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LENGTH AND MEAN FUZZY SUBALGEBRAS IN HILBERT ALGEBRAS OVER HYPERSTRUCTURES

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Abstract. This paper introduces and studies the notions of the length and the mean of a hyperstructure in Hilbert algebras. By employing these notions, we define the concepts of length fuzzy subalgebras and mean fuzzy subalgebras of Hilbert algebras and investigate their fundamental properties. Several characterization theorems are established, describing the necessary and sufficient conditions for a hyperstructure to form a length (resp., mean) k-fuzzy subalgebra for $k \in \{1, 2, 3, 4\}$. Relationships between length (resp., mean) fuzzy subalgebras and hyperfuzzy subalgebras are examined in detail. Furthermore, we provide results connecting these classes of fuzzy subalgebras with their upper-, lower-, and equal-level subsets, and demonstrate that particular combinations of constant and fuzzy structures naturally lead to length- or mean-type fuzzy subalgebras. Illustrative examples and counterexamples are presented to support the main results.

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1. Introduction

Hyperstructure theory, introduced by Marty [17] in 1934, provides a natural generalization of ordinary algebraic operations and has grown into an active area with applications in several branches of mathematics and computer science [5,6]. Corsini [4] discussed historical developments and broader research perspectives on hyperstructures. A further historical survey, with particular attention to the role of hyperstructures in algebra and logic, was later presented by Golzio [12]. Together, these works show that hyperstructure theory has developed into a useful framework for extending classical algebraic ideas in several directions.

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Hilbert algebras arise from the algebraic study of implication-type logical systems. The subject goes back to the work of Henkin [13] and was later developed systematically by Diego [7], who showed that Hilbert algebras form a locally finite variety. Subsequent studies by Busneag [1,2] and Jun [15] examined important structural properties and deductive systems of Hilbert algebras, while Chajda and Halaš [3] investigated congruences and ideals in this setting. From the fuzzy point of view, Dudek and related works contributed to the development of fuzzification and fuzzy ideals in Hilbert algebras, which now provide a useful background for further investigations in this direction [8–10].

The combination of hyperstructures and fuzzy methods is mathematically meaningful, since hyperstructures describe multivalued behavior, while fuzzy sets capture graded membership. Therefore, hyperfuzzy models offer a convenient framework for studying algebraic phenomena under uncertainty. However, quantitative features of a hyperstructure, especially its length and mean, have not yet been systematically studied in the context of Hilbert algebras, to the best of our knowledge.

Some recent works indicate that quantitative measures such as length and mean can yield meaningful algebraic results in fuzzy settings. Tacha et al. [20] introduced length and mean fuzzy UP-subalgebras in UP-algebras. Rajesh et al. [19] later studied length and mean fuzzy ideals in Sheffer stroke Hilbert algebras. In a related direction, Iampan et al. [14] established characterizations of length and mean-fuzzy ideals in Hilbert algebras over hyperstructures, while Rajesh et al. [18] investigated length and mean-fuzzy subalgebras in Sheffer stroke Hilbert algebras. These studies further show that length- and mean-based approaches provide a useful way to examine fuzzy algebraic structures in different logical and hyperstructural settings. Motivated by these developments, in the present paper, we consider the notions of length and mean for hyperstructures over Hilbert algebras, with emphasis on fuzzy subalgebras.

In this paper, we define the length and the mean of a hyperstructure over a Hilbert algebra and use these notions to introduce length fuzzy subalgebras and mean fuzzy subalgebras. After recalling the notion of k-fuzzy subalgebras for $k \in \{1, 2, 3, 4\}$, we establish characterization results for the corresponding length and mean cases. We also investigate their relationships with hyperfuzzy subalgebras and describe them in terms of upper-, lower-, and equal-level subsets. In addition, we show that certain combinations involving constant lower or upper fuzzy structures naturally produce length-type or mean-type fuzzy subalgebras. Several examples and counterexamples are included to illustrate the scope of the results.

2. Preliminaries

In this section, we recall the basic notions and notation needed in the sequel. These concepts will be used in the study of k-fuzzy subalgebras and in the formulations of length and mean fuzzy subalgebras over hyperstructures in Hilbert algebras.

Definition 2.1 ([7]). *A Hilbert algebra is an algebra of the form $X = (X, *, 1)$, where X is a non-empty set, $*$ is a binary operation on X, and $1 \in X$, satisfying the following axioms:*

$$(\forall x, y \in X)\big(x * (y * x) = 1\big), \tag{2.1}$$

$$(\forall x, y, z \in X)\big((x * (y * z)) * ((x * y) * (x * z)) = 1\big), \tag{2.2}$$

$$(\forall x, y \in X)\big(x * y = 1 \text{ and } y * x = 1 \Rightarrow x = y\big). \tag{2.3}$$

The following basic properties are standard and will be used repeatedly.

Lemma 2.1 ([8]). *Let $X = (X, *, 1)$ be a Hilbert algebra. Then:*

(1) $(\forall x \in X)(x * x = 1)$;

(2) $(\forall x \in X)(1 * x = x)$;

(3) $(\forall x \in X)(x * 1 = 1)$;

(4) $(\forall x, y, z \in X)(x * (y * z) = y * (x * z))$;

(5) $(\forall x, y, z \in X)\big((x * z) * ((z * y) * (x * y)) = 1\big)$.

For a Hilbert algebra $X = (X, *, 1)$, define a binary relation $\leqslant$ on X by

$$(\forall x, y \in X)(x \leqslant y \iff x * y = 1).$$

Then $\leqslant$ is a partial order on X, and 1 is the greatest element of X.

Definition 2.2 ([16]). *Let $X = (X, *, 1)$ be a Hilbert algebra. A non-empty subset S of X is called a subalgebra of X if*

$$(\forall x, y \in S)(x * y \in S).$$

Definition 2.3 ([21]). *A fuzzy set in a non-empty set X is a mapping $\mu : X \to [0, 1]$.*

Definition 2.4 ([11]). *Let X be a non-empty set. A mapping $\widetilde{f} : X \to \widetilde{P}([0, 1])$ is called a hyperfuzzy set over X, where $\widetilde{P}([0, 1])$ denotes the family of all non-empty subsets of $[0, 1]$. The ordered pair $(X, \widetilde{f})$ is called a hyperstructure over X.*

Definition 2.5. *Let $(X, \widetilde{f})$ be a hyperstructure over a non-empty set X. We associate with $\widetilde{f}$ two ordinary fuzzy sets, denoted by $\widetilde{f}_{\inf}$ and $\widetilde{f}_{\sup}$, defined by*

$$\widetilde{f}_{\inf} : X \to [0, 1], \qquad \widetilde{f}_{\inf}(x) = \inf \widetilde{f}(x),$$

and

$$\widetilde{f}_{\sup} : X \to [0, 1], \qquad \widetilde{f}_{\sup}(x) = \sup \widetilde{f}(x),$$

for all $x \in X$.

3. On k-Fuzzy Subalgebras

In this section, we introduce the notion of k-fuzzy subalgebras in Hilbert algebras for $k \in \{1, 2, 3, 4\}$. These four types will be used throughout the paper as the basic framework for studying length fuzzy subalgebras and mean fuzzy subalgebras. We also record some simple but useful relations among them that will be used in the later sections.

Definition 3.1. *Let $X = (X, *, 1)$ be a Hilbert algebra, and let $f : X \to [0, 1]$ be a fuzzy set on X. Then (X, f) is called a*

(1) 1-*fuzzy subalgebra of X if*

$$(\forall x, y \in X)\big(f(x * y) \geqslant \min\{f(x), f(y)\}\big); \tag{3.1}$$

(2) 2-*fuzzy subalgebra of X if*

$$(\forall x, y \in X)\big(f(x * y) \leqslant \min\{f(x), f(y)\}\big); \tag{3.2}$$

(3) 3-*fuzzy subalgebra of X if*

$$(\forall x, y \in X)\big(f(x * y) \geqslant \max\{f(x), f(y)\}\big); \tag{3.3}$$

(4) 4-*fuzzy subalgebra of X if*

$$(\forall x, y \in X)\big(f(x * y) \leqslant \max\{f(x), f(y)\}\big). \tag{3.4}$$

Proposition 3.1. *Let (X, f) be a k-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$.*

(1) *If $k \in \{1, 3\}$, then*

$$(\forall x \in X)\big(f(1) \geqslant f(x)\big).$$

(2) *If $k \in \{2, 4\}$, then*

$$(\forall x \in X)\big(f(1) \leqslant f(x)\big).$$

Proof. If (X, f) is a 1-fuzzy subalgebra, then for every $x \in X$,

$$f(1) = f(x * x) \geqslant \min\{f(x), f(x)\} = f(x),$$

since $x * x = 1$. If (X, f) is a 3-fuzzy subalgebra, then

$$f(1) = f(x * x) \geqslant \max\{f(x), f(x)\} = f(x).$$

Hence, $f(1) \geqslant f(x)$ for all $x \in X$.

