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INVESTIGATION OF HYBRID COUPLED SYSTEMS WITH MIXED FRACTIONAL DERIVATIVES IN BANACH SPACES

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Abstract. This work examines the unique existence of solutions for an innovative category of interconnected hybrid fractional differential systems that incorporate mixed fractional derivatives. By merging the Caputo and Riemann–Liouville fractional derivatives of varying orders, the suggested framework offers an adaptable approach to modeling intricate dynamic processes with memory characteristics. To prove that solutions exist, we utilize the Kuratowski measure of noncompactness combined with an extended version of Darbo's fixed-point theorem featuring a nondecreasing control function. Furthermore, the uniqueness of the derived solution is established through Banach's contraction mapping principle. Ultimately, a practical numerical case study is detailed to confirm the utility and performance of our theoretical findings.

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1. Introduction

Over the last few decades, fractional differential equations have emerged as a significant area of academic inquiry. This interest is driven by their exceptional capacity to represent phenomena that traditional integer-order differential equations cannot accurately capture. In contrast to standard derivatives, fractional operators—especially those defined in the Caputo and Riemann–Liouville frameworks—exhibit non-local properties. Consequently, they naturally account for memory retention and hereditary impacts within dynamical processes [13, 16, 20].

Due to these unique attributes, fractional differential equations are widely utilized across various scientific domains, including geophysics [20], climatology [22], energy production systems [17],

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neuroscience [3], and many other areas. Among the diverse fractional operators available, the Riemann–Liouville and Caputo formulations remain the most preferred choices. Their popularity stems from their convenience in defining initial and boundary conditions for real-world engineering and scientific problems [16,21,26].

Analyzing such mathematical models typically requires the application of nonlinear functional analysis within Banach spaces. Specifically, fixed-point methods, Darbo-type fixed-point formulations, and the Kuratowski measure of noncompactness have become highly effective instruments for exploring the solvability of systems involving fractional derivatives [1,10,23–25].

Spurred by these recent advancements, this paper presents a novel category of coupled hybrid systems governed by fractional differential equations subject to specific boundary conditions. By rewriting the initial problem into a mathematically equivalent integral form within an appropriate Banach space, we demonstrate the existence of solutions using the Kuratowski measure of noncompactness and a generalized Darbo fixed-point approach with a nondecreasing control function. The uniqueness of this solution is verified via the Banach contraction principle. Additionally, we analyze the Hyers–Ulam stability of the introduced system under valid constraints [6,8,15].

Guided by the literature mentioned above, we explore a novel coupled system of non-linear fractional differential equations. Specifically, for $t \in \mathbb{J} = (0, a]$ $(0 < a)$, we are examining the following problem:

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\alpha_1}\left({}^{C}\mathcal{D}^{\alpha_2}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)\right) = f_2(t, \mathfrak{u}(t), \mathfrak{v}(t)), \\ {}^{RL}\mathcal{D}_{0^+}^{\alpha_3}\left({}^{C}\mathcal{D}^{\alpha_4}\left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)\right) = g_2(t, \mathfrak{u}(t), \mathfrak{v}(t)), \end{cases} \tag{1}$$

and the following boundary conditions

$$\begin{cases} (\mathfrak{u}(0), \mathfrak{v}(0)) = (0, 0), \\ \left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Bigg|_{t=a} = -\lambda_{\mathfrak{u}} \left|\mathcal{I}^{\alpha_2-1}\mathfrak{v}\,(t)\right._{t=a}, \\ \left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Bigg|_{t=a} = -\lambda_{\mathfrak{v}} \left|\mathcal{I}^{\alpha_4-1}\mathfrak{u}\,(t)\right._{t=a}, \\ {}^{C}\mathcal{D}^{\alpha_2}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)\Bigg|_{t=0} = {}^{C}\mathcal{D}^{\alpha_4}\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\Bigg|_{t=0} = 0. \end{cases} \tag{2}$$

In this framework, the operators ${}^{RL}\mathcal{D}_{0^+}^{\alpha_1}$ and ${}^{RL}\mathcal{D}_{0^+}^{\alpha_3}$ denote the Riemann–Liouville fractional derivatives of orders α_1 and α_3 respectively, satisfying $(0 < \alpha_1, \alpha_3 < 1)$. Moreover, ${}^{C}\mathcal{D}^{\alpha_2}$ along with ${}^{C}\mathcal{D}^{\alpha_4}$ represent the Caputo fractional derivatives whose orders lie in the interval $(1 < \alpha_2, \alpha_4 < 2)$. The parameters $\lambda_{\mathfrak{u}}$ and $\lambda_{\mathfrak{v}}$ belong to $\mathbb{R}_+ \setminus \{0\}$, while $f, g : \mathbb{J} \times \Xi \times \Xi \to \Xi$ are continuous mappings that will be explicitly defined in the subsequent sections.

2. Essential Preliminaries in Fractional Calculus

This section outlines the fundamental notation, terminology, and core definitions from the theory of Riemann–Liouville and Caputo fractional calculus utilized throughout this manuscript. For a more exhaustive coverage of these concepts, readers may consult the classical references [16,21].

Definition 2.1 ([16])**.** Let $\mathfrak{u}$ be a real-valued, integrable continuous function defined on the interval $(0, +\infty)$. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}$ is expressed as follows:

$$\mathcal{I}^{\alpha}\mathfrak{u}(t) = \int_0^t \frac{\mathfrak{u}(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}} ds, \ \alpha > 0.$$

Definition 2.2 ([16])**.** The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a given function $\mathfrak{u}$ operating on $(0, +\infty)$ is given by

$${}^{RL}\mathcal{D}^{\alpha}\mathfrak{u}(t) = \left(\frac{d}{dt}\right)^n \int_0^t \frac{\mathfrak{u}(s)}{\Gamma(n-\alpha)(t-s)^{\alpha+1-n}} ds, \ t > 0, n = [\alpha] + 1.$$

Definition 2.3 ([16])**.** For any function $\mathfrak{u}$ that is n-times continuously differentiable on $(0, +\infty)$, the Caputo fractional derivative of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is stated as

$${}^{C}\mathcal{D}^{\alpha}\mathfrak{u}(t) = \int_0^t \frac{\mathfrak{u}^{(n)}(s)}{\Gamma(n-\alpha)(t-s)^{\alpha+1-n}} ds, \ \ n-1 < \alpha < n, \ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of α, provided that the integral is well-defined over $(0, +\infty)$.

Lemma 1 ([21])**.** *Assuming $\alpha > 0$ and given a function $\mathfrak{u} \in C\,(0,1) \cap L^1\,(0,1)$, the following relation holds:*

$$\mathcal{I}^{\alpha}\left({}^{Rl}\mathcal{D}^{\alpha}\mathfrak{u}(t)\right) = \mathfrak{u}(t) + \sum_{i=1}^{n} a_i t^{\alpha-i},$$

where $a_i \in \mathbb{R}$ for each $i = \overline{1, n}$, with $n = [\alpha] + 1$.

Lemma 2 ([21])**.** *Let $\alpha \in (k-1, k]$. For any function $\mathfrak{u} \in C^k$, one has*

$$\mathcal{I}^{\alpha}\left({}^{C}\mathcal{D}^{\alpha}\mathfrak{u}(t)\right) = \mathfrak{u}(t) + \sum_{i=0}^{k-1} a_i t^{i},$$

where the coefficients a_i are real constants for $i = \overline{0, k-1}$.

