $F(U) \subseteq V \cup R_V$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper rarely $\mu(\sigma_1, \sigma_2)$-continuous if F is upper rarely $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$;*

(2) *for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(F^+(V \cup R_V))$;*

(3) *for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(V) \cap R_V = \emptyset$ such that $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V))$;*

(4) *for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(F^+(V \cup R_V))$;*

(5) *for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a μ-open set U of X containing x such that*

$$\sigma_1\sigma_2\text{-}Int(F(U) \cap (Y - V)) = \emptyset;$$

(6) *for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a μ-open set U of X containing x such that $\sigma_1\sigma_2$-$Int(F(U)) \subseteq \sigma_1\sigma_2$-$Cl(V)$.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since F is upper rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ and a μ-open set U of X containing x such that $F(U) \subseteq V \cup R_V$. Thus, $x \in U \subseteq F^+(V \cup R_V)$ and so $x \in i_\mu(F^+(V \cup R_V))$.

$(2) \Rightarrow (3)$: Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Thus by (2), there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(F^+(V \cup R_V))$. Put $W_V = R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))$. Then, $W_V \cap \sigma_1\sigma_2\text{-Cl}(V) = \emptyset$ and W_V is a $\sigma_1\sigma_2$-rare set. Since

$$\begin{aligned}\sigma_1\sigma_2\text{-Cl}(V) \cup W_V &= \sigma_1\sigma_2\text{-Cl}(V) \cup [R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))] \\ &= \sigma_1\sigma_2\text{-Cl}(V) \cup R_V \\ &\supseteq V \cup R_V,\end{aligned}$$

$x \in i_\mu(F^+(V \cup R_V)) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup W_V))$.

$(3) \Rightarrow (4)$: Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. By (3), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-Cl}(R_V) \cap V = \emptyset$ such that $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V))$. Let $W_V = R_V \cup (\sigma_1\sigma_2\text{-Cl}(V) - V))$. Then by Lemma 2, W_V is a $\sigma_1\sigma_2$-rare set and $W_V \cap V = \emptyset$. Thus, $x \in i_\mu(F^+(V \cup W_V))$.

$(4) \Rightarrow (5)$: Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then,

$$F(x) \subseteq V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$$

and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open. By (4), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(R_V)) \cap V = \emptyset$ and $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V))$. There exists a μ-open set U of X containing x such that $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V)$. Thus, $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V$, by Lemma 2 we have

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(F(U) \cap (Y - V)) &= \sigma_1\sigma_2\text{-Int}(F(U)) \cap \sigma_1\sigma_2\text{-Int}(Y - V) \\ &\subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq (\sigma_1\sigma_2\text{-Cl}(V) \cup \sigma_1\sigma_2\text{-Int}(R_V)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &= \sigma_1\sigma_2\text{-Cl}(V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset\end{aligned}$$

and so $\sigma_1\sigma_2\text{-Int}(F(U) \cap (Y - V)) = \emptyset$.

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then by (5), there exists a μ-open set U of X containing x such that $\sigma_1\sigma_2\text{-Int}(F(U) \cap (Y - V)) = \emptyset$. Thus,

$$\sigma_1\sigma_2\text{-Int}(F(U)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$$

and hence $\sigma_1\sigma_2\text{-Int}(F(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(6) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (6), there exists a μ-open set U of X containing x such that $\sigma_1\sigma_2\text{-Int}(F(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Let $M_V = F(U) \cap (Y - V)$. Then, we have

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(M_V) &\subseteq \sigma_1\sigma_2\text{-Int}(F(U)) \cap \sigma_1\sigma_2\text{-Int}(Y - V)\\ &= \sigma_1\sigma_2\text{-Int}(F(U)) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset.\end{aligned}$$

Therefore, M_V is a $\sigma_1\sigma_2$-rare set and $M_V \cap V = \emptyset$. Let $N_V = \sigma_1\sigma_2\text{-Cl}(V) - V$. Then, N_V is a $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-rare set such that $N_V \cap V = \emptyset$. Thus, $R_V = M_V \cup N_V$ is a $\sigma_1\sigma_2$-rare set and $R_V \cap V = \emptyset$. Then by Lemma 2,

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(R_V) &= \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(R_V))\\ &= \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(M_V \cup N_V))\\ &\subseteq \sigma_1\sigma_2\text{-Int}(N_V) = \emptyset.\end{aligned}$$

Thus,

$$\begin{aligned}F(U) &= [F(U) - \sigma_1\sigma_2\text{-Int}(F(U))] \cup \sigma_1\sigma_2\text{-Int}(F(U))\\ &\subseteq [F(U) - \sigma_1\sigma_2\text{-Int}(F(U))] \cup \sigma_1\sigma_2\text{-Cl}(V)\\ &= [(F(U) \cap (V \cup (Y - V))) - \sigma_1\sigma_2\text{-Int}(F(U))] \cup [(\sigma_1\sigma_2\text{-Cl}(V) - V) \cup V]\\ &= [((F(U) \cap V) \cup (F(U) \cap (Y - V))) - \sigma_1\sigma_2\text{-Int}(F(U))] \cup (N_V \cup V)\\ &\subseteq [V \cup (F(U) \cap (Y - V))] \cup (N_V \cup V)\\ &= V \cup (M_V \cup N_V) = V \cup R_V.\end{aligned}$$