Similarly, if (X, f) is a 2-fuzzy subalgebra, then

$$f(1) = f(x * x) \leqslant \min\{f(x), f(x)\} = f(x),$$

and if (X, f) is a 4-fuzzy subalgebra, then

$$f(1) = f(x * x) \leqslant \max\{f(x), f(x)\} = f(x).$$

Thus $f(1) \leqslant f(x)$ for all $x \in X$. $\square$

Theorem 3.1. *Every 3-fuzzy subalgebra of a Hilbert algebra is a 1-fuzzy subalgebra.*

Proof. Let (X, f) be a 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$. Then, for all $x, y \in X$,

$$f(x * y) \geqslant \max\{f(x), f(y)\} \geqslant \min\{f(x), f(y)\}.$$

Therefore, (X, f) is a 1-fuzzy subalgebra of X. $\square$

Example 3.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set equipped with the binary operation $*$ given by*

*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0	7
2	0	1	0	3	4	5	6	7
3	0	1	0	0	4	5	6	7
4	0	1	0	0	0	5	6	7
5	0	1	0	0	0	0	6	7
6	0	1	0	0	0	0	0	7
7	0	1	0	0	0	0	6	0

Then X is a Hilbert algebra. Define a fuzzy set f on X by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0.7 & 0.5 & 0.1 & 0.4 & 0.5 & 0.1 & 0.4 & 0.2 \end{pmatrix}.$$

One can verify that f is a 1-fuzzy subalgebra of X. However,

$$f(0 * 4) = f(4) = 0.5 < 0.7 = \max\{f(0), f(4)\},$$

so f is not a 3-fuzzy subalgebra of X.

Theorem 3.2. *Every 2-fuzzy subalgebra of a Hilbert algebra is a 4-fuzzy subalgebra.*

Proof. Let (X, f) be a 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$. Then, for all $x, y \in X$,

$$f(x * y) \leqslant \min\{f(x), f(y)\} \leqslant \max\{f(x), f(y)\}.$$

Hence, (X, f) is a 4-fuzzy subalgebra of X. $\square$

Example 3.2. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set equipped with the binary operation* $*$ *given by*

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	7	7
2	0	1	0	3	4	5	6	7
3	0	1	0	0	4	5	6	7
4	0	1	0	0	0	5	6	7
5	0	1	0	0	0	0	6	7
6	0	0	0	0	0	0	0	0
7	0	1	0	0	4	5	1	0

Then X *is a Hilbert algebra. Define a fuzzy set* f *on* X *by*

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0.1 & 0.5 & 0.1 & 0.2 & 0.1 & 0.5 & 0.6 & 0.6 \end{pmatrix}.$$

Then f *is a* 4*-fuzzy subalgebra of* X*. On the other hand,*

$$f(5 * 7) = f(7) = 0.6 > 0.5 = \min\{f(5), f(7)\},$$

and therefore f *is not a* 2*-fuzzy subalgebra of* X*.*

Theorem 3.3. *A fuzzy set* (X, f) *in a Hilbert algebra* $X = (X, *, 1)$ *is a* 2*-fuzzy subalgebra of* X *if and only if* f *is constant.*

Proof. Assume that (X, f) is a 2-fuzzy subalgebra of X. By Proposition 3.1, we have $f(1) \leqslant f(x)$ for all $x \in X$. Also, by Lemma 2.1(2),

$$f(x) = f(1 * x) \leqslant \min\{f(1), f(x)\} = f(1)$$

for every $x \in X$. Thus $f(x) = f(1)$ for all $x \in X$, so f is constant.

Conversely, if f is constant, then condition (3.2) holds trivially. Hence, (X, f) is a 2-fuzzy subalgebra of X. □

Theorem 3.4. *A fuzzy set* (X, f) *in a Hilbert algebra* $X = (X, *, 1)$ *is a* 3*-fuzzy subalgebra of* X *if and only if* f *is constant.*

Proof. Assume that (X, f) is a 3-fuzzy subalgebra of X. By Proposition 3.1, we have $f(1) \geqslant f(x)$ for all $x \in X$. Using Lemma 2.1(2), we obtain

$$f(x) = f(1 * x) \geqslant \max\{f(1), f(x)\} = f(1)$$

for all $x \in X$. Hence, $f(x) = f(1)$ for every $x \in X$, and so f is constant.

Conversely, every constant fuzzy set satisfies (3.3) immediately. Therefore, (X, f) is a 3-fuzzy subalgebra of X. □

As an immediate consequence of Theorems 3.3 and 3.4, the classes of 2-fuzzy subalgebras, 3-fuzzy subalgebras, and constant fuzzy sets on a Hilbert algebra are the same.

4. Length of a Hyperstructure in Hilbert Algebras

In this section, we study the length of a hyperstructure over a Hilbert algebra and use it to define several classes of length fuzzy subalgebras. Our aim is to describe how the induced fuzzy set $\widetilde{f}_{\mathfrak{l}}$ behaves with respect to the binary operation of a Hilbert algebra, and to identify conditions under which it becomes a k-fuzzy subalgebra for $k \in \{1, 2, 3, 4\}$. We also examine its connections with upper-, lower-, and equal-level subsets, together with some relations to hyperfuzzy subalgebras. These results will be used later in the study of mean fuzzy subalgebras.

Definition 4.1. *Let $(X, \widetilde{f})$ be a hyperstructure over a non-empty set X. We define a fuzzy set $(X, \widetilde{f}_{\mathfrak{l}})$ on X by*

$$\widetilde{f}_{\mathfrak{l}} : X \to [0, 1], \qquad x \mapsto \widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x).$$

The fuzzy set $\widetilde{f}_{\mathfrak{l}}$ is called the length of $\widetilde{f}$.

Definition 4.2. *Let $(X, \widetilde{f})$ be a hyperstructure over a Hilbert algebra $X = (X, *, 1)$. Then $(X, \widetilde{f})$ is called a length 1-fuzzy (resp., length 2-fuzzy, length 3-fuzzy, length 4-fuzzy) subalgebra of X if $(X, \widetilde{f}_{\mathfrak{l}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) subalgebra of X.*

Proposition 4.1. *If $(X, \widetilde{f})$ is a length k-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ for $k = 1, 3$, then*

$$(\forall x \in X)\big(\widetilde{f}_{\mathfrak{l}}(1) \geqslant \widetilde{f}_{\mathfrak{l}}(x)\big). \tag{4.1}$$

Proof. It follows directly from Proposition 3.1. □

Proposition 4.2. *If $(X, \widetilde{f})$ is a length k-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ for $k = 2, 4$, then*

$$(\forall x \in X)\big(\widetilde{f}_{\mathfrak{l}}(1) \leqslant \widetilde{f}_{\mathfrak{l}}(x)\big). \tag{4.2}$$

Proof. It follows directly from Proposition 3.1. □

Theorem 4.1. *Every length 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy subalgebra of X.*

Proof. This is a direct consequence of Theorem 3.1. □

Example 4.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $*$ defined by the following Cayley table:*

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	6	0
2	0	1	0	3	4	5	6	7
3	0	1	0	0	4	5	6	7
4	0	1	0	0	0	5	6	7
5	0	1	0	0	0	0	6	7
6	0	1	0	0	0	0	0	0
7	0	1	0	0	0	0	6	0

Then X is a Hilbert algebra. Let $(X, \widetilde{f})$ be a hyperstructure over X given by

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad x \mapsto \begin{cases} (0, 0.9) & \text{if } x = 0, \\ [0.3, 0.5) \cup [0.6, 0.7] & \text{if } x = 1, \\ [0.1, 0.9] & \text{if } x = 2, \\ (0.3, 1) & \text{if } x = 3, \\ (0.5, 0.9] & \text{if } x = 4, \\ (0.2, 0.8) & \text{if } x = 5, \\ [0.4, 0.9] & \text{if } x = 6, \\ (0.2, 0.4) \cup (0.4, 0.6) & \text{if } x = 7. \end{cases}$$

Hence, the length of $\widetilde{f}$ is

$$\widetilde{f}_{\mathfrak{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0.9 & 0.4 & 0.8 & 0.7 & 0.4 & 0.6 & 0.5 & 0.4 \end{pmatrix}.$$

Then $\widetilde{f}_{\mathfrak{l}}$ is a 1-fuzzy subalgebra of X, and therefore $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X. On the other hand,

$$\widetilde{f}_{\mathfrak{l}}(5 * 6) = \widetilde{f}_{\mathfrak{l}}(6) = 0.5 < 0.6 = \max\{\widetilde{f}_{\mathfrak{l}}(5), \widetilde{f}_{\mathfrak{l}}(6)\}.$$

Thus, $\widetilde{f}_{\mathfrak{l}}$ is not a 3-fuzzy subalgebra of X, and so $(X, \widetilde{f})$ is not a length 3-fuzzy subalgebra of X.