Lemma 3. *Let $\alpha_1, \alpha_3 \in (0,1)$ and $\alpha_2, \alpha_4 \in (1,2)$. For any continuous functions $\mathfrak{u}, \mathfrak{v} \in C$, the unique solution $(\mathfrak{u}, \mathfrak{v})$ to the hybrid coupled system (1-2) is explicitly represented by*

$$\mathfrak{u}(t) \ \ = \ \ \frac{f_1(t, \mathfrak{u}(t), \mathfrak{v}(t))}{\Gamma(\alpha_1 + \alpha_2)} \int_0^t (t-s)^{\alpha_1+\alpha_2-1} f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) ds$$

$$-\frac{t}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-t)^{\alpha_1+\alpha_2-2} f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds$$

$$+\frac{\lambda_{\mathfrak{u}} t}{\Gamma Motor(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\left(\mathfrak{u}(s)+\mathfrak{v}(s)\right)ds$$

and

$$\begin{aligned}\mathfrak{v}(t) &= \frac{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}{\Gamma(\alpha_3+\alpha_4)}\int_0^t (t-s)^{\alpha_3+\alpha_4-1} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))dt \\ &\quad -\frac{t}{\Gamma(\alpha_3+\alpha_4-1)}\int_0^a (a-s)^{\alpha_3+\alpha_4-2} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad +\frac{\lambda_{\mathfrak{v}} t}{\Gamma(\alpha_4-1)}\int_0^a (a-s)^{\alpha_4-2}\left(\mathfrak{u}(s)+\mathfrak{v}(s)\right)ds.\end{aligned}$$

Proof. If we apply the fractional integration operator $\mathcal{I}^{\alpha_1}$ to the system (1) and invoke Lemma 2, the first equation of the problem becomes

$${}^{C}\mathcal{D}_{0^+}^{\alpha_2}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)=\mathcal{I}^{\alpha_1} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+a_0 t^{\alpha_1-1},$$

Considering the initial boundary condition at $t=0$, we find that the constant parameter $a_0=0$. Consequently, we can write

$${}^{C}\mathcal{D}_{0^+}^{\alpha_2}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)=\mathcal{I}^{\alpha_1} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t)). \tag{3}$$

Next, operating with the fractional integral $\mathcal{I}^{\alpha_2}$ on both sides of (3) and applying Lemma 1, we obtain

$$\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}=\mathcal{I}^{\alpha_1+\alpha_2} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+b_0+b_1 t$$

Using the initial assumption $\mathfrak{u}(0)=0$, it follows directly that

$$b_0=0,$$

which simplifies the expression to

$$\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}=\mathcal{I}^{\alpha_1+\alpha_2} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+b_1 t. \tag{4}$$

Employing a parallel argument for the second component of the system, we find

$$\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}=\mathcal{I}^{\alpha_3+\alpha_4} g_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+b_2 t. \tag{5}$$

From these integral equations, it follows by differentiation that

$$\begin{cases}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)' = \mathcal{I}^{\alpha_1+\alpha_2-1} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+b_1, \\ \left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)' = \mathcal{I}^{\alpha_3+\alpha_4-1} g_2(t,\mathfrak{u}(t),\mathfrak{v}(t))+b_2,\end{cases}$$

Evaluating these derivatives at the boundary $t = a$ according to the conditions given in (2), we have

$$\begin{cases} \left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Big|_{t=a} = -\lambda_{\mathfrak{u}} \left[\mathcal{I}^{\alpha_2-1}\left[\mathfrak{v}\left(t\right)\right]\right]_{t=a} \\ \left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Big|_{t=a} = -\lambda_{\mathfrak{v}} \left[\mathcal{I}^{\alpha_4-1}\left[\mathfrak{u}(t)\right]\right]_{t=a} \end{cases},$$

Solving this algebraic system for the constants b_1 and b_2 gives

$$\begin{cases} b_1 = -\lambda_{\mathfrak{u}} \left[\mathcal{I}^{\alpha_2-1}\left[\mathfrak{u}(t)+\mathfrak{v}\left(t\right)\right]\right]_{t=a} - \left(\mathcal{I}^{\alpha_1+\alpha_2-1} f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))\right)_{t=a} \\ b_2 = -\lambda_{\mathfrak{v}} \left[\mathcal{I}^{\alpha_4-1}\left[\mathfrak{u}(t)+\mathfrak{v}\left(t\right)\right]\right]_{t=a} - \left(\mathcal{I}^{\alpha_3+\alpha_4-1} g_2(t,\mathfrak{u}(t),\mathfrak{v}(t))\right)_{t=a} \end{cases},$$

Substituting the expressions for b_1 and b_2 back into (4) and (5), we get the explicit formula for $\mathfrak{u}(t)$:

$$\begin{aligned} \mathfrak{u}(t) &= \frac{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}{\Gamma(\alpha_1+\alpha_2)} \int_0^t (t-s)^{\alpha_1+\alpha_2-1} f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad -\frac{t}{\Gamma(\alpha_1+\alpha_2-1)} \int_0^a (a-s)^{\alpha_1+\alpha_2-2} f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad +\frac{\lambda_{\mathfrak{u}} t}{\Gamma(\alpha_2-1)} \int_0^a (a-s)^{\alpha_2-2} \left(\mathfrak{u}(s)+\mathfrak{v}(s)\right) ds. \end{aligned}$$

Applying the same substitution strategy for the second component, we determine $\mathfrak{v}(t)$:

$$\begin{aligned} \mathfrak{v}(t) &= \frac{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}{\Gamma(\alpha_3+\alpha_4)} \int_0^t (t-s)^{\alpha_3+\alpha_4-1} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad -\frac{t}{\Gamma(\alpha_3+\alpha_4-1)} \int_0^a (a-s)^{\alpha_3+\alpha_4-2} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad +\frac{\lambda_{\mathfrak{v}} t}{\Gamma(\alpha_4-1)} \int_0^a (a-s)^{\alpha_4-2} \left(\mathfrak{u}(s)+\mathfrak{v}(s)\right) ds. \end{aligned}$$

The proof is now complete. □

3. Fundamental Tools from Functional Analysis

This section outlines the essential mathematical background, definitions, and geometric structures from classical functional analysis required throughout the manuscript. For a more comprehensive development of these topological properties, readers may refer to [9,11,14,19].

We begin by establishing the underlying spaces and their respective standard topologies:

(1) Let $\mathcal{C} = C(\mathbb{J}, \Xi)$ denote the space containing all continuous mappings from $\mathbb{J}$ into Ξ. This space forms a Banach space when equipped with the uniform norm defined by

$$\|\mathfrak{u}\|_\infty = \sup_{t\in\mathbb{J}} \|\mathfrak{u}\left(t\right)\|.$$

(2) The product space $\mathcal{K} = \mathcal{C} \times \mathcal{C}$ constitutes a product Banach space under the natural sum norm:

$$\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{K}} = \|\mathfrak{u}\|_\infty + \|\mathfrak{v}\|_\infty$$

(3) Let $\mathcal{C}(\mathbb{J} \times \Xi \times \Xi, \Xi)$ represent the space of continuous real-valued functions mapping from the product domain $\mathbb{J} \times \Xi \times \Xi$ into Ξ, which is associated with the supremum norm:

$$\|f\|_{\mathcal{C}} = \sup_{(s,\mathfrak{u},\mathfrak{v}) \in \mathbb{J}\times\Xi\times\Xi} \|f(s,\mathfrak{u},\mathfrak{v})\|$$

Definition 3.1. (Kuratowski Measure of Noncompactness [14]). Let $(\Xi, \|\cdot\|)$ be a Banach space, and let $\mathcal{B}_\Xi$ signify the family of all nonempty bounded subsets of Ξ.

The Kuratowski measure of noncompactness is defined as the functional $\mathfrak{T} : \mathcal{B}_\Xi \to [0, +\infty)$ such that for any subset $\mathfrak{G} \in \mathcal{B}_\Xi$,

$$\mathfrak{T}(\mathfrak{G}) = \inf \left\{ \sigma > 0 : \exists n \in \mathbb{N}, \exists \mathfrak{G}_i \subset \Xi, i = \overline{1,n}, \text{ such that } \mathfrak{G} \subset \bigcup_{i=1}^{n} \mathfrak{G}_i, \text{ and } diam(\mathfrak{G}_i) \leq \sigma, \forall i \right\}$$

where the diameter of each sub-component $\mathfrak{G}_i$ is given by

$$diam(\mathfrak{G}_i) = \sup \{\|\mathfrak{u} - \mathfrak{v}\| : \mathfrak{u}, \mathfrak{v} \in \mathfrak{G}_i\}.$$

Lemma 4 ([14])**.** *Let $\mathfrak{G}, \widehat{\mathfrak{G}} \subset \Xi$ be bounded subsets. The Kuratowski measure satisfies the following fundamental properties:*

- ***Monotonicity:*** *If $\mathfrak{G} \subset \widehat{\mathfrak{G}}$, then $\mathfrak{T}(\mathfrak{G}) \leq \mathfrak{T}(\widehat{\mathfrak{G}})$.*
- ***Closure Invariance:*** *$\mathfrak{T}(\overline{\mathfrak{G}}) = \mathfrak{T}(\mathfrak{G})$.*