Therefore, there exists a $\sigma_1\sigma_2$-rare set $R_V = M_V \cup N_V$ such that $R_V \cap V = \emptyset$ and $F(U) \subseteq V \cup R_V$. This shows that F is upper rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$. □

Definition 2. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower rarely $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ and a μ-open set U of X containing x such that $F(z) \cap (V \cup R_V) \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower rarely $\mu(\sigma_1, \sigma_2)$-continuous if F is lower rarely $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$;*

(2) *for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(F^-(V \cup R_V))$;*

(3) *for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with*

$$\sigma_1\sigma_2\text{-}Cl(V) \cap R_V = \emptyset$$

such that $x \in i_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(V) \cup R_V))$;

(4) *for every $(\sigma_1, \sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(F^-(V \cup R_V))$.*

Proof. The proofs of $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are similar with Theorem 1.

$(4) \Rightarrow (1)$: Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then,

$$F(x) \cap \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \neq \emptyset.$$

By (4), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cap R_V = \emptyset$ such that

$$x \in i_\mu(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V)).$$

Since $\sigma_1\sigma_2\text{-Cl}(V) - V$ is a $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-rare set, by Lemma 2 we have $(\sigma_1\sigma_2\text{-Cl}(V) - V) \cup R_V$ is a $\sigma_1\sigma_2$-rare set. Therefore, $W_V = (\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) - V) \cup R_V$ is a $\sigma_1\sigma_2$-rare set. Thus,

$$\begin{aligned}\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cup R_V &= V \cup (\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) - V) \cup R_V \\ &= V \cup W_V\end{aligned}$$

and so $x \in i_\mu(F^-(V \cup W_V))$. Then, there exists a μ-open set U of X containing x such that $U \subseteq F^-(V \cup W_V)$ and $F(z) \cap (V \cup W_V) \neq \emptyset$ for every $z \in U$. This shows that F is lower rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$. □

Definition 3. *A function $f : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be rarely $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ and a μ-open set U of X containing x such that $f(U) \subseteq V \cup R_V$. A function $f : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be rarely $\mu(\sigma_1, \sigma_2)$-continuous if f is rarely $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Corollary 1. *For a function $f : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is rarely $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$;*

(2) *for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $V \cap R_V = \emptyset$ such that $x \in i_\mu(f^{-1}(V \cup R_V))$;*

(3) *for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-}Cl(V) \cap R_V = \emptyset$ such that $x \in i_\mu(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V) \cup R_V))$;*

(4) *for each* $(\sigma_1, \sigma_2)r$*- open set* V *of* Y *containing* $f(x)$*, there exists a* $\sigma_1\sigma_2$*-rare set* R_V *with* $V \cap R_V = \emptyset$ *such that* $x \in i_\mu(f^{-1}(V \cup R_V))$*;*

(5) *for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* μ*-open set* U *of* X *containing* x *such that*

$$\sigma_1\sigma_2\text{-}Int(f(U) \cap (Y - V)) = \emptyset;$$

(6) *for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* μ*-open set* U *of* X *containing* x *such that* $\sigma_1\sigma_2\text{-}Int(f(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$*.*

Acknowledgements. This research project was financially supported by Mahasarakham University.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] C. Boonpok, C. Viriyapong, Upper and Lower Almost Weak (τ_1, τ_2)-Continuity, Eur. J. Pure Appl. Math. 14 (2021), 1212–1225. https://doi.org/10.29020/nybg.ejpam.v14i4.4072.

[2] C. Boonpok, C. Viriyapong, M. Thongmoon, On Upper and Lower (τ_1, τ_2)-Precontinuous Multifunctions, J. Math. Comput. Sci. 18 (2018), 282–293. https://doi.org/10.22436/jmcs.018.03.04.

[3] M. Caldas, On Rarely $\beta\theta$-Continuous Functions, Pro Math. 26 (2012), 75–85.

[4] M. Caldas, S. Jafari, On Rarely g-Continuous Functions, Glas. Mat. 40 (2005), 317–322. https://doi.org/10.3336/gm.40.2.11.

[5] M. Caldas, S. Jafari, T. Noiri, Characterizations of Rarely g-Continuous Multifunctions, Sarajevo J. Math. 1 (2005), 129–133.

[6] M. Chiangpradit, A. Sama-Ae, C. Boonpok, Upper and Lower Weak $(\tau_1, \tau_2)\mu$-Continuity, Int. J. Math. Comput. Sci. 21 (2026), 87–93. https://doi.org/10.69793/ijmcs/01.2026/mac.

[7] M. Chiangpradit, A. Sama-Ae, C. Boonpok, s-(τ_1, τ_2)-Continuity for Multifunctions, Eur. J. Pure Appl. Math. 18 (2025), 5634. https://doi.org/10.29020/nybg.ejpam.v18i1.5634.