Theorem 4.2. *Every length 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy subalgebra of X.*

Proof. This is a direct consequence of Theorem 3.2. $\square$

Example 4.2. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $*$ *defined by the following Cayley table:*

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	7	7
2	0	1	0	3	4	5	6	7
3	0	1	0	0	4	5	6	7
4	0	1	0	0	0	5	6	7
5	0	1	0	0	0	0	6	7
6	0	0	0	0	0	0	0	0
7	0	1	2	3	4	5	1	0

Then X *is a Hilbert algebra. Let* $(X, \widetilde{f})$ *be a hyperstructure over* X *given by*

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad x \mapsto \begin{cases} (0.8, 0.9) & \text{if } x = 0, \\ [0.4, 0.6) & \text{if } x = 1, \\ (0.7, 1] & \text{if } x = 2, \\ [0.1, 0.5] & \text{if } x = 3, \\ (0.5, 0.6] \cup [0.8, 1) & \text{if } x = 4, \\ [0.3, 0.4) \cup (0.4, 0.9] & \text{if } x = 5, \\ [0, 0.1) \cup [0.7, 0.8) & \text{if } x = 6, \\ (0.2, 0.4] \cup (0.5, 1] & \text{if } x = 7. \end{cases}$$

Hence, the length of $\widetilde{f}$ *is*

$$\widetilde{f}_{\mathfrak{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.8 & 0.8 \end{pmatrix}.$$

Then $\widetilde{f}_{\mathfrak{l}}$ *is a* 4*-fuzzy subalgebra of* X*, and therefore* $(X, \widetilde{f})$ *is a length* 4*-fuzzy subalgebra of* X*. However,*

$$\widetilde{f}_{\mathfrak{l}}(0 * 3) = \widetilde{f}_{\mathfrak{l}}(3) = 0.4 > 0.1 = \min\{\widetilde{f}_{\mathfrak{l}}(0), \widetilde{f}_{\mathfrak{l}}(3)\}.$$

Thus, $\widetilde{f}_{\mathfrak{l}}$ *is not a* 2*-fuzzy subalgebra of* X*, and so* $(X, \widetilde{f})$ *is not a length* 2*-fuzzy subalgebra of* X*.*

Theorem 4.3. *For a hyperstructure* $(X, \widetilde{f})$ *over a Hilbert algebra* $X = (X, *, 1)$*, the following statements are equivalent:*

(1) $(X, \widetilde{f})$ *is a length* 2*-fuzzy subalgebra of* X*;*

(2) $(X, \widetilde{f})$ *is a length* 3*-fuzzy subalgebra of* X*;*

(3) $\widetilde{f}_{\mathfrak{l}}$ *is constant on* X*.*

Proof. This follows directly from Theorems 3.3 and 3.4, applied to the fuzzy set $\widetilde{f}_{\mathfrak{l}}$. □

Theorem 4.4. *Given a subalgebra S of a Hilbert algebra $X = (X, *, 1)$ and $B_1, B_2 \in \widetilde{P}([0,1])$, let $(X, \widetilde{f})$ be a hyperstructure over X defined by*

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{if } x \notin S. \end{cases}$$

Then:

(1) *if $B_1 \subset B_2$, then $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X;*
(2) *if $B_2 \subset B_1$, then $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X.*

Proof. If $x \in S$, then $\widetilde{f}(x) = B_2$ and hence

$$\widetilde{f}_l(x) = \sup B_2 - \inf B_2.$$

If $x \notin S$, then $\widetilde{f}(x) = B_1$ and hence

$$\widetilde{f}_l(x) = \sup B_1 - \inf B_1.$$

(1) Assume that $B_1 \subset B_2$. Then

$$\sup B_2 - \inf B_2 \geqslant \sup B_1 - \inf B_1.$$

Let $x, y \in X$.

If $x, y \in S$, then $x * y \in S$ because S is a subalgebra, and so

$$\widetilde{f}_l(x * y) = \sup B_2 - \inf B_2 = \min\{\widetilde{f}_l(x), \widetilde{f}_l(y)\}.$$

If at least one of x, y does not belong to S, then

$$\min\{\widetilde{f}_l(x), \widetilde{f}_l(y)\} = \sup B_1 - \inf B_1,$$

while $\widetilde{f}_l(x * y)$ is either $\sup B_1 - \inf B_1$ or $\sup B_2 - \inf B_2$, and hence

$$\widetilde{f}_l(x * y) \geqslant \sup B_1 - \inf B_1 = \min\{\widetilde{f}_l(x), \widetilde{f}_l(y)\}.$$

Therefore, $\widetilde{f}_l$ is a 1-fuzzy subalgebra of X.

(2) Assume that $B_2 \subset B_1$. Then

$$\sup B_2 - \inf B_2 \leqslant \sup B_1 - \inf B_1.$$

Let $x, y \in X$.

If $x, y \in S$, then $x * y \in S$, so

$$\widetilde{f}_l(x * y) = \sup B_2 - \inf B_2 = \max\{\widetilde{f}_l(x), \widetilde{f}_l(y)\}.$$

If at least one of x, y does not belong to S, then

$$\max\{\widetilde{f}_l(x), \widetilde{f}_l(y)\} = \sup B_1 - \inf B_1,$$

while $\widetilde{f}_{\mathfrak{l}}(x * y)$ is either $\sup B_1 - \inf B_1$ or $\sup B_2 - \inf B_2$, and hence

$$\widetilde{f}_{\mathfrak{l}}(x * y) \leqslant \sup B_1 - \inf B_1 = \max\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{l}}$ is a 4-fuzzy subalgebra of X. □

Definition 4.3. *Let (X, f) be a fuzzy structure in a non-empty set X. For any $t \in [0, 1]$, define*

$$\mathcal{U}(f, t) = \{x \in X \mid f(x) \geqslant t\},$$

$$\mathcal{L}(f, t) = \{x \in X \mid f(x) \leqslant t\},$$

and

$$\mathcal{E}(f, t) = \{x \in X \mid f(x) = t\}.$$

These are called the upper t-level subset, the lower t-level subset, and the equal t-level subset of f, respectively.

Theorem 4.5. *A hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy subalgebra of X if and only if the set $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$ is a subalgebra of X for all $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t) \neq \varnothing$.*

Proof. Assume that $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X. Let $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t) \neq \varnothing$, and let $x, y \in \mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$. Then

$$\widetilde{f}_{\mathfrak{l}}(x) \geqslant t \qquad \text{and} \qquad \widetilde{f}_{\mathfrak{l}}(y) \geqslant t.$$

Since $\widetilde{f}_{\mathfrak{l}}$ is a 1-fuzzy subalgebra,

$$\widetilde{f}_{\mathfrak{l}}(x * y) \geqslant \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\} \geqslant t.$$

Thus, $x * y \in \mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$, so $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$ is a subalgebra of X.

Conversely, assume that $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$ is a subalgebra of X for every $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t) \neq \varnothing$. Let $x, y \in X$, and put

$$t = \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.$$

Then $x, y \in \mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t) \neq \varnothing$, so $x * y \in \mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$. Hence,

$$\widetilde{f}_{\mathfrak{l}}(x * y) \geqslant t = \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{l}}$ is a 1-fuzzy subalgebra of X. □

Corollary 4.1. *If $(X, \widetilde{f})$ is a length 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t)$ is a subalgebra of X for all $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{l}}, t) \neq \varnothing$.*

Proof. It follows from Theorems 4.1 and 4.5. □

The following example shows that the converse of Corollary 4.1 does not hold in general.

Example 4.3. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ given in Table 1.*

Table 1. Cayley table for the binary operation $*$ on $X = \{1, x, y, z, 0\}$

$*$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	1	1	z	0
z	1	1	1	1	0
0	1	1	1	1	1

Let $(X, \widetilde{f})$ be a hyperstructure over $(X, *, 1)$ defined by

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad u \mapsto \begin{cases} [0.2, 0.3] \cup [0.4, 0.8) & \text{if } u = 1, \\ (0.4, 0.7] & \text{if } u = x, \\ [0.1, 0.6] & \text{if } u = y, \\ [0.3, 0.5) \cup (0.5, 0.7) & \text{if } u = z, \\ (0.6, 0.9] & \text{if } u = 0. \end{cases}$$

Then the length of $\widetilde{f}$ is given by Table 2.

Table 2. The length of $(X, \widetilde{f})$

X	1	x	y	z	0
$\widetilde{f}_\ell$	0.6	0.3	0.5	0.4	0.3

Hence,

$$\mathcal{U}(\widetilde{f}_\ell, t) = \begin{cases} \emptyset & \text{if } t \in (0.6, 1], \\ \{1\} & \text{if } t \in (0.5, 0.6], \\ \{1, y\} & \text{if } t \in (0.4, 0.5], \\ \{1, y, z\} & \text{if } t \in (0.3, 0.4], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Thus, $\mathcal{U}(\widetilde{f}_\ell, t)$ is a subalgebra of $(X, *, 1)$ for every $t \in [0,1]$ with $\mathcal{U}(\widetilde{f}_\ell, t) \neq \emptyset$. However,

$$\widetilde{f}_\ell(1 * x) = \widetilde{f}_\ell(x) = 0.3 < 0.6 = \max\{\widetilde{f}_\ell(1), \widetilde{f}_\ell(x)\}.$$

Therefore, $(X, \widetilde{f})$ is not a length 3-fuzzy subalgebra of $(X, *, 1)$.