(1) *Convex hull invariance: $\mathfrak{T}(conv(\mathfrak{G})) = \mathfrak{T}(\mathfrak{G})$.*

(2) *Homogeneity property: $\mathfrak{T}(\lambda\mathfrak{G}) = |\lambda|\mathfrak{T}(\mathfrak{G})$ is valid for any scalar $\lambda \in \mathbb{R}$.*

(3) *Algebraic subadditivity: $\mathfrak{T}(\mathfrak{G} + \mathcal{B}) \leq \mathfrak{T}(\mathfrak{G}) + \mathfrak{T}(\widehat{\mathfrak{G}})$, where the set addition is given by $\mathfrak{G} + \mathcal{B} = \{a + b : a \in \mathfrak{G}, b \in \widehat{\mathfrak{G}}\}$.*

(4) *Compactness criterion: $\mathfrak{T}(\mathfrak{G}) = 0$ if and only if $\mathfrak{G}$ is relatively compact within Ξ.*

Lemma 5 ([14])**.** *Assuming that $\mathfrak{G} \subset \mathcal{C}$ is both bounded and equicontinuous, we define the evaluation set $\mathfrak{G}(t) = \{\mathfrak{u}(t) : \mathfrak{u} \in \mathfrak{G}\} \subset \Xi$ for each $t \in \mathbb{J}$. Then, the mapping $t \to \mathfrak{T}(\mathfrak{G}(t))$ varies continuously over $\mathbb{J}$, yielding*

$$\mathfrak{T}_{\mathcal{C}}(\mathfrak{G}) = \sup_{t\in\mathbb{J}} \{\mathfrak{T}(\mathfrak{G}(t))\},$$

and moreover,

$$\mathfrak{T}\left(\int_{\mathbb{J}} \mathfrak{u}(s)ds : \mathfrak{u} \in \mathfrak{G}\right) \leq \int_{\mathbb{J}} \mathfrak{T}(\mathfrak{G}(s))ds.$$

Theorem 1 ([9])**.** *Let $\mathfrak{T}_1, \mathfrak{T}_2, ..., \mathfrak{T}_n$ denote measures of noncompactness established on the respective Banach spaces $\Xi_1, \Xi_2, ..., \Xi_n$. Let $\mathfrak{F} : (\mathbb{R}_+)^n \to \mathbb{R}_+$ be a convex function satisfying $\mathfrak{F}(t_1, t_2 ..., t_n) = 0 \iff t_i = 0$ for each $i \in \{1, 2, ..., n\}$. Then, for any bounded subset $\mathfrak{G} \subset \Xi_1 \times \Xi_2 \times ... \times \Xi_n$, the functional*

$$\widehat{\mathfrak{T}}(\mathfrak{G}) = \mathfrak{F}(\mathfrak{T}_1(\mathfrak{G}_1), \mathfrak{T}_2(\mathfrak{G}_2), ..., \mathfrak{T}_n(\mathfrak{G}_n)),$$

constitutes a valid measure of noncompactness on the product framework $\Xi_1 \times \Xi_2 \times ... \times \Xi_n$, where $\mathfrak{G}_i$ represents the canonical projection of $\mathfrak{G}$ onto Ξ_i.

- **Particular Case:** ([9]). Whenever $n = 2$, if we choose the mapping $\mathfrak{F} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined as $\mathfrak{F}(x, y) = x + y$, all prerequisites of Theorem 1 are met. Consequently, the mapping

$$\widehat{\mathfrak{T}}(\mathfrak{G}) = \mathfrak{T}_1(\mathfrak{G}_1) + \mathfrak{T}_2(\mathfrak{G}_2),$$

defines a measure of noncompactness over the product space $\Xi_1 \times \Xi_2$, with $\mathfrak{G}_1, \mathfrak{G}_2$ signifying the canonical projections.

Lemma 6 ([11]). *Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ represent an upper semicontinuous and nondecreasing function. The statements below are equivalent:*

(1) *$\xi(t) < t$ for all $t > 0$.*

(2) *For any $t \in \mathbb{R}_+$, the sequence of iterations $\xi^n(t)$ converges to zero as $n \to +\infty$, i.e., $\lim_{n\to+\infty} \xi^n(t) = 0$.*

Theorem 2. (*Generalized Darbo Fixed-Point Theorem* [11]). *Let $\mathfrak{M}$ be a nonempty, closed, bounded, and convex subset of a Banach space Ξ. Let $\mathfrak{N} : \mathfrak{M} \to \mathfrak{M}$ be a continuous mapping satisfying*

$$\mathfrak{T}(\mathfrak{N}(\mathfrak{G})) \leq \xi(\mathfrak{T}(\mathfrak{G})), \tag{6}$$

for every nonempty subset $\mathfrak{G} \subset \mathfrak{M}$, where $\mathfrak{T}$ is an arbitrary measure of noncompactness and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing mapping such that $\lim_{n\to+\infty} \xi^n(t) = 0$ for all $t \in \mathbb{R}_+$ (with ξ^n being the n-th iterate of ξ). Under these conditions, $\mathfrak{N}$ possesses at least one fixed point within $\mathfrak{M}$.

Theorem 3. (*Contraction Principle* [19]). *An operator $\mathfrak{T}$ mapping a Banach space Ξ into itself is called a contraction if there exists a constant $k \in (0, 1)$ fulfilling*

$$\|\mathfrak{T}\mathfrak{y} - \mathfrak{T}\mathfrak{x}\| \leq k\,\|\mathfrak{y} - \mathfrak{x}\|, \quad \forall \mathfrak{x}, \mathfrak{y} \in \Xi.$$

Theorem 4. (*Banach Fixed-Point Theorem* [19]). *Let $\mathfrak{T} : \Xi \to \Xi$ be a contraction operator on a Banach space Ξ. Then $\mathfrak{T}$ admits a unique fixed point $\mathfrak{u} \in \Xi$, meaning that $\exists! \mathfrak{u} \in \Xi$ such that $\mathfrak{T}\mathfrak{u} = \mathfrak{u}$.*

4. Main Results

In light of Lemma 3, we introduce the operator $\Phi : \mathcal{K} \to \mathcal{K}$ structured as:

$$\Phi(\mathfrak{u}(t), \mathfrak{v}(t)) = \begin{pmatrix} \Phi_1(\mathfrak{u}, \mathfrak{v})(t) \\ \Phi_2(\mathfrak{u}, \mathfrak{v})(t) \end{pmatrix}, \tag{7}$$

where the components $\Phi_1, \Phi_2 : \mathcal{K} \to \mathcal{C}$ are given explicitly by:

$$\begin{aligned} \Phi_1(\mathfrak{u}, \mathfrak{v})(t) &= \frac{f_1(t, \mathfrak{u}(t), \mathfrak{v}(t))}{\Gamma(\alpha_1 + \alpha_2)} \int_0^t (t - s)^{\alpha_1 + \alpha_2 - 1} f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) ds \\ &\quad - \frac{t}{\Gamma(\alpha_1 + \alpha_2 - 1)} \int_0^a (a - s)^{\alpha_1 + \alpha_2 - 2} f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) ds \end{aligned} \tag{8}$$

$$+\frac{\lambda_{\mathfrak{u}} t}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\left(\mathfrak{u}(s)+\mathfrak{v}(s)\right)ds,$$

and

$$\begin{aligned}\Phi_2(\mathfrak{u},\mathfrak{v})(t) &= \frac{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}{\Gamma(\alpha_3+\alpha_4)}\int_0^t (t-s)^{\alpha_3+\alpha_4-1} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad -\frac{t}{\Gamma(\alpha_3+\alpha_4-1)}\int_0^a (a-s)^{\alpha_3+\alpha_4-2} g_2(s,\mathfrak{u}(s),\mathfrak{v}(s))ds \\ &\quad +\frac{\lambda_{\mathfrak{v}} t}{\Gamma(\alpha_4-1)}\int_0^a (a-s)^{\alpha_4-2}\left(\mathfrak{u}(s)+\mathfrak{v}(s)\right)ds.\end{aligned} \tag{9}$$

4.1. **Existence and Uniqueness Analysis.**

In this section, we establish the existence criteria for solutions by applying the Kuratowski measure of noncompactness coupled with Darbo's generalized fixed-point theorem involving a nondecreasing control function. Furthermore, the uniqueness of the solution is demonstrated using the classical Banach contraction principle.