[8] N. Chutiman, A. Sama-Ae, C. Boonpok, Upper and Lower $(\tau_1, \tau_2)\mu$-Continuity, Int. J. Math. Comput. Sci. 21 (2026), 75–80. https://doi.org/10.69793/ijmcs/01.2026/csb.

[9] Á. Császár, Generalized Open Sets in Generalized Topologies, Acta Math. Hung. 106 (2005), 53–66. https://doi.org/10.1007/s10474-005-0005-5.

[10] Á. Császár, Generalized Topology, Generalized Continuity, Acta Math. Hung. 96 (2002), 351–357. https://doi.org/10.1023/a:1019713018007.

[11] E. Ekici, S. Jafari, On Rare s-Precontinuity for Multifunctions, Demonstr. Math. 46 (2013), 395–403. https://doi.org/10.1515/dema-2013-0451.

[12] E. Ekici, J.H. Park, On Weakly s-Precontinuous Multifunctions, Arab. J. Sci. Eng., Sect. A: Sci. 32 (2007), 83–92.

[13] S. Jafari, Rare α-Continuity, Bull. Malays. Math. Sci. Soc. 28 (2005), 157–161.

[14] S. Jafari, On Rarely β-Continuous Functions, J. Inst. Math. Comput. Sci. Math. Ser. 13 (2000), 247–251.

[15] S. Jafari, On Some Properties of Rare Continuity, Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău 7 (1997), 65–73.

[16] S. Jafari, A Note on Rarely Continuous Functions, Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău 5 (1995), 29–34.

[17] C. Klanarong, S. Sompong, C. Boonpok, Upper and Lower Almost (τ_1, τ_2)-Continuous Multifunctions, Eur. J. Pure Appl. Math. 17 (2024), 1244–1253. https://doi.org/10.29020/nybg.ejpam.v17i2.5192.

[18] B. Kong-ied, A. Sama-Ae, C. Boonpok, Upper and Lower Almost Weak $(\tau_1, \tau_2)\mu$-Continuity, Int. J. Math. Comput. Sci. 21 (2026), 95–101. https://doi.org/10.69793/ijmcs/01.2026/ksb.

[19] B. Kong-ied, S. Sompong, C. Boonpok, Rarely s-$(\tau_1, \tau_2)p$-Continuous Multifunctions, Eur. J. Pure Appl. Math. 18 (2025), 5649. https://doi.org/10.29020/nybg.ejpam.v18i1.5649.

[20] P.E. Long, L.L. Herrington, Properties of Rarely Continuous Functions, Glas. Mat. 17(37) (1982), 147–153.

[21] T. Noiri, Weakly α-Continuous Functions, Int. J. Math. Math. Sci. 10 (1987), 483–490. https://doi.org/10.1155/s0161171287000565.

[22] V. Popa, T. Noiri, On s-Precontinuous Multifunctions, Demonstr. Math. 33 (2000), 679–687. https://doi.org/10.1515/dema-2000-0321.

[23] V. Popa, Sur Certaine Décomposition de la Continuité dans les Espaces Topologiques, Glas. Mat. 14(34) (1979), 359–362.

[24] V. Popa, Weakly Continuous Multifunctions, Boll. Unione Mat. Ital. 15-A (1978), 379–388.

[25] P. Pue-on, A. Sama-Ae, C. Boonpok, Upper and Lower Almost $(\tau_1, \tau_2)\mu$-Continuity, Int. J. Math. Comput. Sci. 21 (2026), 81–86. https://doi.org/10.69793/ijmcs/01.2026/psb.

[26] P. Pue-on, A. Sama-Ae, C. Boonpok, Upper and Lower Weakly s-(τ_1, τ_2)-Continuous Multifunctions, Eur. J. Pure Appl. Math. 18 (2025), 5718. https://doi.org/10.29020/nybg.ejpam.v18i1.5718.

[27] P. Pue-on, S. Sompong, C. Boonpok, Upper and Lower (τ_1, τ_2)-Continuous Multifunctions, Int. J. Math. Comput. Sci. 19 (2024), 1305–1310.

[28] M. Thongmoon, S. Sompong, C. Boonpok, Rarely (τ_1, τ_2)-Continuous Functions, Int. J. Math. Comput. Sci. 20 (2025), 423–427. https://doi.org/10.69793/ijmcs/01.2025/chawalit.

[29] M. Thongmoon, S. Sompong, C. Boonpok, Upper and Lower Weak (τ_1, τ_2)-Continuity, Eur. J. Pure Appl. Math. 17 (2024), 1705–1716. https://doi.org/10.29020/nybg.ejpam.v17i3.5238.

[30] N. Viriyapong, S. Sompong, C. Boonpok, Upper and Lower s-$(\tau_1, \tau_2)p$-Continuous Multifunctions, Eur. J. Pure Appl. Math. 17 (2024), 2210–2220. https://doi.org/10.29020/nybg.ejpam.v17i3.5322.

[31] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-Continuity for Multifunctions, J. Math. 2020 (2020), 6285763. https://doi.org/10.1155/2020/6285763.