Theorem 4.6. *A hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy subalgebra of X if and only if the set $\mathcal{L}(\widetilde{f}_\ell, t)$ is a subalgebra of X for all $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_\ell, t) \neq \emptyset$.*

Proof. Assume that $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X. Let $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_\ell, t) \neq \emptyset$, and let $x, y \in \mathcal{L}(\widetilde{f}_\ell, t)$. Then

$$\widetilde{f}_\ell(x) \leqslant t \qquad \text{and} \qquad \widetilde{f}_\ell(y) \leqslant t.$$

Since $\widetilde{f}_\mathfrak{l}$ is a 4-fuzzy subalgebra,

$$\widetilde{f}_\mathfrak{l}(x * y) \leqslant \max\{\widetilde{f}_\mathfrak{l}(x), \widetilde{f}_\mathfrak{l}(y)\} \leqslant t.$$

Thus, $x * y \in \mathcal{L}(\widetilde{f}_\mathfrak{l}, t)$, so $\mathcal{L}(\widetilde{f}_\mathfrak{l}, t)$ is a subalgebra of X.

Conversely, assume that $\mathcal{L}(\widetilde{f}_\mathfrak{l}, t)$ is a subalgebra of X for every $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_\mathfrak{l}, t) \neq \varnothing$. Let $x, y \in X$, and put

$$t = \max\{\widetilde{f}_\mathfrak{l}(x), \widetilde{f}_\mathfrak{l}(y)\}.$$

Then $x, y \in \mathcal{L}(\widetilde{f}_\mathfrak{l}, t) \neq \varnothing$, so $x * y \in \mathcal{L}(\widetilde{f}_\mathfrak{l}, t)$. Hence,

$$\widetilde{f}_\mathfrak{l}(x * y) \leqslant t = \max\{\widetilde{f}_\mathfrak{l}(x), \widetilde{f}_\mathfrak{l}(y)\}.$$

Therefore, $\widetilde{f}_\mathfrak{l}$ is a 4-fuzzy subalgebra of X. □

Corollary 4.2. *If $(X, \widetilde{f})$ is a length 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{L}(\widetilde{f}_\mathfrak{l}, t)$ is a subalgebra of X for all $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_\mathfrak{l}, t) \neq \varnothing$.*

Proof. It follows from Theorems 4.2 and 4.6. □

The following example shows that the converse of Corollary 4.2 does not hold in general.

Example 4.4. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ given in Table 1. Let $(X, \widetilde{f})$ be a hyperstructure over $(X, *, 1)$ defined by*

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad u \mapsto \begin{cases} [0.4, 0.5) \cup (0.5, 0.8) & \text{if } u = 1, \\ (0.2, 0.7] & \text{if } u = x, \\ [0.3, 0.8] & \text{if } u = y, \\ [0.1, 0.5) \cup (0.5, 0.8) & \text{if } u = z, \\ (0.15, 0.9] & \text{if } u = 0. \end{cases}$$

Then the length of $\widetilde{f}$ is given by Table 3.

Table 3. The length of $(X, \widetilde{f})$

X	1	x	y	z	0
$\widetilde{f}_\mathfrak{l}$	0.4	0.5	0.5	0.7	0.75

Hence,

$$\mathcal{L}(\widetilde{f}_\mathfrak{l}, t) = \begin{cases} \varnothing & \text{if } t \in [0, 0.4), \\ \{1\} & \text{if } t \in [0.4, 0.5), \\ \{1, x, y\} & \text{if } t \in [0.5, 0.7), \\ \{1, x, y, z\} & \text{if } t \in [0.7, 0.75), \\ X & \text{if } t \in [0.75, 1]. \end{cases}$$

Thus, $\mathcal{L}(\widetilde{f}_{\mathfrak{l}}, t)$ is a subalgebra of X for every $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_{\mathfrak{l}}, t) \neq \emptyset$. However,

$$\widetilde{f}_{\mathfrak{l}}(1 * x) = \widetilde{f}_{\mathfrak{l}}(x) = 0.5 > 0.4 = \min\{\widetilde{f}_{\mathfrak{l}}(1), \widetilde{f}_{\mathfrak{l}}(x)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{l}}$ is not a 2-fuzzy subalgebra of X. Consequently, $(X, \widetilde{f})$ is not a length 2-fuzzy subalgebra of X.

Theorem 4.7. *A hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$ is a length $2(3)$-fuzzy subalgebra of X if and only if*

$$\mathcal{E}\big(\widetilde{f}_{\mathfrak{l}}, \widetilde{f}_{\mathfrak{l}}(1)\big) = X.$$

Proof. Assume that $(X, \widetilde{f})$ is a length 2-fuzzy or length 3-fuzzy subalgebra of X. Then $\widetilde{f}_{\mathfrak{l}}$ is a 2-fuzzy or 3-fuzzy subalgebra of X. By Theorems 3.3 and 3.4, $\widetilde{f}_{\mathfrak{l}}$ is constant. Hence,

$$\widetilde{f}_{\mathfrak{l}}(x) = \widetilde{f}_{\mathfrak{l}}(1) \qquad \text{for all } x \in X,$$

so $\mathcal{E}(\widetilde{f}_{\mathfrak{l}}, \widetilde{f}_{\mathfrak{l}}(1)) = X$.

Conversely, if $\mathcal{E}(\widetilde{f}_{\mathfrak{l}}, \widetilde{f}_{\mathfrak{l}}(1)) = X$, then $\widetilde{f}_{\mathfrak{l}}$ is constant on X. By Theorems 3.3 and 3.4, $\widetilde{f}_{\mathfrak{l}}$ is both a 2-fuzzy and a 3-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a length $2(3)$-fuzzy subalgebra of X. □

Theorem 4.8. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\inf})$ is constant and $(X, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \widetilde{f}_{\inf})$ is constant, we have

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

Since $(X, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of X,

$$\widetilde{f}_{\sup}(x * y) \geqslant \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.$$

Therefore,

$$\begin{aligned}
\widetilde{f}_{\mathfrak{l}}(x * y) &= \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(x * y) \\
&= \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(1) \\
&\geqslant \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\} - \widetilde{f}_{\inf}(1) \\
&= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(1), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(1)\} \\
&= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\} \\
&= \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\mathfrak{l}})$ is a 1-fuzzy subalgebra of X. □

Corollary 4.3. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\inf})$ is constant and $(X, \widetilde{f}_{\sup})$ is a 3-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.1 and 4.8. □

Corollary 4.4. *For $j \in \{1,3\}$, every $(2(3),j)$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X,*,1)$ is a length 1-fuzzy subalgebra of X.*

Proof. It follows from Theorem 4.8 and Corollary 4.3. □

Theorem 4.9. *If $(X,\widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X,*,1)$ in which $(X,\widetilde{f}_{\inf})$ is constant and $(X,\widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X, then $(X,\widetilde{f})$ is a length 4-fuzzy subalgebra of X.*

Proof. Let $x,y \in X$. Since $(X,\widetilde{f}_{\inf})$ is constant, we have

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

Since $(X,\widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X,

$$\widetilde{f}_{\sup}(x*y) \leqslant \max\{\widetilde{f}_{\sup}(x),\widetilde{f}_{\sup}(y)\}.$$

Therefore,

$$\begin{aligned}
\widetilde{f}_{\mathfrak{l}}(x*y) &= \widetilde{f}_{\sup}(x*y) - \widetilde{f}_{\inf}(x*y)\\
&= \widetilde{f}_{\sup}(x*y) - \widetilde{f}_{\inf}(1)\\
&\leqslant \max\{\widetilde{f}_{\sup}(x),\widetilde{f}_{\sup}(y)\} - \widetilde{f}_{\inf}(1)\\
&= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(1),\widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(1)\}\\
&= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x),\widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\}\\
&= \max\{\widetilde{f}_{\mathfrak{l}}(x),\widetilde{f}_{\mathfrak{l}}(y)\}.
\end{aligned}$$

Hence, $(X,\widetilde{f}_{\mathfrak{l}})$ is a 4-fuzzy subalgebra of X. □

Corollary 4.5. *If $(X,\widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X,*,1)$ in which $(X,\widetilde{f}_{\inf})$ is constant and $(X,\widetilde{f}_{\sup})$ is a 2-fuzzy subalgebra of X, then $(X,\widetilde{f})$ is a length 4-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.2 and 4.9. □

Corollary 4.6. *For $j \in \{2,4\}$, every $(2(3),j)$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X,*,1)$ is a length 4-fuzzy subalgebra of X.*

Proof. It follows from Theorem 4.9 and Corollary 4.5. □

Theorem 4.10. *If $(X,\widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X,*,1)$ in which $(X,\widetilde{f}_{\sup})$ is constant and $(X,\widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X, then $(X,\widetilde{f})$ is a length 1-fuzzy subalgebra of X.*

Proof. Let $x,y \in X$. Since $(X,\widetilde{f}_{\sup})$ is constant, we have

$$\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(1) \qquad \text{for all } x \in X.$$