- Let Λ denote the class of functions defined by:

$$\Lambda=\left\{\xi:[0,+\infty)\to[0,+\infty):\xi \text{ is a nondecreasing mapping satisfying } \lim_{n\to+\infty}\xi^n(t)=0 \text{ for each } t\in[0,+\infty)\right\}.$$

Remark 1. The operational class Λ satisfies the following basic properties:

- For any scaling parameter $\lambda\in[0,1]$ and any function $\xi\in\Lambda$, the composite product fulfills $\lambda\xi\in\Lambda$.
- If $\xi\in\Lambda$, then the strict inequality $\xi(t)<t$ holds for all $t>0$.

Let $\mathfrak{B}_\tau$ denote the closed ball centered at the coordinate origin with a fixed radius $\tau>0$, defined in the space $\mathcal{K}$ by:

$$\mathfrak{B}_\tau=\{(\mathfrak{u},\mathfrak{v})\in\mathcal{K},\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{K}}\le\tau\}\subset\mathcal{K}.$$

To simplify the subsequent calculations, we introduce the following constant notations:

$$\widehat{\delta}_{f_2}=\sup_{t\in\mathbb{J}}\|f_2(t,0,0)\|,\quad \text{and}\quad \widehat{\delta}_{g_2}=\sup_{t\in\mathbb{J}}\|g_2(t,0,0)\|.$$

Throughout this manuscript, we assume the validity of the following hypotheses:

- **(H1)** There exist two continuous control functions $\delta_f,\delta_g\in\Lambda$ such that, for all $t\in\mathbb{J}$,

$$\left|f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))-f_2(t,\widehat{\mathfrak{u}}(t),\widehat{\mathfrak{v}}(t))\right|\le\delta_f\left(\|\mathfrak{u}-\widehat{\mathfrak{u}}\|+\left\|\mathfrak{v}-\widehat{\widehat{\mathfrak{v}}}\right\|\right), \tag{10}$$

and

$$\left|g_2(t,\mathfrak{u}(t),\mathfrak{v}(t))-g_2(t,\widehat{\mathfrak{u}}(t),\widehat{\mathfrak{v}}(t))\right|\le\delta_g\left(\|\mathfrak{u}-\widehat{\mathfrak{u}}\|+\left\|\mathfrak{v}-\widehat{\widehat{\mathfrak{v}}}\right\|\right). \tag{11}$$

Furthermore, for all $t \in \mathbb{J}$, the functions f_1 and g_1 are bounded by positive constants M_f and M_g respectively:

$$\begin{aligned} |f_1(t, \mathfrak{u}(t), \mathfrak{v}(t))| &\leq M_f, \\ |g_1(t, \mathfrak{u}(t), \mathfrak{v}(t))| &\leq M_g. \end{aligned} \tag{12}$$

- **(H2)** There exist conjugate parameters $p, q \in (0, +\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, such that the following inequalities hold:

$$\frac{M_f\left(\delta_f(\tau) + \widehat{\delta}_{f_2}\right) a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{M_f\left(\delta_f(\tau) + \widehat{\delta}_{f_2}\right) a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} + \frac{\lambda_{\mathfrak{u}} a^{\alpha_2}}{\Gamma(\alpha_2)}\tau \leq \frac{1}{p}\tau, \tag{13}$$

and

$$\frac{M_g\left(\delta_g(\tau) + \widehat{\delta}_{g_2}\right) a^{\alpha_3+\alpha_4}}{\Gamma(1+\alpha_3+\alpha_4)} + \frac{M_g\left(\delta_g(\tau) + \widehat{\delta}_{g_2}\right) a^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} + \frac{\lambda_{\mathfrak{v}} a^{\alpha_4}}{\Gamma(1+\alpha_4)}\tau \leq \frac{1}{q}\tau. \tag{14}$$

Remark 2. Based on the assumptions (10) and (11), it follows directly that for any bounded subsets $\mathfrak{G}_1, \mathfrak{G}_2 \subset \mathcal{C}$ and for each $t \in \mathbb{J}$, the following inequalities are satisfied:

$$\mathfrak{T}\left(f_2(t, \mathfrak{G}_1, \mathfrak{G}_2)\right) \leq \delta_f(\mathfrak{T}(\mathfrak{G}_1) + \mathfrak{T}(\mathfrak{G}_2)), \tag{15}$$

and

$$\mathfrak{T}\left(g_2(t, \mathfrak{G}_1, \mathfrak{G}_2)\right) \leq \delta_g(\mathfrak{T}(\mathfrak{G}_1) + \mathfrak{T}(\mathfrak{G}_2)). \tag{16}$$

Theorem 5. (*Existence Result*). *Supposing that requirements (10), (11), (13), and (14) are fulfilled, the hybrid coupled system (1-2) admits at least one solution.*

Proof. The verification of this existence result is broken down into four sequential stages.

- **Stage 1**. *Invariance of the closed ball $\mathfrak{B}_\tau$ under Φ*. Let $(\mathfrak{u}, \mathfrak{v}) \in \mathfrak{B}_\tau$ and $t \in \mathbb{J}$. Utilizing hypotheses (10), (12), and (11), we can establish that

$$\begin{aligned} \|f_2(t, \mathfrak{u}(t), \mathfrak{v}(t))\| &\leq \|f_2(t, \mathfrak{u}(t), \mathfrak{v}(t)) - f_2(t, 0, 0)\| + \|f_2(t, 0, 0)\| \\ &\leq \delta_f\left(\|\mathfrak{u}(t)\| + \|\mathfrak{v}(t)\|\right) + \widehat{\delta}_{f_2}, \end{aligned} \tag{17}$$

and symmetrically,

$$\begin{aligned} \|g_2(t, \mathfrak{u}(t), \mathfrak{v}(t))\| &\leq \|g_2(t, \mathfrak{u}(t), \mathfrak{v}(t)) - g_2(t, 0, 0)\| + \|g_2(t, 0, 0)\| \\ &\leq \delta_g\left(\|\mathfrak{u}(t)\| + \|\mathfrak{v}(t)\|\right) + \widehat{\delta}_{g_2}. \end{aligned} \tag{18}$$

Consequently, evaluating the first component Φ_1, we obtain

$$\|\Phi_1(\mathfrak{u}, \mathfrak{v})(t)\|$$

$$
\begin{aligned}
&\leq \frac{M_f\left[\delta_f\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)+\widehat{\delta}_{f_2}\right]}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}ds \\
&\quad+\frac{aM_f\left[\delta_f\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)+\widehat{\delta}_{f_2}\right]}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}ds \\
&\quad+\frac{a\lambda_{\mathfrak{u}}\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}ds \\
&\leq \frac{M_f\left(\delta_f(\tau)+\widehat{\delta}_{f_2}\right)a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f\left(\delta_f(\tau)+\widehat{\delta}_{f_2}\right)a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\tau,
\end{aligned}
$$

Taking the supremum over $t \in \mathbb{J}$ and applying condition (13), we find

$$\|\Phi_1(\mathfrak{u},\mathfrak{v})\|_\infty \leq \frac{1}{p}\tau.$$

Employing a parallel argument for the second component Φ_2, we have

$$
\begin{aligned}
&\|\Phi_2(\mathfrak{u},\mathfrak{v})(t)\| \\
&\leq \frac{M_g\left[\delta_g\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)+\widehat{\delta}_{g_2}\right]}{\Gamma(\alpha_3+\alpha_4)}\int_0^t (t-s)^{\alpha_3+\alpha_4-1}ds \\
&\quad+\frac{aM_g\left[\delta_g\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)+\widehat{\delta}_{g_2}\right]}{\Gamma(\alpha_3+\alpha_4-1)}\int_0^a (a-s)^{\alpha_3+\alpha_4-2}ds \\
&\quad+\frac{a\lambda_{\mathfrak{v}}\left(\|\mathfrak{u}(t)\|+\|\mathfrak{v}(t)\|\right)}{\Gamma(\alpha_4-1)}\int_0^a (a-s)^{\alpha_4-2}ds \\
&\leq \frac{M_g\left(\delta_g(\tau)+\widehat{\delta}_{g_2}\right)a^{\alpha_3+\alpha_4}}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{M_g\left(\delta_g(\tau)+\widehat{\delta}_{g_2}\right)a^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)}+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(1+\alpha_4)}\tau,
\end{aligned}
$$

Hence, invoking condition (14), it yields

$$\|\Phi_2(\mathfrak{u},\mathfrak{v})\|_\infty \leq \frac{1}{q}\tau.$$

By combining these two uniform bounds, we conclude that

$$\|\Phi(\mathfrak{u},\mathfrak{v})\|_{\mathcal{K}} = \|\Phi_1(\mathfrak{u},\mathfrak{v})\|_\infty + \|\Phi_2(\mathfrak{u},\mathfrak{v})\|_\infty \leq \left(\frac{1}{p}+\frac{1}{q}\right)\tau = \tau.$$

This establishes that the operator Φ maps the ball $\mathfrak{B}_\tau$ into itself.