Since $(X,\widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X,

$$\widetilde{f}_{\inf}(x*y) \leqslant \max\{\widetilde{f}_{\inf}(x),\widetilde{f}_{\inf}(y)\}.$$

Therefore,

$$\begin{aligned}
\widetilde{f}_{\mathfrak{l}}(x * y) &= \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(x * y) \\
&= \widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(x * y) \\
&\geqslant \widetilde{f}_{\sup}(1) - \max\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\} \\
&= \min\{\widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(y)\} \\
&= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\} \\
&= \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\mathfrak{l}})$ is a 1-fuzzy subalgebra of X. □

Corollary 4.7. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\sup})$ is constant and $(X, \widetilde{f}_{\inf})$ is a 2-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.2 and 4.10. □

Corollary 4.8. *For $i \in \{2, 4\}$, every $(i, 2(3))$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a length 1-fuzzy subalgebra of X.*

Proof. It follows from Theorem 4.10 and Corollary 4.7. □

Theorem 4.11. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\sup})$ is constant and $(X, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Since $(X, \widetilde{f}_{\sup})$ is constant, we have

$$\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(1) \qquad \text{for all } x \in X.$$

Since $(X, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X,

$$\widetilde{f}_{\inf}(x * y) \geqslant \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}.$$

Therefore,

$$\begin{aligned}
\widetilde{f}_{\mathfrak{l}}(x * y) &= \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(x * y) \\
&= \widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(x * y) \\
&\leqslant \widetilde{f}_{\sup}(1) - \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\} \\
&= \max\{\widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(y)\} \\
&= \max\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\inf}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\inf}(y)\} \\
&= \max\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\mathfrak{l}})$ is a 4-fuzzy subalgebra of X. □

Corollary 4.9. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\sup})$ is constant and $(X, \widetilde{f}_{\inf})$ is a 3-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.1 and 4.11. □

Corollary 4.10. *For $i \in \{1, 3\}$, every $(i, 2(3))$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a length 4-fuzzy subalgebra of X.*

Proof. It follows from Theorem 4.11 and Corollary 4.9. □

Theorem 4.12. *If $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a length 1-fuzzy subalgebra of X and that $\widetilde{f}_{\inf}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\inf}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{l}}(x * y) = \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(1),$$

so

$$\begin{aligned} \widetilde{f}_{\sup}(x * y) &= \widetilde{f}_{\mathfrak{l}}(x * y) + \widetilde{f}_{\inf}(1) \\ &\geqslant \min\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\} + \widetilde{f}_{\inf}(1) \\ &= \min\{\widetilde{f}_{\mathfrak{l}}(x) + \widetilde{f}_{\inf}(1), \widetilde{f}_{\mathfrak{l}}(y) + \widetilde{f}_{\inf}(1)\} \\ &= \min\{\widetilde{f}_{\mathfrak{l}}(x) + \widetilde{f}_{\inf}(x), \widetilde{f}_{\mathfrak{l}}(y) + \widetilde{f}_{\inf}(y)\} \\ &= \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}. \end{aligned}$$

Hence, $(X, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X. □

Corollary 4.11. *If $(X, \widetilde{f})$ is a length 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.1 and 4.12. □

Theorem 4.13. *If $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 4)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X and that $\widetilde{f}_{\inf}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\inf}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{l}}(x * y) = \widetilde{f}_{\sup}(x * y) - \widetilde{f}_{\inf}(1),$$

so

$$\begin{aligned}\widetilde{f}_{\sup}(x*y) &= \widetilde{f}_{\mathfrak{l}}(x*y)+\widetilde{f}_{\inf}(1)\\ &\leqslant \max\{\widetilde{f}_{\mathfrak{l}}(x),\widetilde{f}_{\mathfrak{l}}(y)\}+\widetilde{f}_{\inf}(1)\\ &= \max\{\widetilde{f}_{\mathfrak{l}}(x)+\widetilde{f}_{\inf}(1),\widetilde{f}_{\mathfrak{l}}(y)+\widetilde{f}_{\inf}(1)\}\\ &= \max\{\widetilde{f}_{\mathfrak{l}}(x)+\widetilde{f}_{\inf}(x),\widetilde{f}_{\mathfrak{l}}(y)+\widetilde{f}_{\inf}(y)\}\\ &= \max\{\widetilde{f}_{\sup}(x),\widetilde{f}_{\sup}(y)\}.\end{aligned}$$

Hence, $(X,\widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X. Therefore, $(X,\widetilde{f})$ is a $(k,4)$-hyperfuzzy subalgebra of X. □

Corollary 4.12. *If $(X,\widetilde{f})$ is a length 2-fuzzy subalgebra of a Hilbert algebra $X=(X,*,1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X,\widetilde{f})$ is a $(k,4)$-hyperfuzzy subalgebra of X for $k\in\{1,2,3,4\}$.*

Proof. It follows from Theorems 4.2 and 4.13. □

Theorem 4.14. *If $(X,\widetilde{f})$ is a length 1-fuzzy subalgebra of a Hilbert algebra $X=(X,*,1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X,\widetilde{f})$ is a $(4,k)$-hyperfuzzy subalgebra of X for $k\in\{1,2,3,4\}$.*

Proof. Assume that $(X,\widetilde{f})$ is a length 1-fuzzy subalgebra of X and that $\widetilde{f}_{\sup}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\sup}$ is a k-fuzzy subalgebra of X for every $k\in\{1,2,3,4\}$. Also,

$$\widetilde{f}_{\sup}(x)=\widetilde{f}_{\sup}(1)\qquad\text{for all } x\in X.$$

For $x,y\in X$, we have

$$\widetilde{f}_{\mathfrak{l}}(x*y)=\widetilde{f}_{\sup}(1)-\widetilde{f}_{\inf}(x*y),$$

so

$$\begin{aligned}\widetilde{f}_{\inf}(x*y) &= \widetilde{f}_{\sup}(1)-\widetilde{f}_{\mathfrak{l}}(x*y)\\ &\leqslant \widetilde{f}_{\sup}(1)-\min\{\widetilde{f}_{\mathfrak{l}}(x),\widetilde{f}_{\mathfrak{l}}(y)\}\\ &= \max\{\widetilde{f}_{\sup}(1)-\widetilde{f}_{\mathfrak{l}}(x),\widetilde{f}_{\sup}(1)-\widetilde{f}_{\mathfrak{l}}(y)\}\\ &= \max\{\widetilde{f}_{\sup}(x)-\widetilde{f}_{\mathfrak{l}}(x),\widetilde{f}_{\sup}(y)-\widetilde{f}_{\mathfrak{l}}(y)\}\\ &= \max\{\widetilde{f}_{\inf}(x),\widetilde{f}_{\inf}(y)\}.\end{aligned}$$

Hence, $(X,\widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X. Therefore, $(X,\widetilde{f})$ is a $(4,k)$-hyperfuzzy subalgebra of X. □

Corollary 4.13. *If $(X,\widetilde{f})$ is a length 3-fuzzy subalgebra of a Hilbert algebra $X=(X,*,1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X,\widetilde{f})$ is a $(4,k)$-hyperfuzzy subalgebra of X for $k\in\{1,2,3,4\}$.*

Proof. It follows from Theorems 4.1 and 4.14. □

Theorem 4.15. *If $(X,\widetilde{f})$ is a length 4-fuzzy subalgebra of a Hilbert algebra $X=(X,*,1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X,\widetilde{f})$ is a $(1,k)$-hyperfuzzy subalgebra of X for $k\in\{1,2,3,4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a length 4-fuzzy subalgebra of X and that $\widetilde{f}_{\sup}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\sup}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{l}}(x * y) = \widetilde{f}_{\sup}(1) - \widetilde{f}_{\inf}(x * y),$$

so

$$\begin{aligned} \widetilde{f}_{\inf}(x * y) &= \widetilde{f}_{\sup}(1) - \widetilde{f}_{\mathfrak{l}}(x * y) \\ &\geqslant \widetilde{f}_{\sup}(1) - \max\{\widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\mathfrak{l}}(y)\} \\ &= \min\{\widetilde{f}_{\sup}(1) - \widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\sup}(1) - \widetilde{f}_{\mathfrak{l}}(y)\} \\ &= \min\{\widetilde{f}_{\sup}(x) - \widetilde{f}_{\mathfrak{l}}(x), \widetilde{f}_{\sup}(y) - \widetilde{f}_{\mathfrak{l}}(y)\} \\ &= \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}. \end{aligned}$$

Hence, $(X, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(1, k)$-hyperfuzzy subalgebra of X. □

Corollary 4.14. *If $(X, \widetilde{f})$ is a length 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X, \widetilde{f})$ is a $(1, k)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 4.2 and 4.15. □

5. Mean of a Hyperstructure in Hilbert Algebras

In this section, we study the mean associated with a hyperstructure over a Hilbert algebra and use it to define several classes of mean fuzzy subalgebras. Our aim is to describe how the induced fuzzy set $\widetilde{f}_{\mathfrak{m}}$ behaves under the binary operation of a Hilbert algebra, and to identify conditions under which it becomes a k-fuzzy subalgebra for $k \in \{1, 2, 3, 4\}$. We also characterize mean fuzzy subalgebras in terms of upper-, lower-, and equal-level subsets, and examine their relations with hyperfuzzy subalgebras under suitable assumptions on $\widetilde{f}_{\inf}$ and $\widetilde{f}_{\sup}$.