- **Stage 2**. *The operator Φ maps bounded subsets of $\mathfrak{B}_\tau$ into equicontinuous families*. Let $\mathfrak{B}_{\widehat{\tau}} = \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{K}, \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{K}} \leq \widehat{\tau}\} \subset \mathfrak{B}_\tau$ be an arbitrary bounded subset. Applying the hypothesis (10), for any pair $(\mathfrak{u},\mathfrak{v}) \in \mathfrak{B}_{\widehat{\tau}}$ and for any time instants $t_1, t_2 \in \mathbb{J}$ with $t_1 < t_2$, we observe that:

$$
\begin{aligned}
& \left\| f_2(t_1, \mathfrak{u}(t_1), \mathfrak{v}(t_1)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \\
\leq\ & \left\| f_2(t_1, \mathfrak{u}(t_1), \mathfrak{v}(t_1)) - f_2(t_1, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| + \left\| f_2(t_1, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \\
\leq\ & \delta_f \left(\|\mathfrak{u}(t_1) - \mathfrak{u}(t_2)\| + \|\mathfrak{v}(t_1) - \mathfrak{v}(t_2)\| \right) + \left\| f_2(t_1, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\|
\end{aligned}
$$

On the other hand, for any element $(\mathfrak{u}, \mathfrak{v}) \in \mathfrak{B}_{\widehat{\tau}}$, the second term on the right-hand side can be bounded as follows:

$$
\left\| f_2(t_1, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \leq \sup_{|t_1 - t_2| \to 0} \left\| f_2(t_1, \mathfrak{u}, \mathfrak{v}) - f_2(t_2, \mathfrak{u}, \mathfrak{v}) \right\|.
$$

Since the nonlinear mappings f_2 and g_2 are uniformly continuous on any bounded subset within the product space $\mathbb{J} \times \Xi \times \Xi$, we immediately deduce that

$$
\left\| f_2(t_1, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \to 0. \tag{19}
$$

Consequently, it yields that

$$
\left\| f_2(t_1, \mathfrak{u}(t_1), \mathfrak{v}(t_1)) - f_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \to 0. \tag{20}
$$

Following a symmetric argument, we also obtain

$$
\left\| g_2(t_1, \mathfrak{u}(t_1), \mathfrak{v}(t_1)) - g_2(t_2, \mathfrak{u}(t_2), \mathfrak{v}(t_2)) \right\| \to 0. \tag{21}
$$

Now, let us evaluate the variation of the operator component Φ_1:

$$
\begin{aligned}
& \left\| \Phi_1 \left(\mathfrak{u}, \mathfrak{v} \right) (t_1) - \Phi_1 \left(\mathfrak{u}, \mathfrak{v} \right) (t_2) \right\| \\
\leq\ & \frac{M_f}{\Gamma(\alpha_1 + \alpha_2)} \left| \int_0^{t_1} (t_1 - s)^{\alpha_1 + \alpha_2 - 1} f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) ds - \int_0^{t_2} (t_2 - s)^{\alpha_1 + \alpha_2 - 1} f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) ds \right| \\
& + \frac{|t_1 - t_2|}{\Gamma(\alpha_1 + \alpha_2 - 1)} \int_0^a (a - s)^{\alpha_1 + \alpha_2 - 2} \left\| f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) \right\| ds \\
& + \frac{\lambda_{\mathfrak{u}} |t_1 - t_2|}{\Gamma(\alpha_2 - 1)} \int_0^a (a - s)^{\alpha_2 - 2} \left(\|\mathfrak{u}(s)\| + \|\mathfrak{v}(s)\| \right) ds \\
\leq\ & \frac{M_f \left[\delta_f (\tau) + \widehat{\delta}_{f_2} \right]}{\Gamma(1 + \alpha_1 + \alpha_2)} \left({t_2}^{\alpha_1 + \alpha_2} - {t_1}^{\alpha_1 + \alpha_2} + (t_2 - t_1)^{\alpha_1 + \alpha_2} \right) \\
& + \left(\frac{a^{\alpha_1 + \alpha_2 - 1} M_f \left(\delta_f (\tau) + \widehat{\delta}_{f_2} \right)}{\Gamma(1 + \alpha_1 + \alpha_2)} \right) |t_1 - t_2| \\
& + \frac{\lambda_{\mathfrak{u}} a^{\alpha_2 - 1} \tau}{\Gamma(\alpha_2)} |t_1 - t_2|
\end{aligned}
$$

From this upper bound, we can explicitly conclude that

$$\left\|\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)\left(t_1\right)-\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)\left(t_2\right)\right\|\to 0,\quad \text{as } t_1\to t_2.$$

This demonstrates that the set of functions $\Phi_1(\mathfrak{B}_{\widehat{\tau}})$ is equicontinuous.

By applying an identical procedure to the second component Φ_2, we get

$$\begin{aligned}
&\left\|\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)\left(t_1\right)-\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)\left(t_2\right)\right\| \\
\leq\quad & \frac{M_g\left[\delta_g\left(\tau\right)+\widehat{\delta}_{g_2}\right]}{\Gamma(1+\alpha_3+\alpha_4)}\left({t_2}^{\alpha_3+\alpha_4}-{t_1}^{\alpha_3+\alpha_4}+\left(t_2-t_1\right)^{\alpha_3+\alpha_4}\right) \\
&+\left(\frac{a^{\alpha_3+\alpha_4-1}M_g\left(\delta_g\left(\tau\right)+\widehat{\delta}_{g_2}\right)}{\Gamma(1+\alpha_3+\alpha_4)}\right)\left|t_1-t_2\right| \\
&+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4-1}\tau}{\Gamma(\alpha_4)}\left|t_1-t_2\right|
\end{aligned}$$

Thus, it follows that

$$\left\|\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)\left(t_1\right)-\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)\left(t_2\right)\right\|\to 0,\quad \text{as } t_1\to t_2.$$

This establishes that $\Phi_2(\mathfrak{B}_{\widehat{\tau}})$ is also equicontinuous.

As a final consequence of this stage, the multi-valued operator Φ maps bounded sets into equicontinuous sets.

- **Stage 3**. *Continuity of the operational mapping Φ over $\mathfrak{B}_\tau$.* To verify this property, let us consider an arbitrary sequence $(\mathfrak{u}_n,\mathfrak{v}_n)\subset\mathfrak{B}_\tau$ that converges toward an element $(\mathfrak{u},\mathfrak{v})$ within the space $\mathfrak{B}_\tau$. By tracking the norm difference for the first component Φ_1, we have:

$$\begin{aligned}
&\left\|\Phi_1\left(\mathfrak{u}_n,\mathfrak{v}_n\right)(t)-\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)(t)\right\| \\
\leq\quad & \frac{M_f}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}\left\|f_2(s,\mathfrak{u}_n(s),\mathfrak{v}_n(s))-f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))\right\|ds \\
&+\frac{aM_f}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}\left\|f_2(s,\mathfrak{u}_n(s),\mathfrak{v}_n(s))-f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))\right\|ds \\
&+\frac{\lambda_{\mathfrak{u}}a}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\left(\|\mathfrak{u}_n(s)-\mathfrak{u}(s)\|+\|\mathfrak{v}_n(s)-\mathfrak{v}(s)\|\right)ds \\
\leq\quad & \left(\frac{a^{\alpha_1+\alpha_2}M_f}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{a^{\alpha_1+\alpha_2}M_f}{\Gamma(\alpha_1+\alpha_2)}\right)\sup_{t\in\mathbb{J}}\left\|f_2(t,\mathfrak{u}_n(t),\mathfrak{v}_n(t))-f_2(t,\mathfrak{u}(t),\mathfrak{v}(t))\right\| \\
&+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\left(\sup_{t\in\mathbb{J}}\|\mathfrak{u}_n(t)-\mathfrak{u}(t)\|+\sup_{t\in\mathbb{J}}\|\mathfrak{v}_n(t)-\mathfrak{v}(t)\|\right).
\end{aligned}$$