Definition 5.1. *Let $(X, \widetilde{f})$ be a hyperstructure over a non-empty set X. We define a fuzzy set $(X, \widetilde{f}_{\mathfrak{m}})$ on X by*

$$\widetilde{f}_{\mathfrak{m}} : X \to [0, 1], \qquad x \mapsto \frac{\widetilde{f}_{\sup}(x) + \widetilde{f}_{\inf}(x)}{2}.$$

The fuzzy set $\widetilde{f}_{\mathfrak{m}}$ is called the mean of $\widetilde{f}$.

Definition 5.2. *Let $(X, \widetilde{f})$ be a hyperstructure over a Hilbert algebra $X = (X, *, 1)$. Then $(X, \widetilde{f})$ is called a mean 1-fuzzy (resp., mean 2-fuzzy, mean 3-fuzzy, mean 4-fuzzy) subalgebra of X if $(X, \widetilde{f}_{\mathfrak{m}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, 4-fuzzy) subalgebra of X.*

Proposition 5.1. *If $(X, \widetilde{f})$ is a mean k-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ for $k = 1, 3$, then*

$$(\forall x \in X)\big(\widetilde{f}_{\mathfrak{m}}(1) \geqslant \widetilde{f}_{\mathfrak{m}}(x)\big).$$

Proof. It follows directly from Proposition 3.1. □

Proposition 5.2. *If $(X, \widetilde{f})$ is a mean k-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ for $k = 2, 4$, then*

$$(\forall x \in X)\big(\widetilde{f}_{\mathfrak{m}}(1) \leqslant \widetilde{f}_{\mathfrak{m}}(x)\big).$$

Proof. It follows directly from Proposition 3.1. □

Theorem 5.1. *Every mean 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy subalgebra of X.*

Proof. This is a direct consequence of Theorem 3.1. □

Example 5.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the Hilbert algebra given in Example 3.1, and let $f : X \to [0, 1]$ be the fuzzy set defined there. Define a hyperstructure $(X, \widetilde{f})$ by*

$$\widetilde{f}(x) = \{f(x)\} \qquad \text{for all } x \in X.$$

Then $\widetilde{f}_{\mathfrak{m}} = f$. Since f is a 1-fuzzy subalgebra of X but not a 3-fuzzy subalgebra of X, it follows that $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X but not a mean 3-fuzzy subalgebra of X. Hence, the converse of Theorem 5.1 does not hold in general.

Theorem 5.2. *Every mean 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy subalgebra of X.*

Proof. This is a direct consequence of Theorem 3.2. □

Example 5.2. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the Hilbert algebra given in Example 3.2, and let $f : X \to [0, 1]$ be the fuzzy set defined there. Define a hyperstructure $(X, \widetilde{f})$ by*

$$\widetilde{f}(x) = \{f(x)\} \qquad \text{for all } x \in X.$$

Then $\widetilde{f}_{\mathfrak{m}} = f$. Since f is a 4-fuzzy subalgebra of X but not a 2-fuzzy subalgebra of X, it follows that $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X but not a mean 2-fuzzy subalgebra of X. Hence, the converse of Theorem 5.2 does not hold in general.

Theorem 5.3. *For a hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$, the following statements are equivalent:*

(1) *$(X, \widetilde{f})$ is a mean 2-fuzzy subalgebra of X;*
(2) *$(X, \widetilde{f})$ is a mean 3-fuzzy subalgebra of X;*

(3) *$\widetilde{f}_{\mathfrak{m}}$ is constant on X.*

Proof. This follows directly from Theorems 3.3 and 3.4, applied to the fuzzy set $\widetilde{f}_{\mathfrak{m}}$. □

Theorem 5.4. *Given a subalgebra S of a Hilbert algebra $X = (X, *, 1)$ and $B_1, B_2 \in \widetilde{P}([0,1])$, let $(X, \widetilde{f})$ be a hyperstructure over X defined by*

$$\widetilde{f} : X \to \widetilde{P}([0,1]), \qquad x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{if } x \notin S. \end{cases}$$

Then:

(1) *if $\sup B_2 \geqslant \sup B_1$ and $\inf B_2 \geqslant \inf B_1$, then $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X;*

(2) *if $\sup B_2 \leqslant \sup B_1$ and $\inf B_2 \leqslant \inf B_1$, then $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X.*

Proof. If $x \in S$, then

$$\widetilde{f}_{\mathfrak{m}}(x) = \frac{\sup B_2 + \inf B_2}{2},$$

while for $x \notin S$,

$$\widetilde{f}_{\mathfrak{m}}(x) = \frac{\sup B_1 + \inf B_1}{2}.$$

(1) Assume that $\sup B_2 \geqslant \sup B_1$ and $\inf B_2 \geqslant \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \geqslant \frac{\sup B_1 + \inf B_1}{2}.$$

If $x, y \in S$, then $x * y \in S$, and so

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\sup B_2 + \inf B_2}{2} = \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

If at least one of x, y does not belong to S, then

$$\min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} = \frac{\sup B_1 + \inf B_1}{2},$$

while $\widetilde{f}_{\mathfrak{m}}(x * y)$ is either $\dfrac{\sup B_1 + \inf B_1}{2}$ or $\dfrac{\sup B_2 + \inf B_2}{2}$. Hence,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \geqslant \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{m}}$ is a 1-fuzzy subalgebra of X.

(2) Assume that $\sup B_2 \leqslant \sup B_1$ and $\inf B_2 \leqslant \inf B_1$. Then

$$\frac{\sup B_2 + \inf B_2}{2} \leqslant \frac{\sup B_1 + \inf B_1}{2}.$$

If $x, y \in S$, then $x * y \in S$, and so

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\sup B_2 + \inf B_2}{2} = \max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

If at least one of x, y does not belong to S, then

$$\max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} = \frac{\sup B_1 + \inf B_1}{2},$$

while $\widetilde{f}_{\mathfrak{m}}(x * y)$ is either $\dfrac{\sup B_1 + \inf B_1}{2}$ or $\dfrac{\sup B_2 + \inf B_2}{2}$. Hence,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \leqslant \max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{m}}$ is a 4-fuzzy subalgebra of X. □

Theorem 5.5. *A hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy subalgebra of X if and only if the set $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for all $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X. Let $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$, and let $x, y \in \mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$. Then

$$\widetilde{f}_{\mathfrak{m}}(x) \geqslant t \qquad \text{and} \qquad \widetilde{f}_{\mathfrak{m}}(y) \geqslant t.$$

Since $\widetilde{f}_{\mathfrak{m}}$ is a 1-fuzzy subalgebra,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \geqslant \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} \geqslant t.$$

Thus, $x * y \in \mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$, so $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X.

Conversely, assume that $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for every $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$. Let $x, y \in X$, and put

$$t = \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Then $x, y \in \mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$, so $x * y \in \mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$. Hence,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \geqslant t = \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{m}}$ is a 1-fuzzy subalgebra of X. □

Corollary 5.1. *If $(X, \widetilde{f})$ is a mean 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for all $t \in [0, 1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$.*

Proof. It follows from Theorems 5.1 and 5.5. □

The following example shows that the converse of Corollary 5.1 does not hold in general.

Example 5.3. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ given in Table 1. Define a fuzzy set g on X by*

$$g(1) = 0.6, \qquad g(x) = 0.3, \qquad g(y) = 0.5, \qquad g(z) = 0.4, \qquad g(0) = 0.3.$$

Let $(X, \widetilde{f})$ be the singleton hyperstructure over $(X, *, 1)$ given by

$$\widetilde{f}(u) = \{g(u)\} \qquad (u \in X).$$

Then

$$\widetilde{f}_{\inf} = g, \qquad \widetilde{f}_{\sup} = g,$$

and hence

$$\widetilde{f}_{\mathfrak{m}} = g.$$

Moreover,

$$\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) = \begin{cases} \varnothing & \text{if } t \in (0.6, 1], \\ \{1\} & \text{if } t \in (0.5, 0.6], \\ \{1, y\} & \text{if } t \in (0.4, 0.5], \\ \{1, y, z\} & \text{if } t \in (0.3, 0.4], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Thus, $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for every $t \in [0,1]$ with $\mathcal{U}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$. However,

$$\widetilde{f}_{\mathfrak{m}}(1 * x) = \widetilde{f}_{\mathfrak{m}}(x) = 0.3 < 0.6 = \max\{\widetilde{f}_{\mathfrak{m}}(1), \widetilde{f}_{\mathfrak{m}}(x)\}.$$

Therefore, $(X, \widetilde{f})$ is not a mean 3-fuzzy subalgebra of X.