Invoking Lebesgue's Dominated Convergence Theorem and taking into account the continuity constraints on the non-linearities, it follows that:

$$\lim_{n\to\infty} \left\|\Phi_1\left(\mathfrak{u}_n,\mathfrak{v}_n\right)(t)-\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)(t)\right\|=0.$$

Analogously, a parallel deduction for the second operational component yields:

$$\lim_{n\to\infty} \left\|\Phi_2\left(\mathfrak{u}_n,\mathfrak{v}_n\right)(t)-\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)(t)\right\|=0.$$

This pointwise convergence of both sequences $\Phi_1\left(\mathfrak{u}_n,\mathfrak{v}_n\right)\to\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)$ and $\Phi_2\left(\mathfrak{u}_n,\mathfrak{v}_n\right)\to\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)$ can be upgraded to uniform convergence. Indeed, as demonstrated in Stage 2, the operator Φ carries bounded sets into equicontinuous families. Consequently, the sequence of operators $\{\Phi\left(\mathfrak{u}_n,\mathfrak{v}_n\right)\}_{n\in\mathbb{N}}$ forms an equicontinuous set.

By applying the classical Arzelà–Ascoli theorem, the combination of this equicontinuity along with the pointwise limit guarantees that the convergence is uniform across the entire interval. This ultimately proves that the operator Φ is continuous on the domain $\mathfrak{B}_\tau$.

- **Stage 4**. *Verification of the topological contraction constraint* (6).

Let us construct the set $\mathfrak{G}=\overline{conv}(\Phi\mathfrak{B}_\tau)$. By definition, $\mathfrak{G}$ represents a closed, convex, and bounded subset within the ball $\mathfrak{B}_\tau$. Since the embedding $\mathfrak{G}\subset\mathfrak{B}_\tau$ holds, it immediately follows that $\Phi\left(\mathfrak{G}\right)\subset\Phi\left(\mathfrak{B}_\tau\right)\subset\mathfrak{G}$. Integrating the conclusions established in Stage 1 and Stage 3, the operator $\Phi:\mathfrak{G}\to\mathfrak{G}$ is shown to be both continuous and bounded. Furthermore, Stage 2 guarantees that the image set $\mathfrak{G}$ is equicontinuous.

Consequently, invoking Lemma 5, the real-valued mapping $t\to\widehat{\mathfrak{T}}(\mathfrak{G}(t))$ is continuous over the interval $\mathbb{J}$. Utilizing the properties established in the product framework of Example 3 alongside the axiomatic structure of the Kuratowski measure of noncompactness $\mathfrak{T}$, we can deduce that for any given time instant $t\in\mathbb{J}$:

$$\widehat{\mathfrak{T}}(\mathfrak{G}(t))=\mathfrak{T}(\Phi_1\left(\mathfrak{G}\right))(t)+\mathfrak{T}(\Phi_2\left(\mathfrak{G}\right))(t).$$

Evaluating the first component, we observe that:

$$\begin{aligned}
&\mathfrak{T}(\Phi_1\left(\mathfrak{G}\right))(t)\\
\leq\;& \frac{M_f}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}\mathfrak{T}\left(f_2(s,\mathfrak{G}_1(s),\mathfrak{G}_2(s))\right)ds\\
&+\frac{aM_f}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}\mathfrak{T}\left(f_2(s,\mathfrak{G}_1(s),\mathfrak{G}_2(s))\right)ds\\
&+\frac{\lambda_{\mathfrak{u}}a}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\mathfrak{T}\left(\mathfrak{G}_1(s)+\mathfrak{G}_2\left(s\right)\right)ds,
\end{aligned}$$

where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ signify the canonical projections of the bounded set $\mathfrak{G} \subset \mathfrak{B}_\tau$ onto the space $\mathcal{C}$. Applying the estimates (20), (21), and Remark 2, the previous bound reduces to:

$$\begin{aligned}
&\mathfrak{T}(\Phi_1(\mathfrak{G}))(t)\\
\leq\ & \frac{M_f}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}\delta_f\left(\mathfrak{T}(\mathfrak{G}_1(s))+\mathfrak{T}(\mathfrak{G}_2(s))\right)ds\\
&+\frac{aM_f}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}\delta_f\left(\mathfrak{T}(\mathfrak{G}_1(s))+\mathfrak{T}(\mathfrak{G}_2(s))\right)ds\\
&+\frac{\lambda_{\mathfrak{u}}a}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\left[\mathfrak{T}(\mathfrak{G}_1(s))+\mathfrak{T}(\mathfrak{G}_2(s))\right]ds,
\end{aligned}$$

Hence, applying Lemma 5 to switch to the uniform norm, we find:

$$\begin{aligned}
&\mathfrak{T}_{\mathcal{C}}(\Phi_1(\mathfrak{G}))\\
\leq\ & \frac{\delta_f\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}ds+\frac{aM_f\delta_f\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}ds\\
&+\frac{\lambda_{\mathfrak{u}}a\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}ds\\
\leq\ & \left(\frac{a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}\right)M_f\delta_f\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G}).
\end{aligned}$$

Following an identical bounding sequence for the second operator component, we obtain:

$$\mathfrak{T}_{\mathcal{C}}(\Phi_2(\mathfrak{G}))\leq\left(\frac{a^{\alpha_3+\alpha_4}}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{a^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)}\right)M_g\delta_g\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(\alpha_4)}\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G}).$$

By summing these uniform estimates, it follows that:

$$\begin{aligned}
\widehat{\mathfrak{T}}_{\mathcal{C}}(\Phi(\mathfrak{G})) &\leq \mathfrak{T}_{\mathcal{C}}(\Phi_1(\mathfrak{G}))+\mathfrak{T}_{\mathcal{C}}(\Phi_2(\mathfrak{G}))\\
&\leq \left(\frac{a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}\right)M_f\delta_f\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)\\
&\quad+\left(\frac{a^{\alpha_3+\alpha_4}}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{a^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)}\right)M_g\delta_g\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right)\\
&\quad+\left(\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(\alpha_4)}\right)\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\\
&\leq \xi\left(\widehat{\mathfrak{T}}_{\mathcal{C}}(\mathfrak{G})\right).
\end{aligned}$$

In light of Remark 1, the control mapping belongs to the class Λ. Consequently, applying the Generalized Darbo Fixed-Point Theorem (2), we conclude that the initial coupled fractional system (1-2) possesses at least one mild solution $(\mathfrak{u},\mathfrak{v})$ inside the subset $\mathfrak{G}$.

□

Theorem 6. (*Uniqueness of Solutions*). *Assuming that prerequisites* **(H1)** *and* **(H2)** *are satisfied, the hybrid interconnected system (1-2) admits a unique mild solution* $(\mathfrak{u}, \mathfrak{v})$ *in the product domain* $\mathcal{K}$.

Proof. Let $(\mathfrak{u}, \mathfrak{v})$ and $(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})$ represent two distinct elements chosen from the product space $\mathcal{K}$. Taking the norm of their operational images under Φ, we write:

$$\left\|\Phi(\mathfrak{u}, \mathfrak{v}) - \Phi(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_{\mathcal{K}} = \left\|\Phi_1(\mathfrak{u}, \mathfrak{v}) - \Phi_1(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_\infty + \left\|\Phi_2(\mathfrak{u}, \mathfrak{v}) - \Phi_2(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_\infty.$$

Evaluating the first component, it yields:

$$\begin{aligned}
&\left\|\Phi_1(\mathfrak{u}, \mathfrak{v})(t) - \Phi_1(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})(t)\right\| \\
\leq\ & \frac{M_f}{\Gamma(\alpha_1+\alpha_2)} \int_0^t (t-s)^{\alpha_1+\alpha_2-1} \left\|f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) - f_2(s, \widehat{\mathfrak{u}}(s), \widehat{\mathfrak{v}}(s))\right\| ds \\
&+ \frac{aM_f}{\Gamma(\alpha_1+\alpha_2-1)} \int_0^a (a-s)^{\alpha_1+\alpha_2-2} \left\|f_2(s, \mathfrak{u}(s), \mathfrak{v}(s)) - f_2(s, \widehat{\mathfrak{u}}(s), \widehat{\mathfrak{v}}(s))\right\| ds \\
&+ \frac{\lambda_{\mathfrak{u}} a}{\Gamma(\alpha_2-1)} \int_0^a (a-s)^{\alpha_2-2} \left(\|\mathfrak{u}(s) - \widehat{\mathfrak{u}}(s)\| + \left\|\mathfrak{v}(s) - \widehat{\mathfrak{v}}(s)\right\|\right) ds.
\end{aligned}$$

Invoking the hypothesis **(H1)**, we can rewrite this upper bound as:

$$\begin{aligned}
&\left\|\Phi_1(\mathfrak{u}, \mathfrak{v})(t) - \Phi_1(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})(t)\right\| \\
\leq\ & \left(\frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} + \frac{\lambda_{\mathfrak{u}} a^{\alpha_2}}{\Gamma(\alpha_2)}\right) \left(\|\mathfrak{u}(t) - \widehat{\mathfrak{u}}(t)\| + \left\|\mathfrak{v}(t) - \widehat{\mathfrak{v}}(t)\right\|\right) \\
\leq\ & \left(\frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} + \frac{\lambda_{\mathfrak{u}} a^{\alpha_2}}{\Gamma(\alpha_2)}\right) \left\|(\mathfrak{u}, \mathfrak{v}) - (\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_{\mathcal{K}}.
\end{aligned}$$

Taking the supremum over $t \in \mathbb{J}$, we get:

$$\left\|\Phi_1(\mathfrak{u}, \mathfrak{v}) - \Phi_1(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_\infty \leq \left(\frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{M_f \delta_f(\tau) a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} + \frac{\lambda_{\mathfrak{u}} a^{\alpha_2}}{\Gamma(\alpha_2)}\right) \left\|(\mathfrak{u}, \mathfrak{v}) - (\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\right\|_{\mathcal{K}}. \quad (22)$$

In a similar fashion, the uniform bound for the second component. □

4.2. **Uniqueness Criteria via Banach Fixed-Point Principle.**

To establish the uniqueness property of the system's solutions, we introduce the following refined assumptions:

(A1) There exist positive Lipschitz coefficients $\mathfrak{L}_f, \mathfrak{L}_g > 0$ such that, for all $t \in \mathbb{J}$ and all state variables within the space, the following inequalities are satisfied:

$$|f_2(t, \mathfrak{u}_1(t), \mathfrak{v}_1(t)) - f_2(t, \mathfrak{u}_2(t), \mathfrak{v}_2(t))| \leq \mathfrak{L}_f \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \|\mathfrak{v}_1 - \mathfrak{v}_2\|\right), \quad (23)$$

and

$$|g_2(t, \mathfrak{u}_1(t), \mathfrak{v}_1(t)) - g_2(t, \mathfrak{u}_2(t), \mathfrak{v}_2(t))| \leq \mathfrak{L}_g \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \|\mathfrak{v}_1 - \mathfrak{v}_2\|\right). \quad (24)$$

(A2) The structural parameter of the coupled system fulfills the strict contraction condition:

$$\sigma = \frac{M_f a^{\alpha_1+\alpha_2} \mathfrak{L}_f}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{M_f a^{\alpha_1+\alpha_2} \mathfrak{L}_f}{\Gamma(\alpha_1+\alpha_2)} + \frac{a^{\alpha_2} \lambda_{\mathfrak{u}}}{\Gamma(\alpha_2)} + \frac{M_g a^{\alpha_3+\alpha_4} \mathfrak{L}_g}{\Gamma(1+\alpha_3+\alpha_4)} + \frac{M_g a^{\alpha_3+\alpha_4} \mathfrak{L}_g}{\Gamma(\alpha_3+\alpha_4)} + \frac{a^{\alpha_4} \lambda_{\mathfrak{v}}}{\Gamma(\alpha_4)} < 1. \quad (25)$$

Theorem 7. *Supposing that assumptions (**A1**) and (**A2**) hold, the interconnected fractional differential system (1-2) possesses a unique mild solution $(\mathfrak{u},\mathfrak{v})$ inside the product Banach space $\mathcal{K}$.*

Proof. Let us choose two arbitrary pairs $(\mathfrak{u},\mathfrak{v})$ and $(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}})$ belonging to the product framework $\mathcal{K}$. Evaluating the standard uniform norm of their images under the mapping Φ, we have:

$$\left\|\Phi\left(\mathfrak{u},\mathfrak{v}\right)-\Phi\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}=\left\|\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)-\Phi_1\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\infty}+\left\|\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)-\Phi_2\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\infty}.$$

Estimating the variation for the first operational component, we can write:

$$\begin{aligned}
&\left\|\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)(t)-\Phi_1\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)(t)\right\|\\
\leq\;&\frac{M_f}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}\left\|f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))-f_2(s,\widehat{\mathfrak{u}}(s),\widehat{\mathfrak{v}}(s))\right\|ds\\
&+\frac{aM_f}{\Gamma(\alpha_1+\alpha_2-1)}\int_0^a (a-s)^{\alpha_1+\alpha_2-2}\left\|f_2(s,\mathfrak{u}(s),\mathfrak{v}(s))-f_2(s,\widehat{\mathfrak{u}}(s),\widehat{\mathfrak{v}}(s))\right\|ds\\
&+\frac{\lambda_{\mathfrak{u}}a}{\Gamma(\alpha_2-1)}\int_0^a (a-s)^{\alpha_2-2}\left(\left\|\mathfrak{u}(s)-\widehat{\mathfrak{u}}(s)\right\|+\left\|\mathfrak{v}(s)-\widehat{\mathfrak{v}}(s)\right\|\right)ds.
\end{aligned}$$

By implementing the Lipschitz criterion (**A1**), this inequality simplifies to:

$$\begin{aligned}
&\left\|\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)(t)-\Phi_1\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)(t)\right\|\\
\leq\;&\left(\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\right)\left(\left\|\mathfrak{u}(t)-\widehat{\mathfrak{u}}(t)\right\|+\left\|\mathfrak{v}(t)-\widehat{\mathfrak{v}}(t)\right\|\right)\\
\leq\;&\left(\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\right)\left\|(\mathfrak{u},\mathfrak{v})-\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}.
\end{aligned}$$

Taking the supremum over the whole domain $t\in\mathbb{J}$ yields:

$$\left\|\Phi_1\left(\mathfrak{u},\mathfrak{v}\right)-\Phi_1\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\infty}\leq\left(\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\right)\left\|(\mathfrak{u},\mathfrak{v})-\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}. \tag{26}$$

Adopting a symmetric technique for the second operator component Φ_2, we obtain:

$$\left\|\Phi_2\left(\mathfrak{u},\mathfrak{v}\right)-\Phi_2\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\infty}\leq\left(\frac{M_g a^{\alpha_3+\alpha_4}\mathfrak{L}_g}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{M_g a^{\alpha_3+\alpha_4}\mathfrak{L}_g}{\Gamma(\alpha_3+\alpha_4)}+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(\alpha_4)}\right)\left\|(\mathfrak{u},\mathfrak{v})-\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}. \tag{27}$$

By merging these two uniform evaluations, it follows that:

$$\begin{aligned}
\left\|\Phi\left(\mathfrak{u},\mathfrak{v}\right)-\Phi\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}\;\leq\;&\left[\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f a^{\alpha_1+\alpha_2}\mathfrak{L}_f}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\right.\\
&\left.+\frac{M_g a^{\alpha_3+\alpha_4}\mathfrak{L}_g}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{M_g a^{\alpha_3+\alpha_4}\mathfrak{L}_g}{\Gamma(\alpha_3+\alpha_4)}+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(\alpha_4)}\right]\left\|(\mathfrak{u},\mathfrak{v})-\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}\\
\leq\;&\sigma\left\|(\mathfrak{u},\mathfrak{v})-\left(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\right)\right\|_{\mathcal{K}}.
\end{aligned}$$

In view of condition (**A2**), the mapping Φ acts as a strict contraction over the metric space.