Theorem 5.6. *A hyperstructure $(X, \widetilde{f})$ over a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy subalgebra of X if and only if the set $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for all $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X. Let $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$, and let $x, y \in \mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$. Then

$$\widetilde{f}_{\mathfrak{m}}(x) \leqslant t \qquad \text{and} \qquad \widetilde{f}_{\mathfrak{m}}(y) \leqslant t.$$

Since $\widetilde{f}_{\mathfrak{m}}$ is a 4-fuzzy subalgebra,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \leqslant \max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} \leqslant t.$$

Thus, $x * y \in \mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$, so $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X.

Conversely, assume that $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for every $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$. Let $x, y \in X$, and put

$$t = \max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Then $x, y \in \mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$, so $x * y \in \mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$. Hence,

$$\widetilde{f}_{\mathfrak{m}}(x * y) \leqslant t = \max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.$$

Therefore, $\widetilde{f}_{\mathfrak{m}}$ is a 4-fuzzy subalgebra of X. □

Corollary 5.2. *If $(X, \widetilde{f})$ is a mean 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$, then the set $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$ is a subalgebra of X for all $t \in [0,1]$ with $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$.*

Proof. It follows from Theorems 5.2 and 5.6. □

The following example shows that the converse of Corollary 5.2 does not hold in general.

Example 5.4. *Let* $X = \{1, x, y, z, 0\}$ *be a Hilbert algebra with the binary operation* $*$ *given in Table 1. Define a fuzzy set* h *on* X *by*

$$h(1) = 0.4, \qquad h(x) = 0.5, \qquad h(y) = 0.5, \qquad h(z) = 0.7, \qquad h(0) = 0.75.$$

Let $(X, \widetilde{f})$ *be the singleton hyperstructure over* $(X, *, 1)$ *given by*

$$\widetilde{f}(u) = \{h(u)\} \qquad (u \in X).$$

Then

$$\widetilde{f}_{\inf} = h, \qquad \widetilde{f}_{\sup} = h,$$

and hence

$$\widetilde{f}_{\mathfrak{m}} = h.$$

Moreover,

$$\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) = \begin{cases} \varnothing & \text{if } t \in [0, 0.4), \\ \{1\} & \text{if } t \in [0.4, 0.5), \\ \{1, x, y\} & \text{if } t \in [0.5, 0.7), \\ \{1, x, y, z\} & \text{if } t \in [0.7, 0.75), \\ X & \text{if } t \in [0.75, 1]. \end{cases}$$

Thus, $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t)$ *is a subalgebra of* X *for every* $t \in [0, 1]$ *with* $\mathcal{L}(\widetilde{f}_{\mathfrak{m}}, t) \neq \varnothing$*. However,*

$$\widetilde{f}_{\mathfrak{m}}(1 * x) = \widetilde{f}_{\mathfrak{m}}(x) = 0.5 > 0.4 = \min\{\widetilde{f}_{\mathfrak{m}}(1), \widetilde{f}_{\mathfrak{m}}(x)\}.$$

Therefore, $(X, \widetilde{f})$ *is not a mean* 2*-fuzzy subalgebra of* X*.*

Theorem 5.7. *A hyperstructure* $(X, \widetilde{f})$ *over a Hilbert algebra* $X = (X, *, 1)$ *is a mean* $2(3)$*-fuzzy subalgebra of* X *if and only if*

$$\mathcal{E}\big(\widetilde{f}_{\mathfrak{m}}, \widetilde{f}_{\mathfrak{m}}(1)\big) = X.$$

Proof. Assume that $(X, \widetilde{f})$ is a mean 2-fuzzy or mean 3-fuzzy subalgebra of X. Then $\widetilde{f}_{\mathfrak{m}}$ is a 2-fuzzy or 3-fuzzy subalgebra of X. By Theorems 3.3 and 3.4, $\widetilde{f}_{\mathfrak{m}}$ is constant. Hence,

$$\widetilde{f}_{\mathfrak{m}}(x) = \widetilde{f}_{\mathfrak{m}}(1) \qquad \text{for all } x \in X,$$

so $\mathcal{E}(\widetilde{f}_{\mathfrak{m}}, \widetilde{f}_{\mathfrak{m}}(1)) = X$.

Conversely, if $\mathcal{E}(\widetilde{f}_{\mathfrak{m}}, \widetilde{f}_{\mathfrak{m}}(1)) = X$, then $\widetilde{f}_{\mathfrak{m}}$ is constant on X. By Theorems 3.3 and 3.4, $\widetilde{f}_{\mathfrak{m}}$ is both a 2-fuzzy and a 3-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a mean $2(3)$-fuzzy subalgebra of X. □

Theorem 5.8. *If* $(X, \widetilde{f})$ *is a hyperstructure over a Hilbert algebra* $X = (X, *, 1)$ *in which* $(X, \widetilde{f}_{\inf})$ *is constant and* $(X, \widetilde{f}_{\sup})$ *is a* 1*-fuzzy subalgebra of* X*, then* $(X, \widetilde{f})$ *is a mean* 1*-fuzzy subalgebra of* X*.*

Proof. Let $c = \widetilde{f}_{\inf}(1)$. Since $(X, \widetilde{f}_{\inf})$ is constant, we have

$$\widetilde{f}_{\inf}(x) = c \qquad \text{for all } x \in X.$$

Let $x, y \in X$. Since $(X, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of X,

$$\widetilde{f}_{\sup}(x * y) \geqslant \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.$$

Therefore,

$$\begin{aligned}
\widetilde{f}_{\mathfrak{m}}(x * y) &= \frac{\widetilde{f}_{\sup}(x * y) + \widetilde{f}_{\inf}(x * y)}{2} \\
&= \frac{\widetilde{f}_{\sup}(x * y) + c}{2} \\
&\geqslant \frac{\min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\} + c}{2} \\
&= \min\left\{\frac{\widetilde{f}_{\sup}(x) + c}{2}, \frac{\widetilde{f}_{\sup}(y) + c}{2}\right\} \\
&= \min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\mathfrak{m}})$ is a 1-fuzzy subalgebra of X. □

Corollary 5.3. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\inf})$ is constant and $(X, \widetilde{f}_{\sup})$ is a 3-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.1 and 5.8. □

Corollary 5.4. *For $j \in \{1, 3\}$, every $(2(3), j)$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy subalgebra of X.*

Proof. It follows from Theorem 5.8 and Corollary 5.3. □

Theorem 5.9. *If $(X, \widetilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \widetilde{f}_{\inf})$ is constant and $(X, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X, then $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X.*

Proof. Let $c = \widetilde{f}_{\inf}(1)$. Since $(X, \widetilde{f}_{\inf})$ is constant, we have

$$\widetilde{f}_{\inf}(x) = c \qquad \text{for all } x \in X.$$

Let $x, y \in X$. Since $(X, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X,

$$\widetilde{f}_{\sup}(x * y) \leqslant \max\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.$$

Therefore,

$$\begin{aligned}\widetilde{f}_{\mathfrak{m}}(x*y) &= \frac{\widetilde{f}_{\sup}(x*y)+\widetilde{f}_{\inf}(x*y)}{2}\\ &= \frac{\widetilde{f}_{\sup}(x*y)+c}{2}\\ &\leqslant \frac{\max\{\widetilde{f}_{\sup}(x),\widetilde{f}_{\sup}(y)\}+c}{2}\\ &= \max\left\{\frac{\widetilde{f}_{\sup}(x)+c}{2},\frac{\widetilde{f}_{\sup}(y)+c}{2}\right\}\\ &= \max\{\widetilde{f}_{\mathfrak{m}}(x),\widetilde{f}_{\mathfrak{m}}(y)\}.\end{aligned}$$

Hence, $(X,\widetilde{f}_{\mathfrak{m}})$ is a 4-fuzzy subalgebra of X. □

Corollary 5.5. *If $(X,\widetilde{f})$ is a hyperstructure over a Hilbert algebra $X=(X,*,1)$ in which $(X,\widetilde{f}_{\inf})$ is constant and $(X,\widetilde{f}_{\sup})$ is a 2-fuzzy subalgebra of X, then $(X,\widetilde{f})$ is a mean 4-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.2 and 5.9. □

Corollary 5.6. *For $j\in\{2,4\}$, every $(2(3),j)$-hyperfuzzy subalgebra of a Hilbert algebra $X=(X,*,1)$ is a mean 4-fuzzy subalgebra of X.*

Proof. It follows from Theorem 5.9 and Corollary 5.5. □

Theorem 5.10. *If $(X,\widetilde{f})$ is a hyperstructure over a Hilbert algebra $X=(X,*,1)$ in which $(X,\widetilde{f}_{\sup})$ is constant and $(X,\widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X, then $(X,\widetilde{f})$ is a mean 4-fuzzy subalgebra of X.*

Proof. Let $c=\widetilde{f}_{\sup}(1)$. Since $(X,\widetilde{f}_{\sup})$ is constant, we have

$$\widetilde{f}_{\sup}(x)=c \qquad \text{for all } x\in X.$$

Let $x,y\in X$. Since $(X,\widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X,

$$\widetilde{f}_{\inf}(x*y)\leqslant\max\{\widetilde{f}_{\inf}(x),\widetilde{f}_{\inf}(y)\}.$$

Therefore,

$$\begin{aligned}\widetilde{f}_{\mathfrak{m}}(x*y) &= \frac{\widetilde{f}_{\sup}(x*y)+\widetilde{f}_{\inf}(x*y)}{2}\\ &= \frac{c+\widetilde{f}_{\inf}(x*y)}{2}\\ &\leqslant \frac{c+\max\{\widetilde{f}_{\inf}(x),\widetilde{f}_{\inf}(y)\}}{2}\\ &= \max\left\{\frac{c+\widetilde{f}_{\inf}(x)}{2},\frac{c+\widetilde{f}_{\inf}(y)}{2}\right\}\\ &= \max\{\widetilde{f}_{\mathfrak{m}}(x),\widetilde{f}_{\mathfrak{m}}(y)\}.\end{aligned}$$

Hence, $(X,\widetilde{f}_{\mathfrak{m}})$ is a 4-fuzzy subalgebra of X. □

Corollary 5.7. *If $(X, \tilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 2-fuzzy subalgebra of X, then $(X, \tilde{f})$ is a mean 4-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.2 and 5.10. □

Corollary 5.8. *For $i \in \{2, 4\}$, every $(i, 2(3))$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a mean 4-fuzzy subalgebra of X.*

Proof. It follows from Theorem 5.10 and Corollary 5.7. □

Theorem 5.11. *If $(X, \tilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy subalgebra of X.*

Proof. Let $c = \tilde{f}_{\sup}(1)$. Since $(X, \tilde{f}_{\sup})$ is constant, we have

$$\tilde{f}_{\sup}(x) = c \qquad \text{for all } x \in X.$$

Let $x, y \in X$. Since $(X, \tilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X,

$$\tilde{f}_{\inf}(x * y) \geqslant \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.$$

Therefore,

$$\begin{aligned}
\tilde{f}_{\mathfrak{m}}(x * y) &= \frac{\tilde{f}_{\sup}(x * y) + \tilde{f}_{\inf}(x * y)}{2} \\
&= \frac{c + \tilde{f}_{\inf}(x * y)}{2} \\
&\geqslant \frac{c + \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}}{2} \\
&= \min\left\{\frac{c + \tilde{f}_{\inf}(x)}{2}, \frac{c + \tilde{f}_{\inf}(y)}{2}\right\} \\
&= \min\{\tilde{f}_{\mathfrak{m}}(x), \tilde{f}_{\mathfrak{m}}(y)\}.
\end{aligned}$$

Hence, $(X, \tilde{f}_{\mathfrak{m}})$ is a 1-fuzzy subalgebra of X. □

Corollary 5.9. *If $(X, \tilde{f})$ is a hyperstructure over a Hilbert algebra $X = (X, *, 1)$ in which $(X, \tilde{f}_{\sup})$ is constant and $(X, \tilde{f}_{\inf})$ is a 3-fuzzy subalgebra of X, then $(X, \tilde{f})$ is a mean 1-fuzzy subalgebra of X.*

Proof. It follows from Theorems 3.1 and 5.11. □

Corollary 5.10. *For $i \in \{1, 3\}$, every $(i, 2(3))$-hyperfuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ is a mean 1-fuzzy subalgebra of X.*

Proof. It follows from Theorem 5.11 and Corollary 5.9. □

Theorem 5.12. *If $(X, \tilde{f})$ is a mean 1-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\tilde{f}_{\inf}$ is constant, then $(X, \tilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X and that $\widetilde{f}_{\inf}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\inf}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\widetilde{f}_{\sup}(x * y) + \widetilde{f}_{\inf}(1)}{2},$$

so

$$\begin{aligned}
\widetilde{f}_{\sup}(x * y) &= 2\widetilde{f}_{\mathfrak{m}}(x * y) - \widetilde{f}_{\inf}(1) \\
&\geqslant 2\min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} - \widetilde{f}_{\inf}(1) \\
&= \min\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\inf}(1),\, 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\inf}(1)\} \\
&= \min\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\inf}(x),\, 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\inf}(y)\} \\
&= \min\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\sup})$ is a 1-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X. □

Corollary 5.11. *If $(X, \widetilde{f})$ is a mean 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 1)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.1 and 5.12. □

Theorem 5.13. *If $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 4)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X and that $\widetilde{f}_{\inf}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\inf}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\inf}(x) = \widetilde{f}_{\inf}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\widetilde{f}_{\sup}(x * y) + \widetilde{f}_{\inf}(1)}{2},$$

so

$$\begin{aligned}
\widetilde{f}_{\sup}(x * y) &= 2\widetilde{f}_{\mathfrak{m}}(x * y) - \widetilde{f}_{\inf}(1) \\
&\leqslant 2\max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} - \widetilde{f}_{\inf}(1) \\
&= \max\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\inf}(1),\, 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\inf}(1)\} \\
&= \max\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\inf}(x),\, 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\inf}(y)\} \\
&= \max\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\sup})$ is a 4-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(k, 4)$-hyperfuzzy subalgebra of X. □

Corollary 5.12. *If $(X, \widetilde{f})$ is a mean 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\inf}$ is constant, then $(X, \widetilde{f})$ is a $(k, 4)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.2 and 5.13. □

Theorem 5.14. *If $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X, \widetilde{f})$ is a $(4, k)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 4-fuzzy subalgebra of X and that $\widetilde{f}_{\sup}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\sup}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\widetilde{f}_{\sup}(1) + \widetilde{f}_{\inf}(x * y)}{2},$$

so

$$\begin{aligned}
\widetilde{f}_{\inf}(x * y) &= 2\widetilde{f}_{\mathfrak{m}}(x * y) - \widetilde{f}_{\sup}(1) \\
&\leqslant 2\max\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} - \widetilde{f}_{\sup}(1) \\
&= \max\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\sup}(1),\ 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\sup}(1)\} \\
&= \max\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\sup}(x),\ 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\sup}(y)\} \\
&= \max\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}.
\end{aligned}$$

Hence, $(X, \widetilde{f}_{\inf})$ is a 4-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(4, k)$-hyperfuzzy subalgebra of X. □

Corollary 5.13. *If $(X, \widetilde{f})$ is a mean 2-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X, \widetilde{f})$ is a $(4, k)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.2 and 5.14. □

Theorem 5.15. *If $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X, \widetilde{f})$ is a $(1, k)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(X, \widetilde{f})$ is a mean 1-fuzzy subalgebra of X and that $\widetilde{f}_{\sup}$ is constant. By Theorems 3.1, 3.2, 3.3, and 3.4, the fuzzy set $\widetilde{f}_{\sup}$ is a k-fuzzy subalgebra of X for every $k \in \{1, 2, 3, 4\}$. Also,

$$\widetilde{f}_{\sup}(x) = \widetilde{f}_{\sup}(1) \qquad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$\widetilde{f}_{\mathfrak{m}}(x * y) = \frac{\widetilde{f}_{\sup}(1) + \widetilde{f}_{\inf}(x * y)}{2},$$

so

$$\begin{aligned}\widetilde{f}_{\inf}(x * y) &= 2\widetilde{f}_{\mathfrak{m}}(x * y) - \widetilde{f}_{\sup}(1)\\ &\geqslant 2\min\{\widetilde{f}_{\mathfrak{m}}(x), \widetilde{f}_{\mathfrak{m}}(y)\} - \widetilde{f}_{\sup}(1)\\ &= \min\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\sup}(1),\ 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\sup}(1)\}\\ &= \min\{2\widetilde{f}_{\mathfrak{m}}(x) - \widetilde{f}_{\sup}(x),\ 2\widetilde{f}_{\mathfrak{m}}(y) - \widetilde{f}_{\sup}(y)\}\\ &= \min\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\}.\end{aligned}$$

Hence, $(X, \widetilde{f}_{\inf})$ is a 1-fuzzy subalgebra of X. Therefore, $(X, \widetilde{f})$ is a $(1, k)$-hyperfuzzy subalgebra of X. □

Corollary 5.14. *If $(X, \widetilde{f})$ is a mean 3-fuzzy subalgebra of a Hilbert algebra $X = (X, *, 1)$ in which $\widetilde{f}_{\sup}$ is constant, then $(X, \widetilde{f})$ is a $(1, k)$-hyperfuzzy subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof. It follows from Theorems 5.1 and 5.15. □

6. Conclusion

This paper introduced the length and the mean of a hyperstructure over a Hilbert algebra and used them to define the corresponding notions of length and mean fuzzy subalgebras. Several characterization results were established for these classes in the setting of k-fuzzy subalgebras, where $k \in \{1, 2, 3, 4\}$, together with their basic relations to upper-, lower-, and equal-level subsets.

We also obtained links between these new fuzzy subalgebras and hyperfuzzy subalgebras under suitable conditions on the associated lower and upper fuzzy structures. The examples and counterexamples given in the paper support the main results and clarify the limitations of the converse statements. Hence, the results provide a workable framework for studying quantitative hyperfuzzy behavior in Hilbert algebras and may be useful for further investigations in related algebraic systems.

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