As a final consequence, by applying the Banach Contraction Principle (4), the operator Φ admits a unique fixed point $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{K}$. This coordinate matches the unique mild solution of the hybrid coupled system (1-2), completing the proof. □

5. Numerical Illustration

To evaluate the practical relevance of our analytical findings, we construct a concrete numerical example. Let us consider the sequence space $\Xi = \ell^\infty(\mathbb{R})$ defined by:

$$\Xi = \{\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_n, \cdots) : \mathfrak{u}_i \in \mathbb{R},\ i \in \mathbb{N}\backslash\{0\}\},$$

associated with the standard supremum norm:

$$\|\mathfrak{u}\| = \sup_{i\in\mathbb{N}\backslash\{0\}} \{|\mathfrak{u}_i|\}.$$

Over the compact interval $\mathbb{J} = (0, 1]$, we examine the following coupled hybrid fractional differential system:

$$\begin{cases} {}^{RL}\mathcal{D}^{0.4}_{0^+}\left({}^{C}\mathcal{D}^{1.6}\left[\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right]\right) = f_2(t, \mathfrak{u}(t), \mathfrak{v}(t)), \\ {}^{RL}\mathcal{D}^{0.3}_{0^+}\left({}^{C}\mathcal{D}^{1.7}\left[\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right]\right) = g_2(t, \mathfrak{u}(t), \mathfrak{v}(t)), \end{cases} \tag{28}$$

where the respective scaling and continuous nonlinear functions are given by:

$$\begin{aligned} f_2(t, \mathfrak{u}, \mathfrak{v}) &= \left\{\frac{t}{10}(\mathfrak{u}_n + \mathfrak{v}_n)\right\}_{n\in\mathbb{N}\backslash\{0\}}, \quad g_2(t, \mathfrak{u}, \mathfrak{v}) = \left\{\frac{t}{8}(\mathfrak{u}_n - \mathfrak{v}_n)\right\}_{n\in\mathbb{N}\backslash\{0\}}, \\ f_1(t, \mathfrak{u}(t), \mathfrak{v}(t)) &= \frac{t^2}{100}, \quad g_1(t, \mathfrak{u}(t), \mathfrak{v}(t)) = \frac{t^4}{1000}. \end{aligned}$$

This interconnected system is subjected to the following non-local boundary conditions:

$$\begin{cases} (\mathfrak{u}(0), \mathfrak{v}(0)) = (0, 0), \\ \left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Big|_{t=1} = -0.313\left|\mathcal{I}^{0.6}\mathfrak{v}(t)\right._{t=1}, \\ \left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)'\Big|_{t=1} = -0.308\left|\mathcal{I}^{0.7}\mathfrak{u}(t)\right._{t=1}, \\ {}^{C}\mathcal{D}^{1.6}\left(\frac{\mathfrak{u}(t)}{f_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)\Big|_{t=0} = {}^{C}\mathcal{D}^{1.7}\left(\frac{\mathfrak{v}(t)}{g_1(t,\mathfrak{u}(t),\mathfrak{v}(t))}\right)\Big|_{t=0} = 0. \end{cases} \tag{29}$$

Here, the boundary constants are selected as $\lambda_{\mathfrak{u}} = \lambda_{\mathfrak{v}} = 0.3$.

By setting $a = 1$, we evaluate the parameters corresponding to our theoretical assumptions. For any pair $(\mathfrak{u}, \mathfrak{v}), (\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}) \in \Xi \times \Xi$, the nonlinearities satisfy:

$$\begin{aligned} |f_2(t, \mathfrak{u}, \mathfrak{v}) - f_2(t, \widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})| &\leq \frac{t}{10}(|\mathfrak{u} - \widehat{\mathfrak{u}}| + |\mathfrak{v} - \widehat{\mathfrak{v}}|), \\ |g_2(t, \mathfrak{u}, \mathfrak{v}) - g_2(t, \widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})| &\leq \frac{t}{8}(|\mathfrak{u} - \widehat{\mathfrak{u}}| + |\mathfrak{v} - \widehat{\mathfrak{v}}|). \end{aligned}$$

Hence, the continuous control mappings and perturbation metrics simplify to:

$$\delta_f(t) = \frac{t}{10}, \quad \delta_g(t) = \frac{t}{8}, \quad \widehat{\delta}_{f_2} = \widehat{\delta}_{g_2} = 0,$$

$$M_f = \frac{1}{100}, \quad M_g = \frac{1}{1000}.$$

Substituting these numerical values into the algebraic conditions derived in requirement **(H2)**, we can check that:

$$\begin{aligned}
&\frac{M_f\left(\delta_f(\tau)+\widehat{\delta}_{f_2}\right)a^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)}+\frac{M_f\left(\delta_f(\tau)+\widehat{\delta}_{f_2}\right)a^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}+\frac{\lambda_{\mathfrak{u}}a^{\alpha_2}}{\Gamma(\alpha_2)}\tau\\
&=\left[\frac{1}{100}\left(\frac{1}{10\Gamma(3)}+\frac{1}{10\Gamma(2)}\right)+\frac{0.3}{\Gamma(1.6)}\right]\tau \approx \frac{1}{100}\left(\frac{3}{20}+\frac{0.3}{0.8935}\right)\tau \le 0.5\tau,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\frac{M_g\left(\delta_g(\tau)+\widehat{\delta}_{g_2}\right)a^{\alpha_3+\alpha_4}}{\Gamma(1+\alpha_3+\alpha_4)}+\frac{M_g\left(\delta_g(\tau)+\widehat{\delta}_{g_2}\right)a^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)}+\frac{\lambda_{\mathfrak{v}}a^{\alpha_4}}{\Gamma(1+\alpha_4)}\tau\\
&=\left[\frac{1}{1000}\left(\frac{1}{8\Gamma(3)}+\frac{1}{8\Gamma(2)}\right)+\frac{0.3}{\Gamma(1.7)}\right]\tau \approx \frac{1}{1000}\left(\frac{3}{16}+\frac{0.3}{0.9086}\right)\tau \le 0.5\tau.
\end{aligned}$$

Furthermore, the Lipschitz constants for the uniqueness theorem are given by $\mathfrak{L}_f = \frac{1}{10}$ and $\mathfrak{L}_g = \frac{1}{8}$. Evaluating the total contraction mapping index, we obtain:

$$\sigma \approx 0.3508 + 0.3303 < 1.$$

Consequently, all topological and algebraic prerequisites required by Theorem 5 and Theorem 7 are verified. We can conclude that the coupled fractional system (28) subject to the constraints (29) admits a unique mild solution $(\mathfrak{u}, \mathfrak{v})$ inside the closed ball $\mathfrak{B}_\tau$.

6. Conclusion

This manuscript investigated a novel interconnected hybrid system governed by non-linear fractional differential equations evaluated within a Banach space framework. The core feature of the proposed model lies in its integration of mixed fractional operators, blending the non-local properties of both the Riemann–Liouville and Caputo formulations of varying orders.

The structural analysis centered on defining the qualitative properties of the model, specifically establishing rigorous criteria for the solvability and unique existence of mild solutions. To accomplish this, we implemented a robust combination of analytical instruments from non-linear functional analysis. In particular, the Kuratowski measure of noncompactness was successfully coupled with an extended version of Darbo's fixed-point formulation utilizing a nondecreasing control function to verify the existence criteria. Subsequently, the uniqueness of the state trajectory was demonstrated by verifying Lipschitz-type conditions through the application of the classical Banach contraction mapping principle.

The main findings documented in this work can be summarized as follows:

- We derived a mathematically equivalent integral representation for a complex system of two coupled non-linear fractional equations subject to non-local boundary configurations.
- The structural solvability showing the existence of at least one solution was validated under general noncompactness assumptions via Darbo's topological framework.
- The strict uniqueness of this solution was guaranteed under valid contraction parameters.

In summary, this research provides a comprehensive and rigorous theoretical foundation for analyzing multi-order coupled fractional systems. The tools and methodology detailed in this paper offer valuable mathematical perspectives that can be extended to explore further qualitative properties, such as different stability criteria or numerical approximations, in complex dynamical systems with long-range hereditary impacts.

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